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Singer Orthogonality and James Orthogonality in the So-Called Quasi-Inner Product Space

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Abstract. In this note we prove that, in a quasi-inner product space, S-orthogonality and J-orthogonality can be defined with the best approximations.

1. Introduction

Let X be a real smooth normed space of dimension greater than 1. It is well known that the functional:

$$(1)\qquad g(x,y) := \|x\| \lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}, \qquad (x, y \in X)$$

always exists (see [3]).

This functional has the following properties:
The functional g is linear in the second argument and we have:

$$(2)\qquad \begin{aligned} g(\alpha x, y) &= \alpha g(x,y), \quad (\alpha \in R); \\ g(x,x) &= \|x\|^2, \qquad |g(x,y)| \le \|x\|\,\|y\|. \end{aligned}$$

Definition 1 ([6]). A normed space X is a quasi-inner product space (*q.i.p.* space) if the equality

$$(3)\qquad \|x+y\|^4 - \|x-y\|^4 = 8\big[\|x\|^2 g(x,y) + \|y\|^2 g(y,x)\big]$$

holds for all $x, y \in X$.

The space of sequences l^4 is a *q.i.p.* space, but l^1 is not a *q.i.p.* space.

It is proved in [6] and [7] that a *q.i.p.* space X is very smooth, uniformly smooth, strictly convex and, in the case of Banach space, reflexive.

The orthogonality of the vector $x \neq 0$ to vector $y \neq 0$ in a normed space X may be defined in several ways. We mention some kinds of orthogonality and their denotations:

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$$x \perp_B y \quad \Leftrightarrow \quad (\forall \lambda \in R) \quad \|x\| \leq \|x + \lambda y\| \quad \text{(Birkhoff orthogonality, brief by } B\text{-orthogonality)},$$

$$x \perp_J y \quad \Leftrightarrow \quad \|x - y\| = \|x + y\| \quad \text{(James orthogonality)},$$

$$x \perp_S y \quad \Leftrightarrow \quad \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \quad \text{(Singer orthogonality)}.$$

In the papers [5], [6] and [7], by the use of functional g, the following orthogonal relations are introduced:

$$x \perp_g y \quad \Leftrightarrow \quad g(x, y) = 0,$$

$$x \overset{g}{\perp} y \quad \Leftrightarrow \quad g(x, y) + g(y, x) = 0,$$

$$x \underset{g}{\perp} y \quad \Leftrightarrow \quad \|x\|^2 g(x, y) + \|y\|^2 g(y, x) = 0.$$

If there exists an inner product $\langle \cdot, \cdot \rangle$ in X^2, then it is easy to see that

$$x \rho y \quad \Leftrightarrow \quad \langle x, y \rangle = 0$$

hold for every

$$\rho \in \left\{ \perp_B, \perp_J, \perp_S, \perp_g, \overset{g}{\perp}, \underset{g}{\perp} \right\}.$$

For more detail on B-orthogonality and g-orthogonality see papers [1], [2], [4], [5], [7], [8] and [9].

B-orthogonality has priority in accordance with above quoted orthogonalities. Namely, in the case of B-orthogonality, the orthogonality of the vector x to the vector y can be defined as

$$P_{[y]} x = 0,$$

i.e., with the best approximation of vector x with vectors from

$$[y] = \text{span}\{y\}.$$

2. Main Result

In the proof of our theorem we shall use the following known assertions:

1) (T.2, [6]). In a smooth space X we have $x \perp_g y \quad \Leftrightarrow \quad x \perp_B y$, i.e., the relation $\perp_g$ is equivalent with the relation $\perp_B$.
2) ([9]). If X is a *q.i.p.* space then $x \underset{g}{\perp} y \quad \Leftrightarrow \quad x \perp_J y$ and $x \overset{g}{\perp} y \quad \Leftrightarrow \quad x \perp_S y$.

The following assertion has important value.

Theorem 1. *Let X be a* q.i.p. *space and $x, y \in X \backslash \{0\}$. Then the following equivalence relations hold:*

a) $x \perp_S y \quad \Leftrightarrow \quad x \perp_B z$,
where is $z = g\big(y, x/\|x\|^2\big) x + y \in \text{span}\{x, y\}$.

b) $x \perp_J y \quad \Leftrightarrow \quad x \perp_B h,$
where is $h = \|x\|^2 y + g\big(\|y\|^2 y, x/\|x\|^2\big)x \in \operatorname{span}\{x, y\}.$

Proof. a) Using 2) we obtain

$$\begin{aligned} g(x,y) + g(y,x) &\equiv g(x,y) + \|x\|^2 g(y, x/\|x\|^2) \\ &\equiv g(x,y) + g(x,x) g(y, x/\|x\|^2) \\ &\equiv g(x, g(y, x/\|x\|^2)x + y) \\ &\equiv g(x,z), \end{aligned}$$

where

$$z = g(y, x/\|x\|^2)x + y.$$

Hence we have

$$x \overset{g}{\perp} y \quad \Leftrightarrow \quad g(x,y) + g(y,x) = 0 \quad \Leftrightarrow \quad g(x,z) = 0.$$

On the other hand by 1) and 2) we have

$$x \perp_g z \quad \Leftrightarrow \quad x \perp_B z, \quad \text{so} \quad x \perp_S y \quad \Leftrightarrow \quad x \perp_B z.$$

b) Using 2) we have

$$\begin{aligned} \|x\|^2 g(x,y) + \|y\|^2 g(y,x) &\equiv \|x\|^2 g(x,y) + \|x\|^2 g(\|y\|^2 y, x/\|x\|^2) \\ &\equiv g(x, \|x\|^2 y) + g(\|y\|^2 y, x/\|x\|^2)\|x\|^2 \\ &\equiv g(x,h), \end{aligned}$$

where

$$h = g(\|y\|^2 y, x/\|x\|^2)x + \|x\|^2 y.$$

Hence

$$x \underset{g}{\perp} y \quad \Leftrightarrow \quad g(x,h) = 0.$$

By 1) and 2) we get

$$x \perp_J y \quad \Leftrightarrow \quad x \perp_B h. \qquad \square$$

Problem. Let X be a smooth and uniformly convex normed space in which the equivalence relations a) and b) hold. Check whether the space X is a $q.i.p.$ space.

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