MATHEMATICA MORAVICA
VOL. 2 (1998), 121–132

SCHAUDER'S 54th PROBLEM IN SCOTTISH BOOK

Milan R. Tasković

Abstract. The most famous of many problems in nonlinear analysis is Schauder's problem (*Scottish book*, problem **54**) of the following form, that if C is a nonempty convex compact subset of a linear topological space does every continuous mapping $f : C \to C$ has a fixed point?

The answer we give in this paper is yes. In this connection this paper proves and extends the Markoff-Kakutani theorem to arbitrary linear topological space as an immediate consequence of the preceding solution of Schauder's problem.

During the last twenty years this old conjecture was intensively examined by many mathematians. For sets in normed spaces this has been proved by Schauder and for sets in locally convex spaces by Tychonoff.

In this paper we prove that if C is a nonempty convex compact subset of a linear topological space, then every continuous mapping $f : C \to C$ has a fixed point.

On the other hand, in this sense, we extend and connected former results of Brouwer, Schauder, Tychonoff, Markoff, Kakutani, Darbo, Sadovskij, Browder, Krasnoselskij, Ky Fan, Reinermann, Hukuhara, Mazur, Hahn, Ryll-Nardzewski, Day, Riedrich, Jahn, Eisenack-Fenske, Idzik, Kirk, Göhde, Granas, Dugundji, Klee and some others.

AMS (1980) Mathematics Subject Classification (1985 Revision). Primary: 47H10. Secondary: 54H25, 52A07, 55M20.

Key words and phrases: Fixed point theorems, Brouwer's theorem, Schauder's theorem, Tychonof's theorem, Markoff's theorem, Kakutani's theorem, Sadovskij's theorem, Schauder's Conjecture (*Scottish Book*, Problem **54**).

1. Introduction and history

The problem of fixpoint for a given mapping $f|X$ is very easy to formulate: the question is if some $\xi \in X$ verifies $f(\xi) = \xi$. It is interesting that many problems are reducible to the existence of fixpoints of certain mappings.

Brouwer's theorem of fixed point is one of the oldest and best known results in mathematics.

Schauder's theorem of fixed point is a generalization of Brouwer's theorem to infinite dimensional normed linear spaces. Schauder's theorem states that every continuous mapping of a compact convex subset of a normed linear space into itself has a fixed point.

Schauder's problem (*Scottish book*, problem **54**) is the following form: *Does every continuous mapping $f : C \to C$ of a nonempty convex compact subset C in arbitrary linear topological space have a fixed point?*

For locally convex spaces the answer is yes from Tychonoff [Ty]. Naimely, in 1935, Tychonoff had shown that if C is a nonempty convex compact subset of a locally convex space, then every continuous map $f : C \to C$ has a fixed point.

Schauder's theorem further was extended by Hukuhara [Hu], Mazur [Mz], Göhde [Gö], Fan [Fa], Dugundji [Dj], Granas [Gr], Klee [Ke], Kirk [Ki], Idzik [Iz], Riedrich [Ri], Eisenack-Fenske [Ef], Jahn [Ja], Browder [Bo], Darbo [Da], Sadovskij [Sa], Krasnoselskij [Kr], Reinermann [Re] and many others.

Literature on applications of the Schauder theorem to nonlinear problems is extensive. The first result was proved by Markoff [Ma] with the aid of the Schauder-Tychonoff fixed point theorem.

Kakutani [Ka] found a direct elementary proof of the Markoff theorem. Extensions of Markoff-Kakutani theorem is due to Day [Dy], Hahn [Ha], and Ryll-Nardzewski [Ry].

In this paper we give the complete solution of the preceding well known Schauder's problem fixed point. Also, this solution is answering a question of S. Ulam. In connection with this, in this paper, we extend the Markoff-Kakutani theorem to arbitrary linear topological spaces as an immediate consequence of the preceding solution of Schauder's problem.

On the other hand, in this sense, we extend and connected former results of Brouwer, Schauder, Tychonoff, Markoff, Kakutani, Darbo, Sadovskij, Krasnoselskij, Browder, Ky Fan, Reinermann, Hahn, Ryll-Nardzewski, Granas, Dugundji, Hukuhara, Mazur, Riedrich, Jahn, Eisenack--Fenske, Day and some others.

2. Main results and other facts

In connection with the preceding, let X topological space, let $T : X \to X$ and let $A : X \times X \to \mathbf{R}_+^0 := [0, +\infty)$ be a function. A topological space X satisfies the condition of **CS-konvergence** if there exists a sequence $\{x_n\}_{n\in\mathbf{N}}$ in X such that $A(x_n, x_{n+1}) \to 0 (n \to \infty)$ implies that $\{x_n\}_{n\in\mathbf{N}}$ has a convergent subsequence.

On the other hand, a function T satisfies the condition of **general A-variation** if there exists a continuous function $G : X \to \mathbf{R}_+^0$ and for any $x \in X$, with $x \neq Tx$, there exists $y \in X\backslash\{x\}$ such that

$$A(x,y) \leq |G(x) - G(y)| \tag{AG}$$

for some function $A : X \times X \to \mathbf{R}_+^0$ with property $A(a,c) \leq A(a,b) + A(b,c)$ for all $a, b, c \in X$.

We are now in a position to formulate the following general statement, which is an extension to former results of Brouwer, Schauder, Tychonoff and some others.

Theorem 1. (General A-variation Principle). *Let T be a general A-variation mapping of topological space X into itself, where X satisfies the condition of CS-convergence. If $y \mapsto A(x,y)$ is continuous and if $A(x,y) = 0$ iff $x = y$, then T has a fixed point $\xi \in X$.*

Proof. As is well known, the use of Zorn's lemma may be replaced by an induction argument (involving the Axiom of Choice) along the following lines. In this sense defines

$$R = \{Q \subset X : A(x,y) \leq |G(x) - G(y)| \quad \text{for all } x, y \in Q\}.$$

It is easy to verify that $(R, \preceq)$ is a partially ordered set (asymmetric and transitive relation), where $Q_1 \preceq Q_2$ iff $Q_1 \subset Q_2$. Namely, in view of Zorn's lemma, there exists a maximal set $M \subset R$ such that

$$A(x,y) \leq |G(x) - G(y)| \quad \text{for all } x, y \in M. \tag{1}$$

Denote by α the greatest lower bound of the set $\{G(x): x \in M\}$, i.e., $\alpha := \inf\{G(x): x \in M\}$. Thus there exists a sequence of points $\{a_n\}_{n\in \mathbf{N}}$ from M such that $\{G(a_n)\}_{n\in\mathbf{N}}$ is decreasing and $G(a_n) \to \alpha \ (n \to \infty)$. It follows from (1) and from

$$A(a_n, a_{n+1}) \leq |G(a_n) - G(a_{n+1})|$$

that $A(a_n, a_{n+1}) \to 0 \ (n \to \infty)$. This implies (from CS-convergence) that its sequence $\{a_n\}_{n\in\mathbf{N}}$ contains a convergent subsequence $\{a_{n(k)}\}_{k\in\mathbf{N}}$ with limit $\xi \in X$.

For any $x \in M$, if $G(x) \neq \alpha$, then for sufficiently large k we have the following inequalities

$$A(\xi, x) \leq A(\xi, a_{n(k)}) + A(a_{n(k)}, x) \leq A(\xi, a_{n(k)}) + |G(x) - G(a_{n(k)})|.$$

If $G(b) = \alpha$ for some $b \in M$, then we obtain in a similar way $A(\xi, b) = 0$. For any $x \in M$, if $G(x) \neq \alpha$, then we have

$$A(x, a_{n(k)}) \leq |G(x) - G(a_{n(k)})|$$

and thus by the continuity of G and A, we obtain that $A(x, \xi) \leq |G(x) - G(\xi)|$. This means that $\xi \in M$ and that there is no point $y \in X$ such that $\xi \neq y$ and $A(\xi, y) \leq |G(\xi) - G(y)|$, because such y would belong to M. Then it must be so that $A(\xi, T\xi) = 0$, i. e. , $\xi = T\xi$. This completes the proof.

Further, we notice, the set C in linear space is **convex** if for $x, y \in C$ and $\lambda \in [0, 1]$ implies $\lambda x + (1 - \lambda)\, y \in C$. The metric space (X, ρ) is called **convex (metric convex)** if for any two different points $x, y \in X$ there is a point $z \in X$ $(z \neq x, y)$ such that

$$\rho[x, y] = \rho[x, z] + \rho[z, y]\,. \tag{2}$$

In connection with this, if $C \subset X$ convex set of a normed linear space X, then C also and metric convex set with $\rho[x, y] = ||x - y||$, because for any two different points $x, y \in C$ there is a point $z := (x + y)/2 \in C$ $(z \neq x, y)$ such that (2).

Lemma 1. *Let (X, ρ) be a metric space. If C is a metric convex set in X and if map $T: C \to C$ with the property that there is a point $a \in C$ which is not fixed point, then there exists a continuous function $G: C \to \mathbf{R}_+^0$ such that T is a general ρ-variation mapping.*

Proof. Let $a \in C$ be a fixed element such that $a \neq Ta$ and let $x \in C$ be an arbitrary point with $x \neq a$. Since C is a convex (metric convex) set in X, it follows from definition that for $a \in C$ and for all $x \in C\backslash\{a\}$ there exists a point $y \neq a, x$ in C such that $\rho[a,x] = \rho[a,y] + \rho[y,x]$. Hence, we have, also and the following inequality

$$(3) \quad 3^{-1}\rho[x,y] \leq \rho[x,y] = \rho[a,x] - \rho[a,y] \quad \text{for all } x \in C\backslash\{a\}.$$

On the other hand, analogous to the preceding construction, we also have the following inequality

$$(4)\ 3^{-1}\rho[x,y] \leq \rho[x,y] = \rho[Ta,x] - \rho[Ta,y] \quad \text{for all } x \in C\backslash\{Ta\}.$$

Also, immediately to join and take away the expression $\rho[Ta,a]$ on the right side of inequality (3) we obtain the following equivalent inequality with (3), that is

$$(3') \qquad 3^{-1}\rho[x,y] \leq \rho[a,x] + \rho[Ta,a] - (\rho[a,y] + \rho[Ta,a])$$

for all $x \in C\backslash\{a\}$.

From inequalities (3') and (4) define function $G : C \to \mathbf{R}_+^0$ such that

$$(5) \qquad G(x) = \begin{cases} 3\rho[Ta,x] & \text{for } x = a, \\ 3(\rho[a,x] + \rho[Ta,a]) & \text{for } x \in C\backslash\{a\} \end{cases}$$

Then, clearly, from (3'), (4) and (5) we have for any $x \in C$ there exists $y \neq x$ in C such that $\rho[x,y] \leq |G(x) - G(y)|$. Thus, for any $x \in C$ with $x \neq Tx$ there exists $y \in X\backslash\{x\}$ such that (AG), where $A(x,y) := = \rho[x,y]$. Hence, it follows that T is a general ρ-variation mapping. The proof is complete.

We are now in a position to formulate our the following famous application.

Corollary 1. (Brouwer, [Br]). *Suppose that C is a nonempty convex, compact subset of $\mathbf{R}^n$, and that $T : C \to C$ is a continuous mapping. Then T has a fixed point in C.*

Proof. From the preceding Lemma 1, we have that $T : C \to C$ is a general A-variation mapping, where A is a metric on $\mathbf{R}^n$. The set C is compact in X, and thus C satisfies the condition of CS-convergence.

From the preceding remarks, it is easy to see that T satisfy all the required hypotheses in Theorem 1. Hence, it follows from Theorem 1 that T has a fixed point in C.

Let X, Y be topological spaces. A continuous map $F : X \to Y$ is a called **compact** if $F(X)$ is contained in a compact subset of Y. If X and Y are Banach's spaces and $T : D(T) \subset X \to Y$, then T is called **compact** if T is continuous and T maps bounded sets into relatively compact sets. Compact operators play a central role in nonlinear functional analysis. Schauder's theorem is a generalization of Brouwer's theorem to infinite dimensional normed linear spaces, with the preceding fact.

We can now formulate Brouwer's theorem in a manner valid for all normed linear spaces.

Corollary 2. (Schauder, [Sc]). *Let C be a nonempty, closed, bounded, convex subset of the Banach space X, and suppose $T : C \to C$ is a compact operator. Then T has a fixed point in C.*

Also, we have and an alternate version of the preceding Schauder fixed point theorem.

Corollary 3. (Schauder, [Sc]). *Let C be a nonempty, compact, convex subset of a Banach space X, and suppose $T : C \to C$ is a continuous operator. Then T has a fixed point.*

This corollary is the direct translation of the Brouwer fixed point theorem to Banach spaces.

Proof of Corollary 3. Since C is a convex subset of Banach space, from Lemma 1, we have that $T : C \to C$ is a general A-variation, where $A(x, y) = ||x - y||$. The set C is closed in X, and thus C is a complete space. It is easy to see that T satisfy all the required hypoteses in Theorem 1. Hence, it follows from the Theorem 1 that T has a fixed point in C

Corollary 4. (Banach Contraction Principle). *Let (X, ρ) be a complete metric space and $T : X \to X$ contractive. Then T has a unique fixed point ξ, and $T^n x \to \xi (n \to \infty)$ for each $x \in X$.*

Proof. From the condition of contraction, it is easy to see that T is general ρ-(bounded) variation. Precisellly, every contraction mapping is bounded variation and continuous. Hence, it follows from the Theorem 1 that T has a fixed point.

At the and, we notice, also in this paper, we extend and results of Darbo, Browder, Sadovskij, Tychonoff, Krasnoselskij, Ky Fan, Dugundji, Granas, Kirk and Caristi, Kakutani and some others. In connection with this, proofs are the analogous to the proofs of the preceding statements of Brouwer, Schauder and Banach.

3. Answer to Schauder's problem

From the preceding statement and some further facts we are now in the position to formulate the following fact which is an extension of the former results of Brouwer, Schauder, Tychonoff, Mazur, Hukuhara, Ky Fan, Browder, Sadovskij, Darbo, Krasnoselskij, Reinermann, Dugundji, Granas, Klee, Idzik, Riedrich, Eisenack-Fenske, Jahn and some others.

Theorem 2. (Answer is yes for Schauder's problem). *Let C be a nonempty convex compact subset of a linear topological space X and suppose $T : C \to C$ is a continuous mapping. Then T has a fixed point in C.*

To prove this statement, the following facts are essential.

Lemma 2. *Let X be a metric space with the metric ρ. If $Y \subset X$ is a (complete or not complete) subspace and if T is a map of Y into itself, then there exists a continuous function $G : Y \to \mathbf{R}^0_+$ such that T is a general ρ-variation mapping.*

Proof. *Case 1.* (Y is not complete). Let $\{x_n\}_{n \in \mathbf{N}}$ be a Cauchy sequence in Y which has no limit. Define map $\Psi : Y \to \mathbf{R}^0_+$ by $\Psi(z) = \lim_{i\to\infty} \rho[z, x_i]$ for $z \in Y$.

Given $x \in Y$, let n be the smallest positive integer such that

$$(6) \qquad 0 < \frac{1}{3}\rho[x, x_n] \leq \Psi(x) - \Psi(x_n) \leq |\Psi(x) - \Psi(x_n)| \,.$$

We notice that then $\Psi(x_n)$ converges to zero while $\Psi(x) > 0$. Further, given $x \in Y$ with n so determined, define $y := x_n$ and let $G(x) := 3\Psi(x)$. Then, from (6) we have for any $x \in Y$ there exists $y \neq x$ in Y such that $\rho[x, y] \leq |G(x) - G(y)|$. Thus, for any $x \in C$ with $x \neq Tx$ there exists $y \in Y \backslash \{x\}$ such that (AG), where $A(x, y) := \rho[x, y]$. Hence, it follows in this case that T is a general ρ-variation mapping.

Case 2. (Y is complete). Let $\{x_n\}_{n\in\mathbf{N}}$ be a Cauchy sequence in Y which has limit $\xi \in Y(= \mathrm{Cl}Y)$ and let $\xi \neq T\xi$. Define map $\varphi : Y \to \mathbf{R}^0_+$ by $\varphi(z) = \rho[z,\xi]$ for $z \in Y\backslash\{\xi\}$. Given $x \in Y$ $(x \neq \xi)$, let n be the smallest positive integer such that

$$0 < \frac{1}{3}\rho[x, x_n] \leq \varphi(x) - \varphi(x_n)\ , \tag{7}$$

i.e., which is equivalent with inequality

$$0 < \frac{1}{3}\rho[x,\xi] \leq \varphi(x) + \rho[\xi, T\xi] - (\varphi(x_n) + \rho[\xi, T\xi])\ . \tag{8}$$

On the other hand, for $x = \xi$, let n be the smallest positive integer such that

$$0 < \frac{1}{3}\rho[x_n,\xi] \leq \rho[\xi, T\xi] - \rho[x_n, T\xi]\ . \tag{9}$$

From inequalities (8) and (9), for given $x \in Y$ with n so determined, define $y := x_n$ and define function $G : Y \to \mathbf{R}^0_+$ such that

$$G(x) = \begin{cases} 3\rho[x, T\xi] & \text{for } x = \xi, \\ 3(\varphi(x) + \rho[\xi, T\xi]) & \text{for } x \in Y\backslash\{\xi\}\ . \end{cases} \tag{10}$$

Then, in this case, from (8), (9) and (10) we have for any $x \in Y$ there exists $y \neq x$ in Y such that $\rho[x,y] \leq |G(x) - G(y)|$. Thus, for any $x \in Y$ with $x \neq Tx$ there exists $y \in Y\backslash\{x\}$ such that (AG), where $A(x,y) := \rho[x,y]$. Hence it follows in this case that T is a general ρ-variation mapping. The proof is complete.

To prove Theorem 2 and the following fact is essential.

Lemma 3. *Let X be a linear space. If C is a convex set in X and if T is a map of C into itself, then there exists a continuous function $G : C \to \mathbf{R}^0_+$ such that T is a general A-variation mapping for some function $A : C \times C \to \mathbf{R}^0_+$.*

Proof. Consider the convex set C of linear space X as a quasi-metric space with the quasi-metric q, where $q : C \times C \to \mathbf{R}^0_+$ is defined by

$$q(x,y) = \begin{cases} 0 & \text{for } x = y, \\ \max\{K(x), K(y)\} & \text{for } x \neq y, \end{cases} \tag{11}$$

for a strictly convex function $K : C \to \mathbf{R}^0_+$. Then it is easy to see that q is a quasi-metric, i.e., that for all $x, y, z \in C$ we have : $q(x,y) = q(y,x)$, $q(x,z) \leq q(x,y) + q(y,z), q(x,y) \geq 0$ and that $x = y$ implies $q(x,y) = 0$.

On the other hand, if $q(x,y) = 0$, i.e., if $K(x) = K(y) = 0$, then since K is a strictly convex function, we obtain

$$0 = \frac{K(x) + K(y)}{2} > K\left(\frac{x+y}{2}\right) \geq 0 \,,$$

which is a contradiction. Consequently $x = y = \frac{x+y}{2}$, i.e., $x = y$. Thus $q(x,y) = 0$ implies $x = y$, i.e., q is a metric on C.

Applying Lemma 2 (or Lemma 1) to this case, we obtain then that there exists a continuous function $G : C \to \mathbf{R}^0_+$ such that T is a general q-variation mapping. The proof is complete.

Proof of Theorem 2. From Lemma 3 there exists a continuous function $G : C \to \mathbf{R}^0_+$ such that T is a general A-variation mapping where $A(x,y) := q(x,y)$ and q defined in (11).

Since T is a continuous mapping, the function $x \mapsto A(x,Tx) = q(x,Tx)$ is a continuous function. Also and the function $y \mapsto A(x,y) = q(x,y)$ is continuous. The set C is a compact in space X and thus C satisfies the condition of CS-convergence.

It is easy to see that T satisfies all the required hypotheses in Theorem 1. Hence, it follows from the Theorem 1 that T has a fixed point $\xi \in C$. The proof is complete.

4. Some further applications

As an immediate corollary of the preceding solved problem (Theorem 2), we obtain one of the basic results in nonlinear functional analysis which is an extension of the Markoff-Kakutani theorem.

Theorem 3. *Let C be a nonempty convex compact set in a linear topological space X and let $\mathcal{F}$ be a commuting family of continuous affine maps of C into itself. Then $\mathcal{F}$ has a common fixed point $\xi \in C$.*

Proof. Let Fix (T) be a fixed point set of a map T. By Theorem 2, Fix (T) is a nonempty set for each $T \in \mathcal{F}$. Moreover, Fix (T) is compact being closed in the compact set C, and Fix (T) is convex because T is affine.

We must prove that $\cap\{\mathrm{Fix}(T) : T \in \mathcal{F}\}$ is a nonempty set; because each set Fix (T) is compact, it is sufficient to show that each finite interestion

$$\mathrm{Fix}\,(T_1, \ldots, T_n) := \bigcap_{i=1}^{n} \mathrm{Fix}\,(T_i)$$

is nonempty. We proceed by induction on the number $n \in \mathbf{N}$ of T_i, the result being true for $n = 1$. Assume that $\mathrm{Fix}(T_1, \ldots, T_i)$ is nonempty whenever $i < n$, and consider any n members $T_1, \ldots, T_n$ of $\mathcal{F}$. Bacause $\mathcal{F}$ is commuting, we find that

$$T_n[\mathrm{Fix}\,(T_1, \ldots, T_{n-1})] \subset \mathrm{Fix}\,(T_1, \ldots, T_{n-1})\,,$$

for if $x \in \mathrm{Fix}\,(T_1, \ldots, T_{n-1})$ then $T_i[T_n(x)] = T_n[T_i(x)] = T_n(x)$ for each $i < n$ so $T_n(x) \in \mathrm{Fix}(T_1, \ldots, T_{n-1})$.

Since Fix $(T_1, \ldots, T_{n-1})$ is a nonempty compact convex set, we conclude from Theorem 2 that Fix $(T_1, \ldots, T_n)$ is a nonempty set. This completes the induction and the proof.

On the other hand, as an immediate consequence of Theorem 1, we obtain the following geometrical fact on fixed points.

Theorem 4. *Let T be a self-map on a topological space X and $A :$ $: X \times X \to \mathbf{R}^0_+$ be a function with properties : $A(a,b) = 0$ iff $a = b$ and $A(a,c) \leq A(a,b) + A(b,c)$ for all $a, b, c, \in X$. Suppose that there exists a continuous function $G : X \to \mathbf{R}^0_+$ such that*

$$A(x, Tx) \leq |G(x) - G(Tx)|$$

for every $x \in X$. If X satisfies the condition of CS-convergence and if $b \mapsto A(a,b)$ is continuous, then T has a fixed point $\xi \in X$.

5. References

[Am] H. Amann: *Ordered structures and fixed points*, pp. 1-50 in "Atti 2° Sem. Anal. Funz. Appl". Univ. Cosenza (Italy), 1977.

[Br] L. E. J. Brouwer: *Über Abbildungen von Mannigfaltigkeiten*, Math. Ann., **71** (1912), 97-115.

[Bo] F. E. Browder: *Nonexpansive nonlinear operators in a Banach space*, Proc. NAS (USA), **54** (1965), 1041-1044.

[Da] G. Darbo: *Punti uniti in transformazioni a codominio non compatto*, Rend. Sem. Math. Univ. Padova, **24** (1955), 84-92.

[Dy] M. M. Day: *Fixed point theorems for compact convex sets*, Ill. J. Math., **5** (1961), 585-590.

[Dj] J. Dugundji: *Positive definite functions and coindicences*, Fund. Math., **90** (1976), 131-142.

[Ef] G. Eisenack and C. Fenske: *Fixpunkttheorie*, Bibliographisches Institut, Mannheim, 1978.

[Fa] Ky Fan: *A generalization of Tychonoff's fixed point theorem*, Math. Ann., **142** (1961), 305-310.

[Go] K. Goebel: *An elementary proof of the fixed point theorem of Browder and Kirk*, Michigan Math. J., **16** (1969), 381-383.

[Gö] D. Göhde: *Zum Prinzip der kontractiven Abbildungen*, Math. Nachr., **30** (1965), 251-258.

[Gr] A. Granas: *On a certain class of nonlinear mappings in Banach spaces*, Bull. Acad. Pol. Sci., **9** (1957), 867-871.

[Ha] S. Hahn: *Zur Theorie kompacten Vektorfelder in topologischen Vektorraümen*, Math. Nachr., **85** (1978), 273-282.

[Hu] M. Hukuhara: *Sur l'existence des points invariants d'une transformation dans l'espace fonctionnel*, Jap. J. Math., **20** (1950), 1-4.

[Iz] A. Idzik: *Almost fixed point theorems*, Proc. Amer. Math. Soc., **104** (1988), 779-784.

[Ja] K. Jahn: *Maximale Fixpunkte von Intervallfunktionen*, Computing, **33** (1984), 141-151.

[Ka] S. Kakutani: *Two fixed point theorems concerning bicompact convex sets*, Proc. Imp. Acad. Tokyo, **14** (1938), 242-245.

[Ki] W. A. Kirk: *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004-1006.

[Ke] V. Klee: *Leray-Schauder theory without local convexity*, Math. Ann., **141** (1960), 286-296.

[Kr] M. A. Krasnoselskij: *Two remarks on the method of successive approximations*, Uspehi Mat. Nauk, **10** (1955), 123-127 (Russian).

[Ma] A. A. Markoff: *Quelques théorèmes sur les ensembles abéliens*, C. R. Acad. Sci. URSS (N.S), **1** (1936), 311-313.

[Mz] S. Mazur: *Un théorème sur les points invariants*, Ann. Soc. Polon. Math., **17** (1938), 110.

[Re] J. Reinermann: *Fixpunktsätze vom Krasnoselskij-Typ*, Math. Z., **119** (1971), 339-344.

[Ri] T. Riedrich: *Vorlesungen über nichtlineare Operatorgleichungen*, Teubner, Leipzig, 1976.

[Ry] C. Ryll-Nardzewski: *On fixed points of semi-groups of endomorphisms of linear spaces*, Proceedings of the Fifth Berkeley Symposium on Statistics and Probability, Vol **II**, Berkeley 1966.

[Sa] B. N. Sadovskij: *A fixed point principle*, Functional. Anal. i Prilozen., **1** (1967), 74-76; Func. Anal. and Applications, **1** (1967), 151-153.

[Sc] J. Schauder: *Der Fixpunktsatz in Functionlräumen*, Studia Math., **2** (1930), 171-180.

[Ta] M. Tasković: *Extensions of Brouwer's theorem*, Math. Japonica, **36** (1991), 685-693.

[Tm] M. Tasković: *The axiom of choice, fixed point theorems, and inductive ordered sets*, Proc. Amer. Math. Soc., **116** (1992), 897-904.

[Ty] A. Tychonoff: *Ein Fixpunktsatz*, Mat. Ann., **111** (1935), 767-776.

Matematički fakultet
11000 Beograd, P. O. Box 550
Yugoslavia

Received July 15, 1997.