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On the Convergence of Ishikawa Iterates to a Common Fixed Point for a Pair of Nonexpansive Mappings in Banach Spaces

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Abstract. In the present paper we prove a common fixed point theorem for Ishikawa iterates of a pair of multivalued mappings on a Banach space, satisfying nonexpansive type condition which extend and generalize the results of Rhoades [16], [17] and others.

1. Introduction and Preliminaries

The Mann iterative scheme was invented in 1953, (see [9]) and was used to obtain convergence to a fixed point for many functions for which the Banach principle fails. For example, Rhoades [14] showed that, for any continuous self-map of a closed and bounded interval, the Mann iteration converges to a fixed point of the function.

In 1974, Ishikawa [4] devised a new iteration scheme to establish convergence for a Lipschitzian pseudo-contractive map in a situation where the Mann iteration process failed to converge. In recent years, a large literature has developed around the themes of establishing convergence of the Mann and Ishikawa for single-valued and multivalued mappings under various contractive conditions [1, 2, 3, 5] and others.

In the present paper, we prove a common fixed point theorem for Ishikawa iterates of a pair of multivalued mappings on a Banach space, satisfying nonexpansive type condition which extend and generalize the results of Rhoades [16], [17], Kubiaczyk and Ali [7], Rashwan[13] and others. To prove our result first we give the following results:

Theorem 1.1. [15] *Let T be a self-map of a closed convex subset K of a real Banach space (X, d). Let $\{x_n\}_{n=1}^{\infty}$ be a general Mann iteration of T that converges to a point $p \in X$. If there exists the contstants $\alpha, \beta, \gamma \geq 0, \delta < 1$*

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such that

$$\begin{aligned} \|Tx_n - Tp\| &\leq \alpha\{|x_n - p\| + \beta\{|x_n - Tx_n\| + \gamma\{|p - Tx_n\| \\ &\quad +\delta \ max\ \{\{|p - Tp\|, \{|x_n - p\|\}, \end{aligned}$$

then p is a fixed point of T.

If T is continuous then Mann iterative process converges to a fixed point of T. But if T is not continuous, then there is no guarantee that, even it the Mann process converges, it will converge to a fixed point of T.

If instead of the Mann iteration, we consider another iterative process, which is in some sense a double Mann iterative process, then it is possible to approximate the fixed point of some other classes of contractive mappings.

In a recent paper Rhoades [16] extended this generic theorem to the Ishikawa iteration process.

Theorem 1.2. [16] *Let K be a convex compact subset of a Hilbert space, $T : K \to K$ a Lipschtzin pseudo-contractive map and $x_1 \in X$. Then the Ishikawa iteration $\{x_n\}$ defined as:*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[(1 - \beta_n)x_n + \beta_n Tx_n],$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers satisfying:

(i) $0 \leq \alpha_n \leq \beta_n \leq 1, n \geq 1,$
(ii) $\lim\limits_{n\to\infty} \beta_n = 0,$
(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty,$

converges strongly to a fixed point of T.

Chidume [2], Reich [19], Chang [1] and Deng and Ding [3] generalized the fundamental results related to Ishikawa iteration.

Throughout this paper let (X, d) be a Banach space, $CB(X)$ the collection of closed, nonempty, bounded subsets of X and $H(A, B)$ the Hausdroff metric on $CB(X)$.

The well known Hausdroff metric on X is defined as:

$$H(A, B) = \ max\ \left\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\right\}$$

for any $A, B \in CB(X)$, where $D(a, B) = \inf_{b\in B} d(a, b)$

We shall need the following results.

Lemma 1.1 ([10]). *If $A, B \in CB(X)$ and $a \in A$, then for $\epsilon > 0$ there exists $b \in B$ such that $d(a, b) \leq H(a, B) + \epsilon$.*

Let K be a nonempty subset of X. The Ishikawa iteration scheme associated with two multivalued mappings $S, t : K \to CB(X)$ are defined as follows:

$$\left\{ \begin{array}{l} x_0 \in K \\ y_n = (1 - \beta_n)x_n + \beta_n a_n, \quad a_n \in Tx_n \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n b_n, \quad b_n \in Sy_n \end{array} \right. \tag{1}$$

where (i) $0 \leq \alpha_n, \beta_n \leq 1$ for all n, additional conditionals will be placed $\{\alpha_n\}$ and $\{\beta_n\}$ as needed.

More recently Rhoades [17] proved a generic theorem for the Ishikawa iterates of a pair of multivalued mappings on a Banach space, and proved that the result has a number of corollaries. Rhoades [17] proved the following theorem.

Theorem 1.3. *Let X be a Banach space, K is a closed, convex subset of X. S and t are multivalued mappings from K to $CB(X)$. Suppose that the Ishikawa scheme (1), with $\{\alpha_n\}$ satisfying:*

(i) $0 \leq \alpha_n, \beta_n \leq 1$ *for all* n,

(ii) $\liminf \alpha_n = d > 0$ *and* $\{a_n\}$, $\{b_n\}$, *satisfying*

$$\|a_n - b_n\| \leq H(Tx_n, Sy_n) + \epsilon_n, \text{ with } lim\ \epsilon_n = 0 \tag{2}$$

converges to a point p. If there exist non-negative numbers $\alpha, \beta, \gamma, \delta$ with $\beta \leq 1$ such that for all sufficiently large n, S and T satisfying

$$H(Tx_n, Sy_n) \leq \alpha \|x_n - b_n\| + \beta \|x_n - a_n\| \tag{3}$$

and

$$\begin{aligned} H(Sp, Tx_n) \leq\ & \alpha \|x_n - p\| + \gamma d(x_n, Tx_n) + \delta d(p, Tx_n) \\ & + \beta\ max\ \{d(p, Sp), d(x_n, Sp)\} \end{aligned} \tag{4}$$

then p is a fixed point of S. If also

$$H(Sp, Tp) \leq \beta[d(p, Tp) + d(p, Sp)], \tag{5}$$

then p is a common fixed point of S and T.

By using the above theorem, Rhoades proved the following corollaries.

Corollary 1.1. *Let X be a normed space, K be a closed convex subset of X. Let $S, T : K \to CB(K)$ be mappings satisfying the following condition:*

$$\begin{aligned} & H(Tx, Sy) \leq \\ & q\ max\{k \|x - y\|, [D(x, Tx) + D(y, Sy)], [D(x, Sy) + D(y, Tx)]\} \end{aligned} \tag{6}$$

for all $x, y \in K$, where $k \geq 0$ and $0 < q < 1$.

If there exists a point $x_0 \in K$ such that $\{x_n\}$, satisfying (1), (2), (i), (ii) and (iii) $\lim \beta_n = 0$, converges to a point p, then p is a common fixed point of S and T.

Corollary 1.2. *On the statement of Corollary 1.1, if we replace the condition (6) by*

$$\begin{aligned} H(Tx, Sy) \leq q\ max\Big\{ & \|x - y\|, \frac{d(y, Sy)[1 + d(x, Tx)]}{1 + \|x - y\|}, \\ & \frac{d(x, Sy)[1 + d(x, Tx) + d(y, Tx)]}{2[1 + \|1 + x - y\|]} \Big\} \end{aligned} \tag{7}$$

then p is a common fixed point of S and T.

Now we prove the result for nonexpansive type condition for multivalued maps.

2. MAIN RESULT

Theorem 2.1. *Let K be a nonempty, closed, convex subset of a Banach space X and $T, S : K \to CB(X)$ satisfying*

(8)

$$H(Tx, Sy) \leq \ max \{\|x-y\|, [d(x,Tx)+d(y,Sy)], [d(x,Sy)+d(y,Tx)]\}$$

for all $x, y \in K$. If there exists an $x_0 \in K$ such that a sequence $\{x_n\}$ satisfying (1), (2), (i), (ii) and (iii) $\beta_n = 0$, converges to a point p, then p is a common fixed point of S and T.

Proof. To prove our result, it is sufficient to show that S and T Satisfy (3), (4), (5). Now by (8), we have

$$\begin{aligned} H(Tx_n, Sy_n) \leq \max\Big\{ &\|x_n - y_n\|, [d(x_n, Tx_n), d(y_n, Sy_n)], \\ &[d(x_n, Sy_n) + d(y_n, Tx_n)]\Big\}. \end{aligned} \tag{9}$$

Also by (1), we have

$$\begin{cases} \|x_n - y_n\| = \beta_n \|x_n - a_n\|, \\ d(x_n, Tx_n) \leq \|x_n - a_n\|, \\ d(y_n, Sy_n) \leq \|y_n - b_n\| = \|y_n - x_n\| + \|x_n - b_n\| \\ \qquad \leq \beta_n \|x_n - a_n\| + \|x_n - b_n\|, \\ d(x_n, Sy_n) \leq \|x_n - b_n\|, \\ d(y_n, Tx_n) \leq \|y_n - a_n\| = \|y_n - x_n\| + \|x_n - a_n\| \\ \qquad \leq (1+\beta_n) \|x_n - a_n\|. \end{cases} \tag{10}$$

Now

$$\begin{aligned} &\leq \|x_n - a_n\| + \|y_n - b_n\| \\ &\leq \|x_n - a_n\| + \beta_n \|x_n - a_n\| + \|x_n - b_n\| \\ &\leq (1+\beta_n)\|x_n - a_n\| + \|x_n - b_n\|. \end{aligned} \tag{11}$$

Also

$$\begin{aligned} &\leq [\|x_n - b_n\| + \|y_n - a_n\|] \\ &\leq [\|x_n - b_n\| + (1+\beta_n)\|x_n - a_n\|]. \end{aligned} \tag{12}$$

Now using (10), (11) and (12) in (9), we have

$$\begin{aligned} H(Tx_n, Sy_n) &\leq \max\Big\{ \beta_n \|x_n - a_n\|, [(1+\beta_n)\|x_n - a_n\| + \|x_n - b_n\|], \\ &\qquad [\|x_n - b_n\| + (1+\beta_n)\|x_n - a_n\|]\Big\} \\ &\leq \max\{\beta_n, (1+\beta_n), (1+\beta_n)\} \|x_n - a_n\| + \|x_n - b_n\|. \end{aligned}$$

Using condition (iii), we have

(13) $$H(Tx_n, Sy_n) \leq \|x_n - a_n\| + \|x_n - b_n\|.$$

It is clear that (3) is satisfied. Again by (8), we have

(14) $$\begin{aligned} H(Tx_n, Sp) &\leq \max\Big\{\|x_n - p\|, [d(x_n, Tx_n) + d(p, Sp)], \\ &\qquad [d(x_n, Sp) + d(p, Tx_n)]\Big\} \\ &\leq \max\Big\{\|x_n - p\|, [\|x_n - a_n\| + d(p, Sp)], \\ &\qquad [d(x_n, Sp) + d(p, a_n)]\Big\}. \end{aligned}$$

Since (3) is satisfied, therefore by (2), we have

$$\begin{aligned} \|x_n - a_n\| &\leq \|x_n - b_n\| + \|b_n - a_n\| \\ &\leq \|x_n - b_n\| + H(Tx_n, Sy_n) + \epsilon_n \\ &\leq \|x_n - b_n\| + \alpha\|x_n - b_n\| + \beta\|x_n - a_n\| + \epsilon_n. \end{aligned}$$

Since $\lim \|x_n - b_n\| = 0$, we obtain

$$\limsup \ \|x_n - a_n\| \leq \beta \limsup \|x_n - a_n\|$$

since $0 \leq \beta \leq 1$, which implies

(15) $$\lim \|x_n - a_n\| = 0.$$

Also

(16) $$\|p - a_n\| \leq \|p - x_n\| + \|x_n - a_n\|.$$

From (14), (15) and (16), we have

$$H(Tx_n, Sp) \leq \|x_n - p\| + \max\{d(p, Sp), d(x_n, Sp)\}.$$

Therefore for all sufficiently large n, (4) is satisfied.

Since (3) and (4) are satisfied, then by Theorem 1.1, p is a fixed point of S. Again by (8), we have

$$h(Tp, Sp) \leq \max\{d(p, Tp) + d(p, Sp), d(p, Sp + d(p, Tp)\}.$$

Hence (5) is satisfied,i.e. p is a common fixed point of S and T. □

Corollary 2.1. *Let K be a nonempty, closed, convex subset of a Banach space X and $T, S : K \to CB(X)$ satisfying*

(17) $$\begin{aligned} H(Tx, Sy) \leq \ max\Big\{\|x - y\|, \frac{1}{2}[d(x, Tx) + d(y, Sy)], \\ \frac{1}{2}[d(x, Sy) + d(y, Tx)]\Big\} \end{aligned}$$

for all $x, y \in K$. If there exists an $x_0 \in K$ such that a sequence $\{x_n\}$ satisfying (1), (2), (i), (ii) and (iii) $\beta_n = 0$, converges to a point p, then p is a common fixed point of S and T.

Corollary 2.2. *Let K be a nonempty, closed, convex subset of a Banach space X and $T, S :\to CB(X)$ satisfying*

$$H(Tx, Sy) \le \max\Big\{\|x-y\|, \frac{d(y,Sy)[1+d(x,Tx)]}{1+\|x-y\|}, \frac{d(x,Sy)[1+d(x,Tx)+d(y,Tx)]}{2[1+\|1+x-y\|]}\Big\} \tag{18}$$

for all $x, y \in K$. If there exists an $x_0 \in K$ such that a sequence $\{x_n\}$ satisfying (1), (2), (i), (ii) and (iii) $\beta_n = 0$, converges to a point p, then p is a common fixed point of S and T.

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