

A Procedure for Obtaining a Family of Iterative Formulas of Higher Order

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ABSTRACT. In this paper a procedure for obtaining iterative formulas of higher order for finding zeros is obtained. The family includes several already known results.

In this paper, starting from Halley's iterative formula (see [1], [2]) for finding zeros of the functions, a family of iterative formulas of the third order has been obtained.

Halley's iterative formula for finding the simple zero of the function $F(x)$ is

$$(1) \quad x_{k+1} = x_k - \frac{2F(x_k)F'(x_k)}{2(F'(x_k))^2 - F(x_k)F''(x_k)}, \quad k = 0, 1, 2, \dots,$$

where $F(x)$ is three times differentiable function.

Formula (1) we can write in the form

$$(2) \quad x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)} \cdot \frac{1}{1 - \frac{F(x_k)F''(x_k)}{2(F'(x_k))^2}}, \quad k = 0, 1, 2, \dots$$

Expression

$$(3) \quad 1 - \frac{F(x_k)F''(x_k)}{2(F'(x_k))^2}$$

can be written in the form

$$(4) \quad 1 - \frac{F(x_k)F''(x_k)}{2(F'(x_k))^2} = 1 - s + s \left(1 - \frac{F(x_k)F''(x_k)}{2s(F'(x_k))^2} \right),$$

where s is a real parameter $\neq 0$.

As it is known, for small values $|h|$ of the real number h and real parameter $v \neq 0$, the following relation is valid

$$(5) \quad 1 - h \approx \left(1 - \frac{h}{v} \right)^v.$$

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If x_k is an approximative value of the simple zero α of the function $F(x)$ that is near enough to the zero α , then the value of $F(x_k)$ is small, and so the absolute value of the expression

$$(6) \quad \frac{F(x_k)F''(x_k)}{2s(F'(x_k))^2}$$

is also small.

If we put expression (6) into (5) instead of h , we will get a relation

$$(7) \quad 1 - \frac{F(x_k)F''(x_k)}{2s(F'(x_k))^2} \approx \left(1 - \frac{F(x_k)F''(x_k)}{2sv(F'(x_k))^2}\right)^v,$$

because of which (4) is reduced to a relation

$$(8) \quad 1 - \frac{F(x_k)F''(x_k)}{2(F'(x_k))^2} \approx 1 - s + s \left(1 - \frac{F(x_k)F''(x_k)}{2sv(F'(x_k))^2}\right)^v.$$

If in the formula (2) instead of expression (3) we put

$$1 - s + s \left(1 - \frac{F(x_k)F''(x_k)}{2sv(F'(x_k))^2}\right)^v$$

we will get an iterative formula

$$(9) \quad x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)} \cdot \frac{1}{1 - s + s \left(1 - \frac{F(x_k)F''(x_k)}{2sv(F'(x_k))^2}\right)^v}, \quad k = 0, 1, 2, \dots,$$

where s and v are real parameters $\neq 0$.

It is not difficult to prove for formula (9) that it represents an iterative formula of the third order.

By formula (9) it is given one family of iterative methods of the third order, where s and v are real parameters $\neq 0$.

The asymptotic error constant for the iterative method (9) is

$$(10) \quad C_3 = \frac{3(3 + 2(v-1)(2sv)^{-1} - 1)(F''(\alpha))^2 - 4F'(\alpha)F'''(\alpha)}{24(F'(\alpha))^2},$$

where α is a simple zero of the function $F(x)$.

Taking for parameters s and v different values, we can obtain from (9) particular iterative formulas. Here we list some of the particular cases:

- 1) For $s = 1, v = 1$, (9) reduces to Halley's method (2).
- 2) For $s = v = \frac{1}{2}$, we have from (9) Euler's method (see [1])

$$(11) \quad x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)} \cdot \frac{2}{1 \pm \left(1 - \frac{2F(x_k)F''(x_k)}{(F'(x_k))^2}\right)^{1/2}}, \quad k = 0, 1, 2, \dots$$

3) For $s = 1, v = -1$, from (9) we obtain

$$(12) \quad x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)} \cdot \frac{2(F'(x_k))^2 + F(x_k)F''(x_k)}{2(F'(x_k))^2}, \quad k = 0, 1, 2, \dots,$$

which represents Chebyshev's method (see [7]).

4) For $s = \frac{n-1}{n}, v = \frac{1}{2}$, when $F(x)$ is a polynomial of degree $n \geq 2$, we have from (9)

$$(13) \quad x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)} \cdot \frac{n}{1 \pm (n-1) \left(1 - \frac{n}{n-1} \cdot \frac{F(x_k)F''(x_k)}{(F'(x_k))^2} \right)^{1/2}}, \quad k = 0, 1, 2, \dots,$$

which is the Laguerre method (see [1], [3]).

5) For $s = 1, v = \frac{1}{2}$, (9) reduces to

$$(14) \quad x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)} \cdot \frac{1}{\pm \left(1 - \frac{F(x_k)F''(x_k)}{(F'(x_k))^2} \right)^{1/2}}, \quad k = 0, 1, 2, \dots,$$

which represents Ostrowski's square root method (see [4]).

6) For $s = \frac{1}{w+1}, v = \frac{1}{2}$, the formula (9) reduces to

$$(15) \quad x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)} \cdot \frac{w+1}{w \pm \left(1 - (w+1) \cdot \frac{F(x_k)F''(x_k)}{(F'(x_k))^2} \right)^{1/2}}, \quad k = 0, 1, 2, \dots,$$

where w is a real parameter $\neq -1$.

The family of iterative formulas (15) are obtained by E. Hansen and M. Patrick [1].

7) For $s = v = -1$, we have from (9) the formula

$$(16) \quad x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)} \cdot \frac{2(F'(x_k))^2 - F(x_k)F''(x_k)}{2(F'(x_k))^2 - 2F(x_k)F''(x_k)}, \quad k = 0, 1, 2, \dots,$$

which is obtained in [5].

8) For $s = 1, v = \frac{n-1}{2n}$, when $F(x)$ is a polynomial of degree $n \geq 2$, we obtain from (9)

$$(17) \quad x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)} \cdot \left(1 - \frac{n}{n-1} \cdot \frac{F(x_k)F''(x_k)}{(F'(x_k))^2} \right)^{\frac{1-n}{2n}}, \quad k = 0, 1, 2, \dots,$$

which represents the method obtained in [6].

The family of iterative method (9) is one particular case of one more general family of iterative methods obtained in [6].

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