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Geometry of semi-Invariant Submanifolds of a Riemannian Product Manifold

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Abstract. In this paper, we show new results on semi-invariant submanifolds of a Riemannian product manifold and introduce equations related with geometry of semi-invariant submanifold in real product space forms. We characterize and study the geometry of the semi-invariant submanifolds in a Riemannian product manifold.

1. Introduction

The notion of a semi-invariant submanifold do not seem to be widely used in the literature and in fact that papers directly related to the problems are scarce so far.

The geometry of invariant submanifolds inherits almost all properties of the ambient manifold and the study of invariant submanifolds is not so interesting from point of view of the geometry of submanifolds. On the other hand, the theory of anti-invariant submanifolds is very nice topic in modern differential geometry and it has been studied by many geometers since 1970.

Generalizing the geometry of invariant and anti-invariant submanifolds ideas, A. Bejancu defined CR-submanifold in almost Hermitian (Kaehlerian) manifolds and defined semi-invariant submanifods of locally product Riemannian manifolds [1]. Similar definitions were applied to submanifolds of almost contact metric manifolds.

Firstly, S. Tachibana [7] introduced and studied a class of locally Riemannian product manifolds. After, A. Bejancu [1] and K. Matsumoto [3] defined and studied the geometry of semi-invariant submanifolds of locally Riemannian product manifolds.

In [6], X. Senlin and N. Yilong defined invariant submanifolds of two Riemannian product manifolds and shown that it could be written as product

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of two manifolds. Also, they researched pseudo-umbilical invariant submanifold of a Riemannian product manifold.

In [5], we defined semi-invariant submanifolds of a Riemannian product manifold and we studied the geometry of these type submanifolds. Because, many papers on these type submanifolds have been published and are going to published proving that the topic is a very interesting in differential geometry. In [5], we defined and studied the geometry of semi-invariant submanifolds of a Riemannian product manifold and obtained many very interesting results. Necessary and sufficient conditions are given on semi-invariant submanifold of a Riemannian product manifold to be a locally Riemannian product manifold. Moreover, on integrability of invariant distribution and anti-invariant distribution were investigated.

In this paper, we get on studying the geometry of semi-invariant submanifolds of a Riemannian product manifold and characterize semi-invariant submanifold. Necessary and sufficient conditions are given on submanifold of a Riemannian product manifold to be semi-invariant submanifold. Moreover, we give two examples for semi-invariant submanifold of Riemannian product manifold to illustrate our results.

2. Preliminaries

In this section, we give the definitions and terminology used throughout this paper. We recall some necessary facts and formulas from the theory of submanifolds in any Riemannian manifold. For an arbitrary submanifold M of any Riemannian manifold $\bar{M}$, the Gauss and Weingarten formulas are, respectively, given by formulas

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2}$$

for any vector fields X, Y tangent to M and V normal to M, where $\bar{\nabla}$, ∇ denote the Levi-Civita connections on $\bar{M}$ and M, respectively. Moreover, $h : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM^{\perp})$ is the second fundamental form of M in $\bar{M}$, where $\Gamma(TM)$ denote the Lie algebra of vector fields on M. $\nabla^{\perp}$ is the normal connection on the normal bundle $\Gamma(TM^{\perp})$ and A_V is the shape operator of M with respect to V. Furthermore, A_V and h are related by formula

$$g(A_V X, Y) = g(h(X, Y), V), \tag{3}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$, where g denotes the Riemannian metric on M as well as $\bar{M}$.

Now, we denote the Riemannian curvature tensors of the connections $\bar{\nabla}$ and ∇ by $\bar{R}$ and R, respectively, then the equations of Gauss, Codazzi and

Ricci are, respectively, given by formulas

$$g(\bar{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \tag{4}$$

$$g(\bar{R}(X,Y)\xi,\eta) = g(\bar{R}(X,Y)^{\perp}\xi,\eta) - g([A_{\xi},A_{\eta}]X,Y) \tag{5}$$

and

$$\{\bar{R}(X,Y)Z\}^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{6}$$

for any vector fields X,Y,Z,W tangent to M and ξ,η, normal to M, where $\{\bar{R}(X,Y)Z\}^{\perp}$ denotes the normal component of $\bar{R}(X,Y)Z$ and the covariant derivative $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\nabla_X Y,Z) - h(\nabla_X Z,Y) \tag{7}$$

for any vector fields X,Y,Z tangent to M [4].

Definition 2.1. Let M be an n-dimensional submanifold of any Riemannian manifold $\bar{M}$. The mean-curvature vector field H of M is defined by formula

$$H = \frac{1}{n}\sum_{j=1}^{n} h(e_j,e_j),$$

where, $\{e_j\}$, $1 \leq j \leq n$, is a locally orthonormal basis of $\Gamma(TM)$. If a submanifold M has one of the conditions

$$h = 0, \quad H = 0, \quad h(X,Y) = g(X,Y)H, \quad \lambda \in C^{\infty}(M,\mathbb{R}),$$

then it is said to be totally geodesic, minimal and totally-umbilical submanifold, respectively, [4].

Furthermore, the norm of h is defined by

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)). \tag{8}$$

3. The Riemannian Product of the Riemannian Manifolds

Let $(\bar{M}_1,\bar{g}_1)$ and $(\bar{M}_2,\bar{g}_2)$ be the Riemannian manifolds with dimensions m_1, m_2, respectively and $\bar{M}_1 \times \bar{M}_2$ be the Riemannian product manifold of the Riemannian manifolds $\bar{M}_1$ and $\bar{M}_2$. We denote the projections mappings of $\Gamma(T(\bar{M}_1 \times \bar{M}_2))$ onto $\Gamma(T\bar{M}_1)$ and $\Gamma(T\bar{M}_2)$ by π_* and σ_*, respectively, then we have

$$\pi_* + \sigma_* = I, \quad \pi_*^2 = \pi_*, \quad \sigma_*^2 = \sigma_*, \quad \text{and} \quad \pi_* o \sigma_* = \sigma_* o \pi_* = 0.$$

The Riemannian metric tensor of the Riemannian product manifold $\bar{M} = \bar{M}_1 \times \bar{M}_2$ is given by

$$g(X,Y) = \bar{g}_1(\pi_* X,\pi_* Y) + \bar{g}_2(\sigma_* X,\sigma_* Y),$$

for any vector fields X, Y tangent to $\bar{M}$. From the definition of g, $\bar{M}_1$ and $\bar{M}_2$ are totally geodesic submanifolds of $\bar{M}_1 \times \bar{M}_2$. Setting $F = \pi_* - \sigma_*$, then we can easily see that $F^2 = I$ and g satisfies

$$g(FX, Y) = g(X, FY), \tag{9}$$

for any vector fields X, Y tangent to $\bar{M}$. Thus F defines a Riemannian almost product structure on $\bar{M}$. Furthermore, we denote the Levi-Civita connection on $\bar{M}$ by $\bar{\nabla}$, then we have

$$(\bar{\nabla}_X F)Y = 0 \tag{10}$$

for any vector fields X, Y tangent to $\bar{M}$. (For the more detail, we refer the readers to [6]).

In the rest of this paper, we denote the Riemannian product manifold $(\bar{M}_1 \times \bar{M}_2, \bar{g}_1 \otimes \bar{g}_2)$ by $(\bar{M}, g)$.

If $\bar{M}_1(c_1)$ is a real space form with sectional curvature c_1 and $\bar{M}_2(c_2)$ is a real space form with sectional curvature c_2, then the Riemannian curvature tensor $\bar{R}$ of $\bar{M} = \bar{M}_1(c_1) \times \bar{M}_2(c_2)$ is given by

(11)
$$\begin{aligned}\bar{R}(X, Y)Z = \frac{1}{4}(c_1 + c_2)\{&g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX \\ &- g(FX, Z)FY\} + \frac{1}{4}(c_1 - c_2)\{g(FY, Z)X - g(FX, Z)Y \\ &+ g(Y, Z)FX - g(X, Z)FY\}\end{aligned}$$

for any vector fields X, Y and Z tangent to $\bar{M}$ [4].

4. SEMI-INVARIANT SUBMANIFOLDS OF A RIEMANNIAN PRODUCT MANIFOLD

Definition 4.1. Let $\bar{M}$ be a Riemannian product manifold with Riemannian almost product structure F. A submanifold M of $\bar{M}$ is called a semi-invariant submanifold of $\bar{M}$ if there exists a differentiable distribution; $D : x \longrightarrow D_x \subset T_x(M)$ on M satisfying the following conditions:

i) D is a invariant distribution, i.e., $F(D_x) \subseteq D_x$, for each $x \in M$, and
ii) the complementary orthogonal distribution of D $D^\perp : x \longrightarrow D_x^\perp \subset T_x(M)$ is an anti-invariant, i.e., $F(D_x^\perp) \subseteq T_x(M)^\perp$, for each $x \in M$.

In the sequel, we put $\dim(\bar{M}) = m$, $\dim(M) = n$, $\dim(D) = p$, $\dim(D^\perp) = q$ and $\operatorname{codim}(M) = m - n$. If $q = 0$, then semi-invariant submanifold M is called an invariant submanifold of $\bar{M}$, and if $p = 0$, then M is called an anti-invariant submanifold of $\bar{M}$. If $pq \neq 0$, then M is said to be proper semi-invariant submanifold.

Now, we suppose that M is a proper semi-invariant submanifold of Riemannian product manifold $\bar{M}$ and denote the orthogonal complementary of

$F(D^\perp)$ in $TM^\perp$ by ν, then we have direct sum

$$TM^\perp = F(D^\perp) \oplus \nu. \tag{12}$$

We can easily see that ν is an invariant vector subbundle with respect to F.

For any vector field X tangent to M, we put

$$FX = fX + \omega X, \tag{13}$$

where fX (resp. ωX) is the tangential (resp. normal) part of FX.

Similarly, for any vector field V normal to M, we put

$$FV = BV + CV, \tag{14}$$

where BV (resp. CV) is the tangential (resp. normal) part of FV.

For any vector fields X and Y tangent to M, we have $g(FX, Y) = g(fX, Y)$ which shows that $g(fX, Y)$ is symmetric. Similarly, for any vector fields U and V normal to M, from (9) and (14), we have $g(FV, U) = g(V, CU)$, which shows that $g(CV, U)$ is also symmetric.

Furthermore, for any vector field X tangent to M, we have

$$X = f^2X + B\omega X \quad \text{and} \quad \omega fX + C\omega X = 0$$

or,

$$I = f^2 + B\omega \quad \text{and} \quad \omega f + C\omega = 0. \tag{15}$$

In the same way, for any vector field V normal to M, we get

$$V = \omega BV + C^2V \quad \text{and} \quad fBV + BCV = 0$$

or,

$$I = \omega B + C^2 \quad \text{and} \quad fB + BC = 0. \tag{16}$$

Example 4.1. Considering in $\mathbb{R}^6 = \mathbb{R}^5 \times \mathbb{R}$ the submanifold M given by the equations

$$x_4 = x_1 + \frac{1}{2}(x_2 + x_3)^2, \quad x_5 + x_6 = 0.$$

Then we have

$$TM = \operatorname{span}\Big\{u_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, u_2 = \frac{\partial}{\partial x_2} + (x_2 + x_3)\frac{\partial}{\partial x_4}, u_3 = \frac{\partial}{\partial x_3} + (x_2 + x_3)\frac{\partial}{\partial x_4}, u_4 = \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6}\Big\}$$

and

$$TM^\perp = \operatorname{span}\Big\{V_1 = \frac{\partial}{\partial x_1} + (x_2 + x_3)\frac{\partial}{\partial x_2} + (x_2 + x_3)\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}, V_2 = \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}\Big\}.$$

It follow that $D = \operatorname{span}\{u_1, u_2, u_3\}$, $D^\perp = \operatorname{span}\{u_4\}$. Thus M is a 4-dimensional semi-invariant submanifold of $\mathbb{R}^6$.

Now, we give a characterization for semi-invariant submanifold in a Riemannian product manifold.

Theorem 4.1. *Let M be a submanifold of a Riemannian product manifold $\bar{M}$. Then M is a semi-invariant submanifold if and only if $\omega f = 0$.*

Proof. We suppose that M is a semi-invariant submanifold of $\bar{M}$ and denote the projection operators on the distributions D and $D^{\perp}$ by P and Q, respectivey, then we have

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q \quad \text{and} \quad PQ = QP = 0.$$

For any $X \in \Gamma(TM)$. For $X = PX + QX$, we can write

$$FX = FPX + FQX$$
$$fX + \omega X = fPX + \omega QX, \quad \omega PX = fQX = 0.$$

Thus we can infer $fX = fPX$, i.e., $fP = f$ which implies that $QfP = Qf = 0$. By using (15), we have $\omega fP + C\omega P = 0$. Since $\omega P = 0$, we conclude

$$\omega f = 0. \tag{17}$$

Conversely, Let us suppose that a submanifold M of a Riemannian product manifold $\bar{M}$ and $\omega f = 0$. So from the right side of (15) we have

$$C\omega = 0. \tag{18}$$

Furthermore, for any vector field X tangent to M and vector field V normal to M, we have

$$g(X, BV) = g(\omega X, V)$$

and

$$g(X, FBV) = g(F\omega X, V)$$
$$g(X, fBV) = g(C\omega X, V) = 0,$$

which gives us

$$fB = 0. \tag{19}$$

From (16) and (19), we can derive

$$BC = 0. \tag{20}$$

From the equations (15) and (16), respectively, we get

$$f^3 = f \quad \text{and} \quad C^3 = C. \tag{21}$$

If we put

$$P = f^2, \quad Q = I - P, \tag{22}$$

then we can easily to see that

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0,$$

which show that P and Q are orthogonal complementary projection operators and define complementary distributions such as D and $D^{\perp}$, respectively.

Since ωf is identically zero, from the equations (21) and (22) we can derive

$$fP = f, \quad fQ = 0, \quad QfP = 0 \;\; and \;\; \omega P = 0.$$

These equations show that the distributions D is an invariant and the distribution $D^{\perp}$ is an anti-invariant. This completes the proof of the theorem. □

Now, for any vector fields X, Y tangent to M, we have

$$\begin{aligned}
\bar{\nabla}_X FY &= F\bar{\nabla}_X Y \\
\bar{\nabla}_X fY + \bar{\nabla}_X \omega Y &= F\bar{\nabla}_X Y + Fh(X,Y) \\
\bar{\nabla}_X fY + h(X,fY) - A_{\omega Y}X + \nabla^{\perp}_X \omega Y &= f(\nabla_X Y) + \omega(\nabla_X Y) + Bh(X,Y) \\
&\quad + Ch(X,Y)
\end{aligned}$$

$$h(X,fY) + (\nabla_X f)Y - A_{\omega Y}X + (\nabla^{\perp}_X \omega)Y = Bh(X,Y) + Ch(X,Y).$$

Comparing the tangential and normal parts of the both sides of this last equation, we infer

$$(\nabla_X f)Y = A_{\omega Y}X + Bh(X,Y) \tag{23}$$

and

$$(\nabla_X \omega)Y = Ch(X,Y) - h(X,fY), \tag{24}$$

where the derivations of f and ω are, respectively, defined by

$$\begin{aligned}
(\nabla_X f)Y &= \nabla_X fY - f(\nabla_X Y), \\
(\nabla_X \omega)Y &= \nabla^{\perp}_X \omega Y - \omega(\nabla_X Y).
\end{aligned}$$

Definition 4.2. The Riemannian almost product manifold $\bar{M}$ is said to be a locally Riemannian product manifold if the Riemannian almost product structure F on $\bar{M}$ has no torsion, that is, $(\bar{\nabla}_X F)Y = \bar{\nabla}_X FY - F\bar{\nabla}_X Y = 0$. In this case F is called integrable, In other words the tensor $[F,F]$ vanishes identically on $\bar{M}$, where $[F,F]$ is the Nijenhuis tensor of F.

On the other hand, for a semi-invariant submanifold M of $\bar{M}$, If the distribution D is integrable and the Riemannian almost product structure f induced on each integral submanifold of D is integrable, then we say that structure f is partially integrable.

The vector fields X, Y tangent to D, if $[X,Y] \in \Gamma(D)$, then D is called completely integrable.

Lemma 4.1. *Let M be a semi-invariant submanifold of a Riemannian product manifold $\bar{M}$. For any vector fields X, Y tangent to $D^{\perp}$, we have*

$$A_{\omega X}Y = -A_{\omega Y}X. \tag{25}$$

Proof. For any vector field Z tangent to M, we have

$$\begin{aligned}
g((\nabla_Z f)X, Y) &= g(\nabla_Z fX, Y) - g(f\nabla_Z X, Y) \\
&= 0 - g(\nabla_Z X, fY) = 0,
\end{aligned}$$

for any vector fields $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(TM)$. By using (23), we have

$$0 = g((\nabla_Z f)X, Y) = g(A_{\omega X}Z + Bh(X,Z), Y) = g(A_{\omega X}Z, Y) + g(Bh(X,Z), Y).$$

Since $A_{\omega X}$ is self-adjoint, we have

$$g(A_{\omega X}Y, Z) = -g(Bh(X,Z), Y) = -g(h(X,Z), \omega Y) = -g(A_{\omega Y}X, Z)$$

which proves our assertion. □

Thus we have the following theorem.

Theorem 4.2. *Let M be a semi-invariant submanifold of a Riemannian product manifold $\bar{M}$. Then the distribution anti-invariant $D^\perp$ is completely integrable if and only if $A_{F(D^\perp)}D^\perp = \{0\}$.*

Proof. For any vector fields X and Y tangent to $\Gamma(D^\perp)$, by using (23) we have

$$\begin{aligned} f[X,Y] &= f(\nabla_X Y) - f(\nabla_Y X) = \nabla_X fY - (\nabla_X f)Y + (\nabla_Y f)X - \nabla_Y fX \\ &= (\nabla_Y f)X - (\nabla_X f)Y \\ &= A_{\omega X}Y + Bh(Y,X) - A_{\omega Y}X - Bh(X,Y) \\ &= A_{\omega X}Y - A_{\omega Y}X = 2A_{\omega X}Y, \end{aligned}$$

which gives the proof of Theorem. □

Example 4.2. Let M_1 be $\mathbb{R}^3$ with g_1 given in the canonical coordinates (x,y,z) by matrix, $\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ y & 0 & 1+y^2 \end{pmatrix}$ and M_2 be $\mathbb{R}^3$ with its canonical euclidian metric as g_2. Let us denote by (f_1, f_2, f_3) the canonical basis of $M_1 = \mathbb{R}^3$ and by (f_1^*, f_2^*, f_3^*) the canonical basis of $M_2 = \mathbb{R}^3$. Let us set $M = \mathrm{span}\{e_1, e_2, e_3\} \subset M_1 \times M_2$ with $e_1 = f_1, e_2 = f_2 + f_2^*$ and $e_3 = f_3 + f_3^*$. Let us set, at each point of M, $D = \mathrm{span}\{e_1\}$. Then M semi-invariant submanifold, as $F(D) = D$ and $D^\perp = \mathrm{span}\{e_2, e_3 - ye_1\} = \mathrm{span}\{f_2 + f_2^*, f_3 + f_3^* - yf_1\}$. Thus $F(D^\perp) = \mathrm{span}\{f_2 - f_2^*, f_3 - f_3^* - yf_1\}$. We can easily see that

$$\begin{aligned} &g(f_2 - f_2^*, e_1) = g_1(f_2, f_1) = 0, \\ &g(f_2 - f_2^*, e_2) = g_1(f_2, f_2) - g_2(f_2^*, f_2^*) = 0, \\ &g(f_2 - f_2^*, e_3) = g_1(f_2, e_3) = 0, \\ &g(f_3 - f_3^* - yf_1, e_1) = g_1(f_3 - yf_1, f_1) = y - y = 0, \\ &g(f_3 - f_3^* - yf_1, e_2) = g_1(f_3 - yf_1, f_2) = 0, \\ &g(f_3 - f_3^* - yf_1, e_3) = g_1(f_3 - yf_1, f_3) - g_2(f_3^*, f_3^*) = (1+y^2) - y.y - 1 = 0. \end{aligned}$$

Now, (x,y,z) are also coordinate functions on M and $D^\perp = ker\alpha$ with $\alpha = dx + ydz$. As $d\alpha = dy \wedge dz$, $\alpha \wedge d\alpha \neq 0$. Thus $D^\perp$ is not integrable.

Remark 4.1. It is well known that in the complex geometry, the invariant distributions of a CR-manifold are always even dimensional and the anti-invariant distribution $D^{\perp}$ is completely integrable. However, in the Riemannian product manifolds, these cases are quite different from complex case. For instance, in the Riemannian product manifold, the invariant distribution D and the anti-invariant distribution $D^{\perp}$ may be even or odd dimensional and $D^{\perp}$ is not necessarily integrable. For instance, in Example 4.1, semi-invariant submanifolds is even dimensional($\dim(D) = 3$, $\dim(D^{\perp}) = 1$), while in Example 4.2, semi-invariant submanifold is odd dimensional($\dim(D) = 1$, $\dim(D^{\perp}) = 2$). Thus we conclude that there are even and odd dimensional semi-invariant submanifolds in Riemannian product manifolds.

Theorem 4.3. *Let M be a semi-invariant submanifold of a Riemannian product manifold $\bar{M}$. Then the structure f is partially integrable if and only if the second fundamental form of M satisfies*

$$h(X, fY) = h(fX, Y), \tag{26}$$

for any vector fields X and Y tangent to D.

Proof. Let X and Y be vector fields in D. By using (25), we have

$$\begin{aligned}\omega[X, Y] &= \omega(\nabla_X Y) - \omega\nabla_Y X = (\nabla^t_Y \omega)X - (\nabla^t_X \omega)Y \\ &= Ch(Y, X) - h(Y, fX) - Ch(X, Y) + h(X, fY) \\ &= h(X, fY) - h(Y, fX).\end{aligned}$$

Thus the distribution D is integrable if and only if (26) holds. In this case, the integral submanifold M_1 of D is an invariant submanifold and M_1 is also a Riemannian product manifold, that is, f define a Riemannian almost product structure on M_1 and it is integrable on M_1. Consequently, f is partially integrable if and only if (26) holds. □

Taking into account that the curvature tensor field of $\bar{M}_1(c_1) \times \bar{M}_2(c_2)$ is given by (11), we have special forms for the structure equations of Gauss, Codazzi and Ricci the submanifold in $\bar{M} = \bar{M}_1(c_1) \times \bar{M}_2(c_2)$. By direct calculations, the equation of Gauss is given by

(27)

$$\begin{aligned}R(X, Y)Z = &\frac{1}{4}(c_1 + c_2)\{g(Y, Z)X - g(X, Z)Y + g(fY, Z)fX - g(fX, Z)fY\} \\ &+ \frac{1}{4}(c_1 - c_2)\{g(fY, Z)X - g(fX, Z)Y + g(Y, Z)fX - g(X, Z)fY\} \\ &+ A_{h(Y,Z)}X - A_{h(X,Z)}Y\end{aligned}$$

for any vector fields X, Y and Z tangent to M, where R is the Riemannian curvature tensor of M. The equation of Codazzi for semi-invariant submanifold M is given by

(28)
$$(\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) = \frac{1}{4}(c_1+c_2)\{g(fY,Z)\omega X - g(fX,Z)\omega Y\} + \frac{1}{4}(c_1-c_2)\{g(Y,Z)\omega X - g(X,Z)\omega Y\}.$$

Finally, the Ricci equation of M becomes

(29)
$$g(R^{\perp}(X,Y)V,U) + g([A_U,A_V]X,Y) = \frac{1}{4}(c_1+c_2)\{g(\omega Y,V)g(\omega X,U) - g(\omega X,V)g(\omega Y,U)\}$$

for any vector fields X, Y tangent to M and U, V normal to M.

Theorem 4.4. *Let M be a semi-invariant submanifolds of a Riemannian product manifold $\bar{M}$. Then M is a semi-Riemannian product if and only if the second fundamental form of M satisfies*

$$h(fX,U) = Ch(X,U), \tag{30}$$

for any $X \in \Gamma(D)$ and $U \in \Gamma(TM)$.

Proof. Let M be a semi-invariant product in $\bar{M}$. Then the leaves of distributions D and $D^{\perp}$ are total geodesic in M. Thus for any vector fields $X \in \Gamma(D)$ and $U \in \Gamma(TM)$, we have

$$\begin{aligned} \bar{\nabla}_U FX &= F\bar{\nabla}_U X \\ \nabla_U fX + h(fX,U) &= F\nabla_U X + Fh(U,X) \\ &= f\nabla_U X + \omega\nabla_U X + Bh(U,X) + Ch(U,X). \end{aligned} \tag{31}$$

Comparing the tangent and normal parts of the equation (31), respectively, we get

$$(\nabla_U f)X = Bh(U,X), \quad \omega(\nabla_U X) = 0 \tag{32}$$

and

$$h(fX,U) = Ch(U,X). \tag{33}$$

Conversely, we assume that (30) is satisfied. Then for any vector fields $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, we have

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\bar{\nabla}_X Y, Z) = g(\bar{\nabla}_X FY, FZ) = g(\bar{\nabla}_X fY, \omega Z) \\ &= g(h(X,fY),\omega Z) = g(Ch(X,Y),\omega Z) = 0. \end{aligned}$$

Similarly, for any vector fields $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$, we get

$$\begin{aligned} g(\nabla_W Z, X) &= g(\bar{\nabla}_W Z, X) = -g(\bar{\nabla}_W X, Z) = -g(\bar{\nabla}_W FX, FZ) \\ &= -g(\bar{\nabla}_W fX, \omega Z) = -g(h(fX, W), \omega Z) \\ &= -g(Ch(X, W), \omega Z) = 0, \end{aligned}$$

that is, $\nabla_X Y \in \Gamma(D)$ and $\nabla_W Z \in \Gamma(D^{\perp})$. Thus M is a semi-Riemannian product. □

Theorem 4.5. *Let M be a semi-invariant submanifolds of a Riemannian product manifold $\bar{M} = \bar{M}_1(c_1) \times \bar{M}_2(c_2)$. Then Ricci tensor and scalar curvature of M satisfy*

$$\begin{aligned} S(X, X) &= \frac{1}{4}(c_1 + c_2)\{(n-2)\|X\|^2 + tr(F)g(FX, X)\} \\ &+ \frac{1}{4}(c_1 - c_2)\{(n-2)g(FX, X) + tr(F)\|X\|^2\} \\ &+ ng(h(X, X), H) - \|h(X, e_i)\|^2 \end{aligned} \tag{34}$$

and

$$\begin{aligned} \tau = \frac{1}{4}(c_1 + c_2)\{n(n-2) + (trF)^2\} + \frac{1}{2}(c_1 - c_2)(n-1)tr(F) \\ + n^2\|H\|^2 - \|h\|^2, \end{aligned} \tag{35}$$

respectively, where τ is the scalar curvature of M.

Proof. By using (27), we get

$$S(X, X) = g(R(e_i, X)X, e_i), \quad \text{for} \quad i \in \{1, 2, ..., n\},$$

which gives us (34). From $\tau = S(e_j, e_j)$, for $j \in \{1, 2, ..., n\}$, we get (35). □

Thus we have the following proposition.

Proposition 4.1. *Let M be a n-dimensional semi-invariant submanifold of a Riemannian product manifold $\bar{M} = \bar{M}_1(c_1) \times \bar{M}_2(c_2)$. If M is totally geodesic and $tr(F) = 0$, then the Ricci tensor and scalar curvature of M satisfy*

1) $S(\cdot, \cdot) = \frac{1}{4}(n-2)\{(c_1 + c_2)g(\cdot, \cdot) + (c_1 - c_2)g(F\cdot, \cdot)\}$,
2) $\tau = \frac{1}{4}(c_1 + c_2)n(n-2)$.

Theorem 4.6. *Let M be a n-dimensional submanifold of a Riemannian product manifold $\bar{M} = \bar{M}_1(c_1) \times \bar{M}_2(c_2)$. If M is anti-invariant and 2dim(M)=dim($\bar{M}$), then M has constant sectional curvature with $\frac{1}{4}(c_1 + c_2)$.*

Proof. For any $X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned} g(A_{\omega X}Y - A_{\omega Y}X, Z) &= g(h(Y,Z), \omega X) - g(h(X,Z), \omega X) \\ &= g(Fh(Y,Z), X) - g(Fh(X,Z), Y) \\ &= g(F(\bar{\nabla}_Y Z - \nabla_Y Z), X) - g(F(\bar{\nabla}_X Z - \nabla_X Z), Y) \\ &= g(\bar{\nabla}_Y FZ, X) - g(\bar{\nabla}_X FZ, Y) \\ &= g(\bar{\nabla}_Y \omega Z, X) - g(\bar{\nabla}_X \omega Z, Y) \\ &= -g(A_{\omega Z}Y, X) + g(A_{\omega Z}X, Y) = 0, \end{aligned}$$

which is equivalent to

$$A_{\omega X}Y = A_{\omega Y}X. \tag{36}$$

From (25) and (36), we conclude $A_{\omega X}Y = 0$, for any $X, Y \in \Gamma(TM)$.

Since $2\dim(M) = \dim(\bar{M})$, the normal space $T^{\perp}M$ is spanned by $\{\omega X : X \in \Gamma(TM)\}$. Thus M is totally geodesic in $\bar{M}$. By using (27), we obtain

$$R(X,Y)Z = \frac{1}{4}(c_1 + c_2)\{g(Y,Z)X - g(X,Z)Y\}, \tag{37}$$

for any $X, Y, Z \in \Gamma(TM)$, which proves our assertion. □

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