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Warped Product Semi-Invariant Submanifolds in Almost Paracontact Metric Manifolds

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Abstract. In this paper we have investigated the existence of warped product semi-invariant submanifolds in almost para contact metric manifolds. Finally, we see that there exists no any warped product semi-invariant submanifold in almost para contact metric manifolds such that the contravariant vector field tangent to submanifold.

1. Introduction

It is well known that the notations of warped product are widely used in differential geometry as well as physics. The study of warped product manifolds was initiated by R.L. Bishop and B. O'Neill with differential geometric point of view[8]. After then several papers appeared which have dealt with various geometric aspects of warped product submanifolds[2, 5, 7, 11].

The notation of CR-warped product were first introduced by B-Y. Chen. Recently, he studied warped product CR-submanifolds in Kaehler manifolds and shown that there exist no warped product CR-submanifolds in the form $N_\perp \times_f N_T$ in Kaehler manifolds. Therefore he considered warped product CR-submanifolds in the form $N_T \times_f N_\perp$ called CR-warped product by reversing factor manifolds. He established a relationship between the warping function f and the second fundamental form of CR-warped product submanifold in Kaehler manifolds[2, 3].

I. Hasegawa and I. Mihai obtained a similarly inequality for the squared norm of the second fundamental form in terms of the warping function for contact CR-warped products in Sasakian manifolds[9].

In [10], Authors studied the geometry of anti-invariant submanifolds of almost para contact metric manifolds. They obtained a necessary and sufficient condition for a submanifold to be T-invariant.

In this paper we investigate warped products $N = N_1 \times_f N_2$ which are warped product semi-invariant submanifolds with respect to case of ξ in

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almost para contact metric manifolds. Our aim in this paper is to study the warped product manifolds in the setting of almost para contact metric manifold.

2. PRELIMINARIES

Let M be a m-dimensional differentiable manifold, φ, ξ and η be a tensor field of type $(1,1)$, a contravariant field and a 1-form on M, respectively, satisfying

$$\begin{aligned} \varphi^2 X = X - \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1 \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi) \end{aligned} \tag{1}$$

for any vector fields X and Y on M, then M is called almost para contact metric manifold with structure (φ, ξ, η, g)[6].

By $\bar{\nabla}$ we denote the Levi-Civita connection on almost para contact metri manifold M, then the almost para contact metric manifold M is said to be normal if

$$(\bar{\nabla}_X \varphi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2}$$

and

$$\bar{\nabla}_X \xi = \varphi X \tag{3}$$

for any $X, Y \in \Gamma(TM)$[10]. The covariant derivative of 1-form η is defined by

$$(\bar{\nabla}_X \eta)Y = X\eta(Y) - \eta(\bar{\nabla}_X Y).$$

Thus we have

$$(\bar{\nabla}_X \eta)Y = g(\varphi X, Y). \tag{4}$$

Now let N be an n-dimensional immersed submanifold in M. Then the Gauss and Weingarten formulas are, respectively, given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{5}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X Y \tag{6}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$, where ∇ and $\nabla^{\perp}$ denote the connections on N and $TN^{\perp}$, respectively, and A_V and h are called the shape operator and second fundamental form of N in M, respectively[1]. The shape operator A_V and the second fundamental for h are related by

$$g(h(X,Y),V) = g(A_V X, Y) \tag{7}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$.

For any vector field X tangent to N we can write

$$\varphi X = tX + nX, \tag{8}$$

where tX and nX denote tangential and normal components of φX, respectively. In the same way, for any vector field V normal to N we put

$$\varphi V = BV + CV, \tag{9}$$

where BV and CV are also tangential and normal components of φV, respectively.

The submanifold N is said to be invariant if n is identically zero. On the other hand, N is said to be anti-invariant submanifold if t is identically zero.

For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds. Now we will mention the following:

1) A submanifold N tangent to ξ is called an invariant submanifold if φ preserves the tangent space of N, i.e., $\varphi(T_xN) \subseteq T_xN$ for each $x \in N$.
2) A submanifold N tangent to ξ is called an anti-invariant submanifold if φ maps the tangent space of N into the normal space, i.e., $\varphi(T_xN) \subset T_x^{\perp}N$ for each $x \in N$.
3) A submanifold N tangent to ξ is called semi-invariant submanifold if it admits an invariant distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, i.e., $TN = D \oplus D^{\perp} \oplus \{\xi\}$ with $\varphi(D_x) \subseteq D_x$ and $\varphi(D_x^{\perp}) \subset T_x^{\perp}N$ for each $x \in N$.

Definition 2.1. Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and f is a positive definite differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$ equipped with the Riemannian metric tensor such that

$$g(X,Y) = g_1(\pi_{1*}X, \pi_{1*}Y) + f^2(\pi_1(x))g_2(\pi_{2*}X, \pi_{2*}Y)$$

for any $X, Y \in \Gamma(TN)$, where π_1 and π_2 are the canonical projections of $N_1 \times N_2$ onto N_1 and N_2, respectively, and $*$ is the symbol for the tangent map. Thus we have $g = g_1 + f^2 g_2$, where f is called warping function of the warped product. The warped product manifold $N = N_1 \times_f N_2$ is characterized by N_1 and N_2 are totally geodesic and totally umbilical submanifolds of N, respectively[4].

Now we recall the following lemma from [4] for later use.

Lemma 2.1. *Let $N = N_1 \times_f N_2$ be a warped product manifold with warped function f. Then we have*

1) $\nabla_X Y \in \Gamma(TN_1)$ *for any* $X, Y \in \Gamma(TN_1)$,
2) $\nabla_X U = \nabla_U X = X(\ln f)Z$ *for any* $X \in \Gamma(TN_1)$ *and* $U \in \Gamma(TN_2)$,
3) $\nabla_U W = \nabla_U^{N_2} W - g(U,W)\frac{1}{f}\mathrm{grad} f$ *for any* $W, V \in \Gamma(TN_2)$,

where ∇ *and* ∇^{N_2} *denote the Levi-Civita connections on* N *and* N_2*, respectively.*

If the manifolds N_T and $N_\perp$ are invariant and anti-invariant submanifolds of almost para contact metric manifold M, then their warped products are in the form 1.) $N_\perp \times_f N_T$ and 2.) $N_T \times_f N_\perp$.

3. WARPED PRODUCT SEMI-INVARIANT SUBMANIFOLDS OF AN ALMOST PARA CONTACT METRIC MANIFOLD

Throughout this section we assume that M is an almost para contact metric manifold with structure (φ, ξ, η, g) and $N = N_1 \times_f N_2$ be a warped product semi-invariant submanifold of M. Such submanifolds are always tangent to the structure vector field ξ. In case $N = N_\perp \times_f N_T$, there are two subcases.

1) ξ is tangent to N_T,
2) ξ is tangent to $N_\perp$.

First we start with ξ is tangent to N_T.

Theorem 3.1. *Let M be an almost para contact metric manifold. Then there exist no warped product semi-invariant submanifolds in the form $N = N_\perp \times_f N_T$ such that N_T is an invariant submanifold tangent ξ and $N_\perp$ is an anti-invariant submanifold of M.*

Proof. We suppose that $N = N_\perp \times_f N_T$ is a warped product semi-invariant submanifold of almost para contact metric manifold M such that N_T is invariant submanifold tangent to ξ and $N_\perp$ is anti-invariant submanifold. Then from Lemma 2.1 we have

$$\nabla_X Z = \nabla_Z X = Z(\ln f)X, \quad X \in \Gamma(TN_T), \quad \textit{and} \quad Z \in \Gamma(TN_\perp). \tag{10}$$

In particular, replacing X by ξ in (10) and by using (3), (5) we obtain

$$\begin{aligned} \bar{\nabla}_Z \xi &= \nabla_Z \xi + h(Z, \xi) \\ \varphi Z &= Z(\ln f)\xi + h(Z, \xi). \end{aligned} \tag{11}$$

The normal and tangential components of (11), respectively, we find

$$\varphi Z = h(\xi, Z) \quad \textit{and} \quad Z(\ln f) = 0 \quad \text{for any} \quad Z \in \Gamma(TN_\perp).$$

Thus we conclude that f is a constant function on $N_\perp$. This completes the proof. □

Now we consider ξ is tangent $N_\perp$.

Theorem 3.2. *Let M be an almost para contact metric manifold. Then there exist no warped product semi-invariant submanifolds in the form $N = N_\perp \times_f N_T$ such that N_T is an invariant submanifold and $N_\perp$ is an anti-invariant submanifold tangent to ξ of M.*

Proof. We suppose that $N = N_\perp \times_f N_T$ is warped product semi-invariant submanifold such that $N_\perp$ is an anti-invariant submanifold tangent to ξ and N_T is an invariant submanifold of M. Then we have

$$\nabla_X Z = \nabla_Z X = Z(\ln f)X \quad \text{for any} \quad Z \in \Gamma(TN_\perp), X \in \Gamma(TN_T). \tag{12}$$

In particular, replacing Z by ξ and by using (3), (5) we get

$$\begin{aligned} \bar{\nabla}_X \xi &= \nabla_X \xi + h(X, \xi) \\ \varphi X &= \xi(\ln f)X + h(X, \xi). \end{aligned} \tag{13}$$

The tangential components of (13) we conclude that

$$\varphi X = \xi(\ln f)X \tag{14}$$

for any $X \in \Gamma(TN_T)$. Applying φ to (14) again and consider $\xi \in \Gamma(TN_\perp)$, we get $[\xi(\ln f)]^2 = 1$. This means that $\xi(f) = f$ or $\xi(f) = -f$. Since $N = N_\perp \times_f N_T$ is a warped product, we have $\nabla_X \xi = \nabla_\xi X = (\xi \ln f)X = \varphi\xi = 0$, this is impossible. This proves our result. □

Next we research the existence of warped product semi-invariant submanifolds in the form $N = N_T \times_f N_\perp$ in almost para contact metric manifolds. Here there are two subcases such as

1) ξ is tangent to N_T,
2) ξ is tangent to $N_\perp$.

We start with case 1).

Theorem 3.3. *Let M be an almost para contact metric manifold. Then there exist no warped product semi-invariant submanifolds in the form $N = N_T \times_f N_\perp$ such that N_T is invariant submanifold tangent to ξ and $N_\perp$ is anti-invariant submanifold of M.*

Proof. We suppose that $N = N_T \times_f N_\perp$ be a warped product semi-invariant submanifold such that N_T is an invariant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of M. Then taking account that Lemma 2.1 and by using (3), (5) we have

$$\varphi Z = \xi(\ln f)Z + h(Z, \xi) \tag{15}$$

for any $Z \in \Gamma(TN_\perp)$. From the normal and tangential components of (15) we have

$$\varphi Z = h(Z, \xi) \quad \text{and} \quad \xi(\ln f)Z = 0. \tag{16}$$

Making use of (7) and (16) we arrive at

$$g(\varphi Z, \varphi Z) = g(h(Z, \xi), \varphi Z) = g(A_{\varphi Z}\xi, Z). \tag{17}$$

On the other, taking account of φ being symmetric and using (2), (5) and Lemma 2.1, we infer

$$
\begin{aligned}
g(\varphi Z, \varphi Z) &= g(\varphi h(Z,\xi), Z) = g(\varphi(\bar{\nabla}_\xi Z - \nabla_\xi Z), Z) \\
&= g(\bar{\nabla}_\xi \varphi Z - (\bar{\nabla}_\xi \varphi)Z - \varphi(\xi(\ln f)Z), Z) \\
&= g(\bar{\nabla}_\xi \varphi Z, Z) - g(-g(\xi, Z)\xi - \eta(Z)\xi + 2\eta(\xi)\eta(Z)\xi, Z) \\
&= -g(A_{\varphi Z}\xi, Z).
\end{aligned} \tag{18}
$$

From (17) and (18) we conclude that $\varphi Z = 0$. Since $N_\perp \neq 0$ and it is anti-invariant, this is a contradiction which proves our assertion. □

Now we consider the subcase 2).

Theorem 3.4. *Let M be an almost para contact metric manifold. Then there exist no warped product semi-invariant submanifolds in the form $N = N_T \times_f N_\perp$ such that N_T is invariant submanifold and $N_\perp$ is anti-invariant submanifold tangent to ξ of M.*

Proof. By using (3) and (5) and consider Lemma 2.1 we get

$$\varphi X = X(\ln f)\xi + h(X, \xi)$$

for any $X \in \Gamma(TN_T)$. Since N_T is an invariant submanifold and $\xi \in \Gamma(TN_\perp)$ we have

$$\varphi X = X(\ln f)\xi \quad \textit{and} \quad h(X, \xi) = 0. \tag{19}$$

From the tangential components of (19) we get

$$X(\ln f) = g(\varphi X, \xi) = \eta(\varphi X) = 0.$$

This means that the warping function f is constant on N_T. Thus the proof is complete. □

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