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Some New 2-Designs from a Wreath Product on 18 Points*

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Abstract. The total of 22 2-designs on 18 points have been found. All these designs have the same group as an automorphism group. This group can be represented as the wreath product of G and H, where G denotes the subgroup of order 3 of PSL(2,2) and H denotes the subgroup of order 12 of PSL(2,5).

The 2-(18,7,42·s) designs for $s \in \{15, 19, 25, 27, 30, 37, 38, 50\}$ and the 2-(18,8,28·s) designs for $s \in \{27, 44, 46, 48, 50, 53, 54, 57, 59, 61, 73, 77, 80, 81\}$ have been detected. Up to our knowledge, 16 of these 22 found designs are new.

1. Introduction

A t-(v, k, λ) design is a collection $\mathbf{B}$ of k-subsets (called *block*) of a v-element set Δ of *points*, which satisfies the property that each t-element subset of Δ is in exactly λ blocks. We also require that blocks are not repeated.

Given a group M acting on Δ, the Kramer-Mesner method [7] searches for t-(v, k, λ) designs having M as an automorphism group. The group M is a subgroup of the full automorphism group and the collection $\mathbf{B}$ is a union of M-orbits of k-subsets (shortly: k-M-orbits).

The method includes a construction of t-M-orbits and k-M-orbits, computation of the orbit incidence matrix $\Lambda(t, k) = \lambda_{i,j}$ (where $\lambda_{i,j}$ denotes the number of blocks from the j-th k-M-orbit, containing a specified set from the i-th t-M-orbit), and design recognition (by finding those proper sets of the column-set of $\Lambda(t, k)$, that have the uniform row sum λ).

In this paper we are going to apply the Kramer-Mesner method to the wreath product of some groups. This product will be described and discussed in the following section.

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2. Construction

Let G and H be two groups acting on the ground-sets Γ and Ω respectively. The wreath product $G \wr H$ is the group which acts on $\Gamma \times \Omega$ as follows ([6], Ch. I, Th. 15.3.):

$$(i, j)(\mathbf{f}, h) = (i^{\mathbf{f}(j)}, j^h),$$

where $h \in H$, $\mathbf{f}$ is a mapping from Ω into G, $(\mathbf{f}, h) \in G \wr H$, $i \in \Gamma$, $j \in \Omega$.

Groups G and H will be defined as some transitive subgroups of $PSL(2, 2)$ and $PSL(2, 5)$, respectively. The group $PSL(2, 2)$ acts 2-transitively on the projective plane Γ of order 2 and is isomorphic to the group $GL(2, 2)$ of all regular 2×2 matrices over $GF(2)$. Similarly, the group $PSL(2, 5)$ acts on the projective plane Ω of order 6 and is isomorphic to the groups $GL(2, 5)$ of all regular 2×2 matrices over $GF(5)$.

The group $PSL(2, 2)$ is also isomorphic to the symmetric group S_3. We choose G to be its alternating subgroup A_3, which is known to act transitively on Γ. We choose H to be the normalizer of a Klein subgroup of $PSL(2, 5)$. This normalizer is known ([6], Ch. II, Lemma 8.16) to be of order 12.

The group $PSL(2, 2) \wr PSL(2, 5)$ of order $6^6 \cdot 60$ is not computationally tractable. Combining the facts that

- the Kramer-Mesner method searches (for) designs as some unions of orbits;
- orbits by action of a group are partitioned into orbits by action of its subgroups ([2], Lemma 1),

it follows that no design arising by action of $PSL(2, 2) \wr PSL(2, 5)$ will be missed by considering the action of $G \wr H$.

Since we found that the wreath product is reach in designs [4], [3],[1] we were motivated to continue this investigation.

3. Designs with wreath product on 18 points

Throughout the remaining part of this paper, considerations will be restricted to the automorphism group $G \wr H$. Therefore, the notation "$k - (G \wr H)$-orbit" will be abbreviated to "$k-$orbit".

Design recognition, i.e. search over the matrices $\Lambda(2, k)$, has been very facilitated by the fact that the matrices have many repeated columns. We use these repetitions to abbreviate notations for $\Lambda(2, k)$ by writing only the non-repeating columns, with the additional fourth row containing data on multiplicity (on the number of repetitions). The abbreviated tables will be denoted by $T(k)$; the first and the fourth row in such a table will be separated by a horizontal line. The ordinal numbers of columns of matrix $\Lambda(2, k)$ are listed in the first row on the $T(k)$.

We have performed the complete search for 2-designs with the automorphism group $G \wr H$.

3.1. **Matrices $\Lambda(2,k)$.** Since we found only 2-(18,7,λ) and 2-(18,8,λ) designs with the automorphism group equals to the wreath product $G \wr H$, we list only the matrices $\Lambda(2,k)$ $k = 7, 8$. It turns out that there exist 3 2-orbits, 44 7-orbits and 52 8-orbits. We list the matrices $\Lambda(2,k)$ $k = 7, 8$ in their abbreviated forms $T(k)$ (frequencies of the columns are listed in the last row of the tables):

TABLE 1. $T(7)\lambda_{\max}/2 = 2184$.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20
1	4	10	6	15	36	54	5	16	33	42	45	48	144	126	189	405	72	432	648
6	4	24	8	42	60	108	0	0	72	36	24	12	72	144	270	432	36	324	648
6	12	30	15	36	72	81	12	30	72	72	72	72	162	162	162	324	81	32	243
2	2	2	2	2	2	2	2	2	2	4	4	2	2	4	2	2	1	2	1

TABLE 2. $T(8)$ $\lambda_{\max}/2 = 2285$.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12
1701	405	270	567	189	180	513	171	153	63	60	21
1620	486	270	432	108	144	648	180	252	24	36	4
972	243	243	486	216	216	486	216	216	90	90	36
1	1	3	2	2	3	2	4	2	4	2	1

13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	23.	24.
57	51	19	13	7	36	24	45	16	3	5	2
48	72	12	36	0	72	42	96	24	10	8	6
90	90	36	36	14	54	45	90	36	9	14	7
2	4	4	1	2	1	2	2	1	2	2	2

3.2. **2-(18,7,λ) designs.** We searched over the matrix $T(7)$ without using multiplicity. In this way we obtained parameters $\lambda \in \{630, 798, 1050, 1554\}$. We additionally searched over the matrix $T(7)$ using multiplicity. In this manner we found λ parameters 1134, 1360,1596, 2100.

In the following table we present choice of the index of columns (without using multiplicity) of the matrix $T(7)$ for designs 2-(18,7,42$\cdot s$) $s \in \{15, 19, 25, 37\}$:

TABLE 3.

$\lambda = 630 = 42 \cdot 15$	1. 2. 4. 7. 9. 14. 17.
$\lambda = 798 = 42 \cdot 19$	2. 3. 4. 7. 9. 11. 12. 16. 19.
$\lambda = 1050 = 42 \cdot 25$	1. 2. 3. 4. 6. 7. 8. 9. 10. 13. 17. 19.
$\lambda = 1554 = 42 \cdot 37$	3. 6. 7. 8. 10. 13. 14. 15. 16. 17. 18. 19.

Since the supposed columns for parameters $\lambda = 630$, 798, 1050 have multiplicity greater than 1, we found the designs with parameters $\lambda = 1360, 1596, 2100$, using multiplicity 2.

In the following table we give the choice of columns of designs 2-$(18, 7, 42 \cdot 27)$ together with its multiplicity.

TABLE 4.

$\lambda = 1134 = 42 \cdot 27$			
14. $\cdot$ 2	15. $\cdot$ 2	16. $\cdot$ 1	17. $\cdot$ 1

3.3. **2-(18,8,λ) designs.** We searched over the matrix $T(8)$ without using multiplicity. In this way we obtained parameters $\lambda \in \{28 \cdot s,\ s = 27, 44, 46, 48, 50, 53, 57, 59, 61, 73, 77, 80, 81\}$. In [5] the results for λ are found: $\lambda \in \{28 \cdot s,\ s \bmod 2 = 0\}$. In the following table the choice of index of columns for new λ parameters is given:

TABLE 5.

$\lambda = 756 = 28 \cdot 27$	4. 9. 18.
$\lambda = 1484 = 28 \cdot 53$	2. 4. 5. 9. 10. 15. 16. 17. 20. 21. 23. 24.
$\lambda = 1596 = 28 \cdot 57$	2. 4. 5. 6. 9. 12. 16. 20. 21. 23. 24.
$\lambda = 1652 = 28 \cdot 59$	3. 4. 5. 7. 15. 16. 18. 19. 21. 23.
$\lambda = 1708 = 28 \cdot 61$	2. 3. 4. 9. 10. 11. 12. 13. 14. 15. 19. 21. 24.
$\lambda = 2044 = 28 \cdot 73$	2. 3. 4. 5. 6. 8. 9. 12. 16. 17. 18. 19. 22. 23.
$\lambda = 2156 = 28 \cdot 77$	2. 4. 5. 6. 7. 10. 11. 12. 14. 17. 18. 20. 21. 22.
$\lambda = 2268 = 28 \cdot 81$	2. 4. 5. 6. 7. 8. 10. 11. 14. 19. 20.

Theorem. *Let G be a subgroup of order 3 of $PSL(2, 2)$ and let H be the normalizer of a Klein subgroup of $PSL(2, 5)$. Then there exist* 2-$(18, 7, \lambda)$ *designs for*

$$\lambda \in \{42 \cdot s,\ s = 15, 19, 25, 27, 30, 37, 38, 50\}$$

and the 2-$(18, 8, \lambda)$ *designs for*

$$\lambda \in \{28 \cdot s,\ s = 27, 44, 46, 48, 50, 53, 54, 57, 59, 61, 73, 77, 80, 81\}$$

with automorphism group which is equal to the wreath product $G \wr H$.

The direct action of this wreath product on the Cartesian product of the projective line of order 2 and the projective plane of order 6 does not give 2-(18,k, λ) designs with other values of λ.

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