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On a Result of W.A. Kirk and L.M. Saliga

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Abstract. We prove that a result of Kirk and Saliga [J. Comput. Appl. Math., **113** (2000), 141-152, Theorem 4.2., p. 149] has been for the first time proved before 25 years in Tasković [Publ. Inst. Math., **41** (1980), 249–258, Theorem 1, p. 250]. But the authors neglected and ignored this historical fact.

1. Introduction

In recent years a great number of papers have presented generalizations of the well-known Banach-Picard contraction principle.

Recently, Kirk and Saliga have proved the following statement (see [1, Theorem 4.2., p. 149]).

Theorem 1. (Kirk-Saliga [1], Walter [8]). *Let (X, ρ) be a complete metric space and suppose $T : X \to X$ has bounded orbits and satisfies the following condition:*

$$\text{(K)} \qquad \rho[Tx, Ty] \le \Phi\Big(\operatorname{diam}\{x, y, Tx, Ty, T^2x, T^2y, \ldots\}\Big)$$

for all $x, y \in X$, where $\Phi : \mathbb{R}^0_+ \to \mathbb{R}^0_+ := [0, +\infty)$ is a continuous nondecreasing function and satisfies $\Phi(t) < t$ for every $t > 0$. Then T has a unique fixed point $\xi \in X$ and $\{T^n(a)\}_{n\in\mathbb{N}}$ converges to ξ for every $a \in X$.

In connection with this, in 1980 I have proved the following result of fixed point on metric spaces which has a best long of all known sufficiently conditions (linear and nonlinear) for the existing unique fixed point, cf. Tasković [3], [4] and [5].

We notice that the manuscript of [3] was received by the editors January 27, in 1979, but published in 1980. This result generalizes a great number of known results.

In this sense, first, let (X, ρ) be a metric space and T a mapping of X into itself. A metric space X is said to be T*-orbitally complete* iff every Cauchy sequence which is contained in *orbit* $O(x) = \{x, Tx, T^2x, \ldots\}$ for some $x \in X$ converges in X.

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Theorem 2. (Tasković [3]). *Let T be a mapping of a metric space (X, ρ) into itself and let X be T-orbitally complete. Suppose that there exists a function $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+ := [0, +\infty)$ satisfying*

$$(\mathrm{I}\varphi) \qquad \Big(\forall t \in \mathbb{R}_+ := (0, +\infty)\Big)\Big(\varphi(t) < t \text{ and } \limsup_{z \to t+0} \varphi(z) < t\Big)$$

such that

$$(\mathrm{A}) \qquad \rho[Tx, Ty] \le \varphi\Big(\operatorname{diam}\{x, y, Tx, Ty, T^2x, T^2y, \ldots\}\Big)$$

and $\operatorname{diam} O(x) \in \mathbb{R}^0_+$ *for all $x, y \in X$. Then T has a unique fixed point $\xi \in X$ and $\{T^n(a)\}_{n\in\mathbb{N}}$ converges to ξ for every $a \in X$.*

A brief first proof of this statement may be found in 1980 from Tasković [3]. Other brief proofs for this we can see in Tasković [4], [5], [6] and [7].

Annotation 1. We notice that Theorem 1 is a very special case of Theorem 2. Indeed, since $\Phi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ of Theorem 1 satisfy all required hypothesis $(\mathrm{I}\varphi)$ for the function $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ in Theorem 2 (other conditions are equal, an example (K) and (A) and completeness), directly applying Theorem 2 we obtain Theorem 1.

Annotation 2. De facto, in [3] I have introduced the concept of a *diametral φ-contraction T* of a metric space (X, ρ) into itself, i.e., there exists a function $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ satisfying $(\mathrm{I}\varphi)$ and (A).

Annotation 3. In the preceding sense, since the function $\Phi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ (in Theorem 1) is continuous nondecreasing and satisfies $\Phi(t) < t$ for every $t > 0$, we directly obtain that the conditions $(\mathrm{I}\varphi)$ in Theorem 2 hold. Thus we directly obtain Theorem 1.

Annotation 4. The main part of the first written proof of Theorem 2 (on diametral φ-contractions) may be found in 1978 as a frame for general convergence of real sequences. The proof of Theorem 2 is based upon the following fundamental lemma in 1978.

Lemma 1. (Tasković [2]). *Let the mapping $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ have the property $(\mathrm{I}\varphi)$. If the sequence (x_n) of nonnegative real numbers satisfies the condition of the form*

$$x_{n+1} \le \varphi(x_n), \text{ for } n \in \mathbb{N},$$

then the sequence (x_n) tends to zero. The velocity of this convergence is not necessarily geometrical.

We notice that the manuscript [2] was received by the editors on October 15, 1975, but published in 1978.

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