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MEASURE OF NONCOMPACTNESS ON UNIFORM SPACES

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Dedicated to professor Dušan Adamović, on the occasion of his 70th birthday

Abstract. In this paper we define measure of noncompactness on arbitrary uniform spaces, and give some their properties. We also give one characterization of complete uniform spaces.

1. Introduction

There are many nonequivalent definitions of measures of noncompactness on metric spaces. First of them was introduced by Kuratowski [8]. In this paper we consider following axiomatic approach to this notion which include the most important definitions.

By $\mathcal{P}(X)$ we denote set of all subsets of space X, and by diam(.) diametar of given subsets of metric space.

Definition 1. *Let (X, d) be a metric space. Measure of noncompactness on X is an arbitrary function $\phi : \mathcal{P}(X) \to [0, \infty]$ which satisfies following conditions:*

1) $\phi(A) = \infty$ if and only if A is an unbounded set;

2) $\phi(A) = \phi(\overline{A})$;

3) $\phi(A) = 0$ if and only if A is a totally bounded set;

4) from $A \subseteq B$ follows $\phi(A) \leq \phi(B)$;

5) if X is a complete, and if $\{B_n\}_{n\in N}$ is sequence of closed subsets of X such that $B_{n+1} \subseteq B_n$ for each $n \in N$ and $\lim_{n\to\infty} \phi(B_n) = 0$, then $K = \bigcap_{n\in N} B_n$ is a nonempty compact set.

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The most important examples of measures of noncompactness are:

1. function of Kuratowski
$\alpha(A) = \inf\{r > 0 \mid A \subseteq \bigcup_{i=1}^{n} S_i,\ S_i \subseteq X,\ \text{diam}(S_i) < r;\ 1 \leq i \leq n,\ n \in N\}$;
2. function of Hausdorff
$\chi(A) = \inf\{\varepsilon > 0\ :\ A$ has a finite $\varepsilon-$ net in $X\}$;
3. inner function of Hausdorff
$\chi_i(A) = \inf\{\varepsilon > 0\ :\ A$ has a finite $\varepsilon-$ net in $A\}$;
4. function of Istratescu
$I(A) = \inf\{\varepsilon > 0\ :\ A$ is no contains a infinite $\varepsilon-$ discrete set in $A\}$.

Relations beetwen this functions are given by following inequality, which are obtain by Danes [4]:

$$\chi(A) \leq \chi_i(A) \leq I(A) \leq \alpha(A) \leq 2\chi(A).$$

Measures of noncompactness have many applications in topology, nonlinear analysis and theory of differential equations (see [1], [3], [10] and [11]). We can not generalize this notion to arbitrary topological space because completness is defined only for class of uniform spaces. In this paper we introduce notion of the measure of noncompactness on arbitrary uniform space (not necessery Haussdorff) and give an example of such function. We also give an characterization of complete uniform spaces and generalisations of some results from [6].

2. Main results

Let X be a nonempty set and $d : X \times X \to [0, \infty)$ function which satisfies following conditions:

1. $d(x, x) = 0$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq d(x, z) + d(z, y)$

for each $x, y, z \in X$ then ordered pair (X, d) is caled *pseudometric space.*

Let (X, d) be a pseudometric space. We defined equivalence relation $\sim$ on X by $x \sim y$ if and only if $d(x, y) = 0$. Then quotient space $\tilde{X} \equiv X/\sim$ is the metric space with quotient metric. Image of $x \in X$ by quotient mapping $p : X \to \tilde{X}$, we denote by $\tilde{x}$, image of set $A \subseteq X$ by $\tilde{A}$, and quotient metric by $\tilde{d}$. If $\{x_n\}$ is a Cauchy sequence in (X, d) then $\{\tilde{x}_n\} \subseteq \tilde{X}$ is also Cauchy sequence, because $d(x_i, x_j) = \tilde{d}(\tilde{x}_i, \tilde{x}_j)$ for each $i, j \in \mathcal{N}$. So pseudometric space (X, d) is complete if and only if its quotient space $(\tilde{X}, \tilde{d})$ is a complete space.

Definition 1'. *Let (X, d) be a pseudometric space. Measure of noncompactness on X, is an arbitrary function $\phi : \mathcal{P}(X) \to [0, \infty]$ which satsfies following conditions:*

1) $\phi(A) = \infty$ if and only if A is an unbounded set;
2) $\phi(A) = \phi(\overline{A})$;
3) $\phi(A) = 0$ if and only if A is a totally bounded set;
4) from $A \subseteq B$ follows $\phi(A) \leq \phi(B)$;
5) if X is a complete, and if $\{B_n\}_{n \in N}$ is sequence of closed subsets of X such that $B_{n+1} \subseteq B_n$ for each $n \in N$ and $\lim_{n\to\infty} \phi(B_n) = 0$, then $K = \bigcap_{n \in N} B_n$ is a nonempty compact set.

Proposition. *Let (X, d) be a pseudometric space. Function $\phi : \mathcal{P}(X) \to [0, \infty]$ is a measure of noncompactnes on X if and only if function $\tilde{\phi} : \mathcal{P}(\tilde{X}) \to [0, \infty]$ defined by*

$$\tilde{\phi}(\tilde{A}) = \phi(A)$$

is the measure of noncompactness on $\tilde{X}$.

Proof. $\operatorname{diam}(A) = \operatorname{diam}(\tilde{A})$ implies 1). F is a closed (totally bounded) set in pseudometric space (X, d) if and only if $\tilde{F}$ is such set in space $(\tilde{X}, \tilde{d})$. This implies 2) and 3). 4) follows from: $A \subseteq B$ if and only if $\tilde{A} \subseteq \tilde{B}$.

Let (X, d) be a complete pseudometric space, and $\{B_n\}_{n \in N}$ be a sequence of closed subsets of X such that $B_{n+1} \subseteq B_n$ for each $n \in N$ and $\lim_{n\to\infty} \phi(B_n) = 0$. Then $(\tilde{X}, \tilde{d})$ is a complete metric space and $\{\tilde{B}_n\}_{n \in N}$ be a sequence of closed subsets of $\tilde{X}$ such that $\tilde{B}_{n+1} \subseteq \tilde{B}_n$ for each $n \in N$ and $\lim_{n\to\infty} \tilde{\phi}(\tilde{B}_n) = 0$. So $\tilde{K} = \bigcap_{n \in N} \tilde{B}_n$ is a nonempty compact set. So, $\bigcap_{n \in N} B_n$ is nonempty closed totally bounded set. This implies that this set is compact.

Now we give definition of measure of noncompactness on arbitrary uniform space.

Definition 2. *Let X be a uniform space. Measure of noncompactness on X is an arbitrary function $\Phi : \mathcal{P}(X) \to [0,\infty]$, which satisfies following conditions:*

1) $\Phi(A) = \infty$ if and only if set A is unbounded;

2) $\Phi(A) = \Phi(\overline{A})$;

3) from $\Phi(A) = 0$ follows that A is totally bounded set;

4) from $A \subseteq B$ follows $\Phi(A) \leq \Phi(B)$;

5) if X is complete space, and if $\{B_n\}_{n\in N}$ is sequence of closed subsets of X such that $B_{n+1} \subseteq B_n$ for each $n \in N$ and $\lim_{n\to\infty} \Phi(B_n) = 0$, then $K = \bigcap_{n\in N} B_n$ is a nonempty compact set.

Remark. We can see that definition given on uniform spaces coincides with definition given for metric spaces. This follows from characterization of uniform spaces obtained by Đ. Kurepa [9].

A family of sets has the *finite intersection property* if and only if the intersection of each its finite subfamily is nonempty.

Now we give following matching theorem on complete uniform spaces, which generalize earlier results of Horvath [5,6].

Theorem 1. *Let X be a complete uniform space, Φ measure of noncompactness on X and $\{G_j \mid j \in J\} \subseteq \mathcal{P}(\mathcal{X})$ family of its closed subsets which has finite intersection property such that for all $t > 0$ there exists finite set $A \subseteq J$ such that $\Phi(\bigcap_{j\in A} G_j) < t$. Then*

$$\bigcap_{j\in J} G_j \neq \emptyset.$$

Proof. Let X be a complete uniform space and $\{G_j | j \in J\}$ family of closed subsets which satisfies conditions of this statement. Then, for each $n \in \{1,2,...\}$ there exists finite set $F(n) \subseteq J$ such that $\Phi(\bigcap_{j\in F(n)} G_j) < \frac{1}{n}$. Now we define sequence of sets B_n by:

$$B_1 = \bigcap_{j\in F(1)} G_j; \quad \dots \quad ; B_{n+1} = B_n \bigcap \left(\bigcap_{j\in F(n+1)} G_j \right); \dots$$

Finite intersection property implies that $B_n \neq \emptyset$ for each $n \in \{1,2,...\}$. Every B_n is a closed set, $\alpha(B_n) < \frac{1}{n}$ and $B_{n+1} \subseteq B_n$ for each n. From definition follows that $K = \bigcap_{n\in N} B_n$ is a nonempty compact set. For each finite $F \subseteq J$ we have $C_{F,n} = (\bigcap_{j\in F} G_j \bigcap B_n) \neq \emptyset$ for each n. Sequence $\{C_{F,n}\}_{n\in N}$, satisfies conditions of Theorem 1 and so $\bigcap_{n\in N} C_{F,n}$ is a nonempty subset of K. Hence $(\bigcap_{j\in F} G_j) \bigcap K \neq \emptyset$ for each finite set $F \subseteq J$. Family of closed sets $\{G_j \bigcap K\}_{j\in J}$ has finite intersection property, by compactness of K follows $\bigcap_{j\in J}(G_j \bigcap K) \neq \emptyset$, which implies $\bigcap_{j\in J} G_j \neq \emptyset$.

Let $f : X \longrightarrow \mathcal{R}$, where X is an arbitrary topological space. Function f is upper (lower) semicintinuous at point $x_0 \in X$ if:

$$\limsup_{x\to x_0} f(x) \leq f(x_0)$$
$$\left(\liminf_{x\to x_0} f(x) \geq f(x_0)\right).$$

An f is upper (lower) semicontinuous on set $A \subseteq X$ if it has this property at each point of A. $f : X \longrightarrow R$ is upper (lower) semicontinuous function on X if and only if set $\{x \in X : f(x) < r\}$ ($\{x \in X : f(x) > r\}$) is open for each $r \in R$. If X is the compact topological space and $f : X \longrightarrow R$ upper (lower) semicontinuous function on X, then f has maximum (minimum) on X. In next result we extend this statements to the complete uniform spaces. Our result generalize earlier result obtained by Horvath [6] for complete metric spaces.

Theorem 2. *Let X be a complete uniform space, Φ measure of noncompactness on X and $f : X \to \mathcal{R}$ upper (lower) semicontinuous function such that*

$$\inf_{x\in X} \Phi(\{y \in X \mid f(x) \leq f(y)\}) = 0$$
$$\left(\inf_{x\in X} \Phi(\{y \in X \mid f(y) \leq f(x)\}) = 0\right)$$

Then f has a maximum (minimum) on X.

Proof. Let $\{G_x\}_{x\in X}$ be the family of subsets defined by $G_x = \{y \in X : f(x) \leq f(y)\}$ ($G_x = \{y \in X : f(y) \leq f(x)\}$). From Theorem 1 follows that

$$\bigcap_{x\in X} G_x \neq \emptyset$$

and so there exists $x_0 \in X$ such that $x_0 \in G_x$ for each $x \in X$.

Theorem 3. *Let X be a uniform space and $\{d_i|i \in I\}$ a family of pseudometric which defined topology on X. Denote by ϕ_i arbitrary measure of noncompactness on pseudometric space (X, d_i) for each $i \in I$. Then the function $\Phi^* : X \to [0, \infty]$ defined by*

$$\Phi^*(Q) = \sup_{i \in I} \phi_i(Q)$$

for each $Q \subseteq X$, is the measure of noncompactness on X.

Proof. A is an unbounded set if and only if $\sup_{i\in I} \operatorname{diam}_i(Q) = \infty$. This implies 1). From $\phi_i(A) = \phi_i(\overline{A})$ follows 2). If $\Phi^*(A) = 0$ by 2) we have $\Phi^*(\overline{A}) = 0$, and so $\phi_i(\overline{A}) = 0$ for each $i \in I$. This implies that set $\overline{A}$ in compact in space (X, d_i) for $i \in I$, because it is closed and totally bounded. So, $\overline{A}$ is a compact set ([7]), and we have 3). Now, A is totally bounded because it is subset of compact set $\overline{A}$. $A \subseteq B$ follows $\phi_i(A) \leq \phi_i(B)$ for each $i \in I$. This implies 4).

Completness of space X implies that space (X, d_i) is a complete for each $i \in I$. Now we have $\lim_{n\to\infty} \phi_i(B_n) = 0$ and so K is a nonempty compact set in space (X, d_i), for each $i \in I$. This implies that K is a compact set in the topology of uniform space X.

Now we give following characterization of complete uniform spaces, which generalize Theorem 23 from [7] - page 193.

Theorem 4. *Let X be a uniform space and $\{d_i|i \in I\}$ a family of pseudometic which defined the topology on X. Then X is a complete space if and only if*

$$\bigcap_{j\in J} G_j \neq \emptyset$$

for each family of closed sets $\{G_j \mid j \in J\} \in \mathcal{P}(X)$ which satisfies:

1) $\{G_j \mid j \in J\}$ has finite intersection property;

2) for all $t > 0$ there exists a finite set $A \subseteq J$ such that $\Phi^*(\bigcap_{j\in A} G_j) < t$.

Proof. Necessity follows from Theorem 1.

Now we shall proved sufficiency. Let B_n be a familly of closed subsets of X such that $B_n \subseteq B_{n+1}$ for each $n \in \mathcal{N}$ and $\lim \operatorname{diam}_i(B_n) = 0$ (diam_i denote diametar in (X, d_i)) for each $i \in I$. Then $\cap B_n$ is a closed totally bounded set and $\lim \Phi^*(B_n) = 0$. From the preceding fact of statement follows that $\bigcap_{n\in\mathcal{N}} B_n \neq \emptyset$ and so, from famous result of Kuratowski [8] follows, that each of spaces (X, d_i) is complete and so X is complete.

Corollary 1. *Let (X, d) be a uniform space and $\{d_i | i \in I\}$ a family of pseudometrics which defined topology on X. Then X is a complete space if and only if*

$$\bigcap_{j \in J} G_j \neq \emptyset$$

for each family of closed sets $\{G_j \mid j \in J\} \in \mathcal{P}(X)$ which satisfies:

1) $\{G_j \mid j \in J\}$ has finite intersection property;

2) for each $t > 0$ there exists $j \in J$ such that $\Phi^*(G_j) < t$.

Following fixed point result generalize earlier result of Horvath [6].

Corollary 2. *Let X be a complete uniform Hausdorff space and $g : X \to X$ be a function such that:*

1) function $x \mapsto \sup_{i \in I} d_i(x, g(x))$ is lower semicontinuous;

2) $\inf_{x \in X} \sup_{i \in I} d_i(x, g(x)) = 0$;

3) $\inf_{x \in X} \Phi^*(\{y \in X \mid \sup_{i \in I} d_i(y, g(y)) \leq \sup_{i \in I} d_i(x, g(x))\}) = 0$.

Then there exists $x_0 \in X$ such that $x_0 = g(x_0)$.

Proof. Let $f(x) = \sup_{i \in I} d_i(x, g(x))$. From Theorem 2 follows that there exists x_0 such that $f(x_0) = 0$, because $inf_{x \in X} f(x) = 0$. This implies $d_i(x_0, g(x_0)) = 0$ for each $i \in I$ and so $x_0 = g(x_0)$.

At the end of this paper we give one example for calculating of function Φ^*.

Example. Let X be an Hausdorff topological vector space and $\{d_i | i \in I\}$ family of pseudometrics which defined the topology on X. Let $\{r_i\}_{i \in I} \subseteq \mathcal{R}_+$ be a bounded set of nonnegative reals, $\{x_i\}_{i \in I} \subseteq X$ and $A = \{x \in X \mid d_i(x, x_i) \leq r_i \text{ for some } i \in I\}$. If ϕ_i is a Hausdorff measure of noncompactness on the quotient space (X, d_i) and (X, d_i) is a locally bounded metric linear space for each $i \in I$, then from results obtained in [2] follows:

$$\Phi^*(A) = \sup_{i \in I} r_i.$$

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