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SOME MINIMAX THEOREMS ON ORDERED SETS

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Dedicated to Professor Dušan D. Adamović on his 70th birthday.

Abstract. In this paper we continue the study and considerations of some minimax statements on ordered sets.

1. Introduction

John von Neumann's minimax theorem can be stated as follows: if X and Y are finite demensional simplices and f is a bilinear function on $X \times Y$, then f has a saddle point, i.e.,

$$\max_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\max_{x\in X} f(x,y)\ .$$

There have been several generalizations of this theorem. The result of Sion [7] is the best representive of von Neumann's theory.

In this paper we prove some general minimax theorems on partially ordered sets which are other type. On these theorems role of saddle point play transversal point (see: Tasković [9]).

In connection with the preceding, the following our former results allows us to prove the basic statements for further facts.

Let $(P, \preceq)$ be a partially ordered set by the ordering relation $\preceq$. The function $g : P^k \to P$ (k is a fixed positive integer) is **decreasing** on the

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ordered set P if $a_i, b_i \in P$ and $a_i \preceq b_i$ $(i = 1, \dots, k)$ implies $g(b_1, \dots, b_k) \preceq$ $\preceq g(a_1 \dots, a_k)$.

Let L be a lattice and g a mapping from L^2 into L. For any $g : L^2 \to L$ it is natural to consider the following property of **local comparability**, which means, if $J \in L$ is comparable with $g(J, J) \in L$ then J is comparable with every $t \in L$.

We begin with the following essential statements from Tasković [10].

Lemma 1. (Sup-Inf Inequalities). *Let $(L \preceq)$ be a lattice and let $g : L^2 \to L$ be a decreasing mapping. If L has property of local comparability, then for arbitrary functions $p : X \to L$ and $g : Y \to L$ (X and Y are arbitrary nonempty sets) the following relations are valid:*

$$\text{(S)} \qquad \xi \preceq g(\xi, \xi) \quad \textit{implies} \quad \xi \preceq \sup\{p(x), q(x), g(p(x), q(y))\},$$

and

$$\text{(I)} \qquad g(\xi, \xi) \preceq \xi \quad \textit{implies} \quad \inf\{p(x), q(x), g(p(x), q(y))\} \preceq \xi,$$

for all $x \in X$ and for all $y \in Y$. Hence, in particular, $\xi = g(\xi, \xi)$ implies

$$\text{(U)} \quad \inf\{p(x), q(y), g(p(x), q(y))\} \preceq \xi \preceq \sup\{p(x), q(y), g(p(x), q(y))\},$$

for all $x \in X$ and for all $y \in Y$.

A brief proof of this statement based on the former facts may be found in Tasković [10].

An immediate consequence (special case for totally ordered sets) of the preceding Lemma 1 is the following its form.

Lemma 1a. (Minimax Inequalities). *Let P be a totally ordered set by the order relation $\preceq$, and let $g : P^2 \to P$ be a decreasing mapping. Then for functions p, $g : X \to P$ (X is a nonempty set) the following relations are valid:*

$$\text{(a)} \qquad \xi \preceq g(\xi, \xi) \quad \textit{implies} \quad \xi \preceq \max\{p(x), q(y), g(p(x), q(y))\},$$

and

$$\text{(b)} \qquad g(\xi, \xi) \preceq \xi \quad \textit{implies} \quad \min\{p(r), q(s), g(p(r), q(s))\} \preceq \xi,$$

for all $x, y, r, s \in X$. Hence, in particular, $\xi = g(\xi, \xi)$ implies

$$\text{(c)} \quad \min\{p(r), q(s), g(p(r), q(s))\} \preceq \xi \preceq \max\{p(x), q(y), g(p(x), q(y))\} ,$$

for all $x, y, r, s \in X$.

We notice, quantifying the assertions (S), (I) and (U) we obtain the following interesting conclusions (which, incidentally are their equivivalent formulations for $X = Y$):

$$\text{(ES)} \quad \xi \preceq g(\xi, \xi) \quad \textit{implies} \quad \xi \preceq \inf_{x,y \in X} \sup\{p(x), q(y), g(p(x), q(y))\},$$

and

$$\text{(EI)} \quad g(\xi, \xi) \preceq \xi \quad \textit{implies} \quad \sup_{x,y \in X} \inf\{p(x), q(y), g(p(x), q(y))\} \preceq \xi \,;$$

and $g(\xi, \xi) = \xi$ implies the following inequalities:

$$\text{(EU)} \quad \sup_{x,y \in X} \inf\{p(x), q(y), g(p(x), q(y))\} \preceq \xi \preceq$$
$$\preceq \inf_{x,y \in X} \sup\{p(x), q(y), g(p(x), q(y))\} .$$

Remark. The above statements (Lemma 1) still hold when $g : L^k \to L$ (k is a fixed positive integer) is a decreasing function. The proof is quite similar; the assertions corresponding to (S) and (I) look as follows:

$$\text{(S')} \quad \xi \preceq g(\xi, \ldots, \xi) \quad \textit{implies} \quad \xi \preceq \sup\{\lambda_1, \ldots, \lambda_k, g(\lambda_1, \ldots, \lambda_k)\}$$

and

$$\text{(I')} \quad g(\xi, \ldots, \xi) \preceq \xi \quad \textit{implies} \quad \inf\{\lambda_1, \ldots, \lambda_k, g(\lambda_1, \ldots, \lambda_k)\} \preceq \xi$$

for arbitrary functions $\lambda_1, \ldots, \lambda_k : X \to L$, where X is an arbitrary nonempty set. Also, in particular, $\xi = g(\xi, \ldots, \xi)$ implies

$$\text{(U')} \quad \inf\{\lambda_1, \ldots, \lambda_k, g(\lambda_1, \ldots, \lambda_k)\} \preceq \xi \preceq \sup\{\lambda_1, \ldots, \lambda_k, g(\lambda_1, \ldots, \lambda_k)\}$$

for arbitrary functions $\lambda_i \in X$ ($i = 1, \ldots, k$), where X is an arbitrary nonempty set. To simplify the notation we will give the proof only for the case $k = 2$.

2. Sup-Inf Equalities

With the help of the preceding statements we now obtain the fundamental fact of tihis section. With this statements we precision, correction and expand our the former results of new minimax theory (Theorems 1 and 3 in [11]).

Theorem 1. (Sup-Inf Theorem). *Let $(L \preceq)$ be a lattice and let $g : L^2 \to L$ be a decreasing mapping. If L has property of local comparability, then for some arbitrary functions $p : X \to L$ and $q : X \to L$ (X is an arbitrary nonempty set) the equality*

$$\text{(SI)} \qquad \max_{x,y\in X} \inf\{p(x), q(y), g(p(x), q(y))\} = \\ = \min_{x,y\in X} \sup\{p(x), q(y), g(p(x), q(y))\}$$

holds if and only if

$$\text{(Si)} \quad \inf\{p(x_0), q(y_0), g(p(x_0), q(y_0))\} = \sup\{p(r_0), q(z_0), g(p(r_0), q(z_0))\}$$

for some $x_0, y_0, r_0, z_0 \in X$.

Proof. This follows at once form (EU) of Lemma 1 and the trivial fact that the strict inequality cannot hold in (EU).

In this sense, the necessity of the condition being trivial, we only prove its sufficiency. If (Si) holds, then we have the following relations

$$(1) \qquad p(r_0), q(z_0), g(p(r_0), q(z_0)) \preceq s = i \preceq p(x_0), q(y_0), g(p(x_0), q(y_0))$$

for $s := \sup\{p(r_0), q(z_0), g(p(r_0), q(z_0))\}$, $i := \inf\{p(x_0), q(y_0), g(p(x_0), q(y_0))\}$, and for some $x_0, y_0, r_0, z_0 \in X$. Since $g : L^2 \to L$ is decreasing, from (1) we obtain

$$(1') \qquad g(i,i) = g(s,s) \preceq g(p(r_0), q(z_0)) \preceq s = i \preceq \\ \preceq g(p(x_0), q(y_0)) \preceq g(s,s) = g(i,i)\,,$$

i.e., $i = s = g(i,i) = g(s,s)$. Applying Lemma 1 (case (U)) from local comparability we have

$$\inf\{p(x), q(y), g(p(x), q(y))\} \preceq i = s \preceq \sup\{p(x), q(y), g(p(x), q(y))\}$$

for all $x, y \in L$. Therefore, we have (SI). The proof is complete.

An immediate consequence (special case) of the preceding statement is the following principle.

Theorem 1a. (Minimax Principle). *Let P be a totally ordered set by the order relation $\preceq$, and let $g : P^2 \to P$ be a decreasing mapping. Then for some arbitrary functions $p : X \to P$ and $q : X \to P$ (X is an arbitrary nonempty set) the equality*

$$\text{(MM)} \qquad \max_{x,y\in X} \min\{p(x), q(y), g(p(x), q(y))\} = \\ = \min_{x,y\in X} \max\{p(x), q(y), g(p(x), q(y))\}$$

holds if and only if

$$\text{(Mm)} \qquad p(x_0) = q(y_0) := \xi = g(\xi, \xi) \quad \textit{for some } x_0, y_0 \in X\ .$$

Proof. Applying Theorem 1 we obtain that (MM) is an equivalent with (Si), i.e., since P is a totally ordered set the equality (Si) is in the following form

$$\min\{p(x_0), q(y_0), g(p(x_0), q(y_0))\} = \max\{p(r_0), q(z_0), g(p(r_0), q(z_0))\}$$

for some totally comparable elements $p(x_0), q(y_0), g(p(x_0), q(y_0)), p(r_0)$, $q(z_0)$ and $g(p(r_0), q(z_0))$ on P. Hence, we get that $p(x_0) = q(y_0) = p(r_0) = = q(z_0)$, i.e., from (1') we have $\xi := p(x_0) = q(y_0) = g(\xi, \xi)$, i.e., (Mm). The proof is complete.

The statement above still holds when $g : P^k \to P$ (k is a fixed positive integer). The proof is quite similar. Therefore, let $(P, \preceq)$ be a totally ordered set by the order relation $\preceq$, and $g : P^k \to P$ ($k \in \mathbf{N}$) be a decreasing mapping. Then, the equality

$$\text{(Uk)} \qquad \max_{\lambda_1,\ldots,\lambda_k\in P} \min\{\lambda_1, \ldots, \lambda_k, g(\lambda_1, \ldots, \lambda_k)\} = \\ = \min_{\lambda_1,\ldots,\lambda_k\in P} \max\{\lambda_1, \ldots, \lambda_k, g(\lambda_1, \ldots, \lambda_k)\}$$

holds if and only if

$$\lambda_1(a_1) = \cdots = \lambda_k(a_k) := \xi = g(\xi, \ldots, \xi) \quad \text{for some } a_1, \ldots, a_k \in X\ ,$$

where $\lambda_i : X \to P (i = 1, \ldots, k)$ are arbitrary functions and X is a nonempty set.

We remark that when $X = P$, $p(x) = x$ and $q(y) = y$ Theorem 1a reduces to that of our the following former result.

Corollary 1. (Tasković, [10]). *Let P be a totally ordered set by the order relation $\preceq$, and let $g : P^2 \to P$ be a decreasing mapping. Then the equality*

$$\max_{x,y\in P} \min\{x, y, g(x, y)\} = \min_{x,y\in P} \max\{x, y, g(x, y)\}$$

holds if and only if there is $\xi \in P$ such that $g(\xi, \xi) = \xi$.

In connection with the preceding, we note that we can give an extension of the preceding Theorem 1, as a direct consequence of the preceding facts, in the following sense.

Theorem 2. (General Sup-Inf Theorem). *Let $(L, \preceq)$ be a lattice and let $g : L^2 \to L$ be a mapping. Then for some arbitrary $p : X \to L$ and $q : X \to L$ (X is an arbitrary nonemptu set) the following equality holds*

$$\text{(SI')} \qquad \max_{x,y\in X} \inf\{p(x), q(y), g(p(x), q(y))\} = \\ = \min_{x,y\in X} \sup\{p(x), q(y), g(p(x), q(y))\}$$

if and only if the following inequalities hold

$$\text{(DI)} \quad \inf\{p(x), q(y), g(p(x), q(y))\} \preceq \inf\{p(x_0), q(y_0), g(p(x_0), q(y_0))\} = \\ = \sup\{p(r_0), q(z_0), g(p(r_0), q(z_0))\} \preceq \sup\{p(x), q(y), g(p(x), q(y))\}$$

for some $x_0, y_0, r_0, z_0 \in X$ and for all $x, y \in X$.

On the other hand, if L is a totally ordered set, then condition (DI) an equivalent with the following equality

$$\max_{x,y\in X} \min\{p(x), q(y), g(p(x), q(y))\} = \min_{x,y\in X} \max\{p(x), q(y), g(p(x), q(y))\}\,.$$

Also, in connection with the preceding equality (Uk), if $g : P^k \to P$ (k is a fixed positive integer) is not decreasing mapping, we can extension equality (Uk). In this sense, if $g : P^k \to P$ (k is a fixed positive integer) some arbitrary mapping then equality (Uk) holds if and only if the following inequalities hold

$$\begin{aligned}\min\{\lambda_1,\dots,\lambda_k,g(\lambda_1,\dots,\lambda_k)\} &\preceq \\ \preceq \min\{\lambda_1(a_1),\dots,\lambda_k(a_k),g(\lambda_1(a_1),\dots,\lambda_k(a_k))\} &= \\ = \max\{\lambda_1(b_1),\dots,\lambda_k(b_k),g(\lambda_1(b_1),\dots,\lambda_k(b_k))\} &\preceq \\ \preceq \max\{\lambda_1,\dots,\lambda_k,g(\lambda_1,\dots,\lambda_k)\}\end{aligned}$$

for some $a_1,b_1,\dots,a_k,b_k \in X$, where $\lambda_i : X \to P$ $(i = 1,\dots,k)$ are arbitrary functions and X is a nonempty set.

On the other hand, the next result follows form the preceding statements.

Corollary 2. *Let L be a lattice with the order relation $\preceq$. Then for some arbitrary mappings $p : X \to L$ and $q : X \to L$ (X is an arbitrary nonempty set) the following equality holds*

$$\max_{x,y\in X} \inf\{p(x),q(y)\} = \min_{x,y\in X} \sup\{p(x),q(y)\}$$

if and only if the following inequalities hold

$$\inf\{p(x),q(y)\} \preceq \inf\{p(x_0),q(y_0)\} = \sup\{p(r_0),q(z_0)\} \preceq \sup\{p(x),q(y)\}$$

for some $x_0,y_0,r_0,z_0 \in X$ and for all $x,y \in X$.

We note, in the preceding statements (as in Corollary 2) we can defined the preceding functions $p,q : X \to L$ and different sets, in sense that $p : X \to L$ and $q : Y \to L$ (X and Y are arbitrary nonempty sets). Then the preceding statements hold too. In this sense, for some arbitrary functions $f_i : X_i \to L (i = 1,\dots,k)$ the following equality holds

$$\begin{aligned}&\max_{x_1\in X_1,\dots,x_k\in X_k} \inf\{f_1(x_1),\dots,f_k(x_k)\} = \\ &= \min_{x_1\in X_1,\dots,x_k\in X_k} \sup\{f_1(x_1),\dots,f_k(x_k)\}\end{aligned}$$

if and only if the following inequalities hold

$$\begin{aligned}\inf\{f_1(x_1),\dots,f_k(x_k)\} &\preceq \inf\{f_1(a_1),\dots,f_k(a_k)\} = \\ = \sup\{f_1(b_1),\dots,f_k(b_k)\} &\preceq \{f_1(x_1),\dots,f_k(x_k)\}\end{aligned}$$

for some $a_i,b_i \in X_i$ $(i = 1,\dots,k)$ and for all $x_i \in X_i$ $(i = 1,\dots,k)$.

In this part of this section, we show that the existence of a separation in the preceding sense, is essential for applications of the preceding statements. This is a separation for the preceding equalities of Minimax type.

In this sense we give a characterization of general variational equality. With this result we precision, correction and expand our the former statement of separation (Theorem 4 in [11]). It is the following result.

Theorem 3. (Statement of Separation). *Let L be a lattice with the order relation $\preceq$, and with local comparability. Then for some arbitrary mappings $p : X \to L$ and $q : Y \to L$ (X and Y are two arbitrary nonempty sets) the following equality holds*

$$\text{(IS)} \qquad \operatorname*{Max}_{x \in X} p(x) = \operatorname*{Min}_{y \in Y} q(y)$$

if and only if there exists a decreasing function $g : L^2 \to L$ such that the following inequalities hold

$$\text{(PQ)} \qquad p(x) \preceq g(p(x), q(y)) \preceq q(y)$$

for all $x \in X$ and $y \in Y$, and if there is $\xi \in L$ such that the $\xi \cap p(X)$ and $\xi \cap q(Y)$ are nonempty sets.

Proof. *Necessity.* Let the einequalities (PQ) hold and let, from the conditions, there exist points $x_0 \in X$ and $y_0 \in Y$ such that $\xi = p(x_0) = = q(y_0)$. Thus, we obtain the following inequalities and equality of form (from Lemma 1)

$$\inf\{p(x), q(y), g(p(x), q(y))\} \preceq \xi = g(\xi, \xi) \preceq \sup\{p(x), q(y), g(p(x), q(y))\}$$

for some $x_0 \in X$ and $y_0 \in Y$, and for all $x \in X$ and $y \in Y$. This means, from Theorem 1 and from (PQ), that the equality (MM) holds, which give the equality (IS) of this statement.

Sufficiently. Assume that equality (IS) holds. Thus, there is $\xi \in L$ such that $p(x) \preceq \xi \preceq q(y)$ for all $x \in X$ and $y \in Y$, where $p(x_0) = q(y_0) = \xi$ for some $x_0 \in X$ and $y_0 \in Y$. If a decreasing function $g : L^2 \to L$ defined by $g(s, t) = \xi$, then, directly, we obtain inequalities (PQ). The proof is complete.

At the end of this section, based on the preceding statements, as an immediate consequence we have the following fact.

Corollary 3. *Let P be a set totally ordered by the order relation $\preceq$, and let $g : P^2 \to P$ be a decreasing mapping. Then the following equality holds*

$$\max_{\xi \leq x} \min_{y \leq \xi} g(x,y) = \min_{y \leq \xi} \max_{\xi \leq x} g(x,y)$$

if and only if there is $\xi \in P$ such that $g(\xi,\xi) = \xi$.

3. Sup-Inf Inequalities

We give now some immediate applications of the preceding statements to sup-inf inequalities.

As an immediate consequence of Lemma 1a we obtain the following inequalities.

Lemma 3. *Let P be a totally ordered set by the order relation $\preceq$, and let $g : P^2 \to P$ be a decreasing mapping. If for some arbitrary mapping $f : P^2 \to P$ is $f(\xi,\xi) \preceq \xi$, and $f(\xi,\xi) \preceq g(\xi,\xi)$, then*

$$\text{(Sf)} \qquad f(\xi,\xi) \preceq \max\{p(x), q(y), g(p(x), q(y))\}$$

for all $x, y \in X$, where $p, q : X \to P$ and X is an arbitrary nonempty set.

Quantifying the precedintg assertion (Sf) we obtain the following conculsion that $f(\xi,\xi) \preceq \xi$ and $f(\xi,\xi) \preceq g(\xi,\xi)$ implies

$$f(\xi,\xi) \preceq \min_{x,y \in X} \max\{p(x), q(y), g(p(x), q(y))\} .$$

With the following statements we precision, correction and expand our the former results of Sup-Inf Inequalities (Theorems 6, 7 and 8 in [11]).

In connection with this, we now obtain the fundamental fact of this section, which is essential for inequalities.

Theorem 4. (Sup-Inf Inequality). *Let $(L, \preceq)$ be a lattice with zero and unit, and let $A, B : X \times Y \to L$ (X and Y are arbitrary nonempty sets). Then for arbitrary mappings $a, c : X \to L$ and $b, d : Y \to L$ with $a(x), b(y), A(x,y) \preceq c(x), d(y), B(x,y)$ for all $x \in X$ and $y \in Y$, the following inequality holds*

$$\text{(IN)} \qquad \inf_{x \in X, y \in Y} \sup\{a(x), b(y), A(x,y)\} \preceq \sup_{x \in X, y \in Y} \inf\{c(x), d(y), B(x,y)\}$$

if and only if the following inequality holds

$$\text{(OI)} \qquad \sup\{a(x), b(y), A(x,y)\} \preceq \inf\{c(x), d(y), B(x,y)\}$$

for all $x \in X$ and $y \in Y$.

Proof. Since inequality (OI) holds for all $x \in X$ and $y \in Y$, directly, quantifying this inequality we obtain the preceding inequality (IN). On the other hand, if (IN) holds, we assume that inequality (OI) not hold. Then there is some $x_0 \in X$ and $y_0 \in Y$ such that

$$\alpha := \inf\{c(x_0), d(y_0), B(x_0, y_0)\} \prec \sup\{a(x_0), b(y_0), A(x_0, y_0)\} := \beta\ ,$$

which a contradiction with inequality $\beta \preceq \inf\{c(x), d(y), B(x,y)\}$ for all $x \in X$ and $y \in Y$, i.e., with $\beta \preceq \alpha$. The proof is complete.

As an immediate consequence of the preceding statement we obtain the following statement.

Theorem 4a. *Let $L, \preceq)$ be a lattice with zero and unit, and let $A, B : X \times Y \to L$ (X and Y are arbitrary nonempty sets). Then for arbitrary mappings $a, c : X \to L$ and $b, d : Y \to L$ with $a(x), b(y), A(x,y) \preceq$ $\preceq c(x), d(y), B(x,y)$ for all $x \in X$ and $y \in Y$, the following inequality holds*

$$\inf_{x \in X, y \in Y} \sup\{a(x), b(y), A(x,y)\} \preceq \sup_{x \in X, y \in Y} \sup\{c(x), d(y), B(x,y)\}$$

if and only if the following inequality holds

$$\text{(SS)} \qquad \sup\{a(x), b(y), A(x,y)\} \preceq \sup\{c(x), d(y), B(x,y)\}$$

for all $x \in X$ and $y \in Y$.

At the end of this section, we give a separation of statement for separation of the preceding inequalities.

Theorem 4b. (Separation of Inequalities). *Let L be a conditionally complete lattice with the order relation $\preceq$, and let the functions $c : X \to L$ and $b : X \to L$ (X and Y are two arbitrary nonempty sets) satisfies the inequality $b(y) \preceq c(x)$ for all $x \in X$ and $y \in Y$. Then the following inequality holds*

$$\text{(NT)} \qquad \operatorname*{Inf}_{y \in Y} b(y) \preceq \operatorname*{Sup}_{x \in X} c(x)$$

if and only if there exist functions $A, B : X \times Y \to L$, $a : X \to L$ *and* $d : Y \to L$ *such that the following inequalities hold*

$$\text{(NI)} \qquad a(x) \preceq A(x,y) \preceq b(y) \preceq c(x) \preceq B(x,y) \preceq d(y)$$

for all $x \in X$ *and for all* $y \in Y$.

Proof. Let the inequality (NT) holds, and let $\alpha := \operatorname*{Inf}_{y\in Y} b(y)$ and $\beta := \operatorname*{Sup}_{x\in X} c(x)$. Defined functions $A(x,y) = a(x) = \alpha$ and $B(x,y) = d(y) = \beta$, we obtain, directly, that inequalities (NI) hold. On the other hand, since L is conditionally complete if inequalities (NI) hold, from Theorem 4 and the inequality (OI), we directly obtain the inequality (IN), i.e., the inequality (NT) of this statement.

Finally, we give the following characterization a min-sup (max-inf) inequality via finite sets in the following form.

Theorem 5. *Let* S *be a conditionally complete lattice by the order relation* $\preceq$, *and* $f, g : X \times Y \to S$ *(*X *and* Y *are nonempty sets) such that* $x \mapsto f(x,y)$ *has a minimum on* X *and* $y \mapsto g(x,y)$ *has a maximum on* Y. *Then the inequallity*

$$\text{(2)} \qquad \min_{x\in X} \sup_{y\in Y} f(x,y) \preceq \max_{y\in Y} \inf_{x\in X} g(x,y)$$

holds if and only if for any two finite sets $\{x_1, \dots, x_n\} \subset X$ *and* $\{y_1, \dots, y_n\} \subset \subset Y$ *there exist* $x_0 \in X$ *and* $y_0 \in Y$ *such that*

$$\text{(3)} \qquad f(x_0, y_k) \preceq g(x_i, y_0) \quad \textit{for} \quad 1 \le i \le n, \quad 1 \le k \le m\,.$$

Proof. Let the inequality (2) holds. Then there exist $x_0 \in X$ and $y_0 \in Y$ such that

$$f(x_0, y_k) \preceq \sup_{y\in Y} f(x_0, y) \preceq \inf_{x\in X} g(x, y_0) \preceq g(x_i, y_0)$$

for all $i = 1, 2, \dots, n$ and for all $k = 1, 2, \dots, m$. This means that (3) holds.

Conversely, according to this condition, from (3),

$$\text{(4)} \qquad \sup_{1\le k\le m} f(x_0, y_k) \preceq \inf_{1\le i\le n} g(x_i, y_0)$$

holds for any two finite sets $\{x_1, x_2 \ldots, x_n\} \subset X$ and $\{y_1, y_2 \ldots, y_m\} \subset Y$. Since S is a conditionally complete lattice, from (4) we have

$$\sup_{m \leq \rho \leq Card\, Y} \sup_{1 \leq k \leq m} f(x_0, y_k) \preceq \inf_{n \leq \rho \leq Card\, X} \inf_{1 \leq i \leq n} f(x_i, y_0) .$$

Hence we obtain the following inequality

$$\sup_{y \in Y} f(x_0, y) \preceq \inf_{x \in X} g(x, y_0) ,$$

which is an equivalent with the inequality (2). The proof is complete.

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