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On e^*-Open Sets and $(\mathcal{D}, \mathcal{S})^*$-Sets

Erdal Ekici

Abstract. The aim of this paper is to introduce some new classes of sets and some new classes of continuity namely e^*-open sets, $(\mathcal{D}, \mathcal{S})$-sets, $(\mathcal{DS}, \mathcal{E})$-sets, $(\mathcal{D}, \mathcal{S})^*$-sets, $(\mathcal{DS}, \mathcal{E})^*$-sets, e^*-continuity, $(\mathcal{D}, \mathcal{S})^*$-continuity and $(\mathcal{DS}, \mathcal{E})^*$-continuity. Properties of these new concepts are investigated. Moreover, some new decompositions of continuity are provided.

1. Introduction and preliminaries

After Levine's paper [4], mathematicians introduced different new decompositions of continuous functions and some weaker forms of continuous functions. The main purpose of this paper is to establish some new decompositions of continuous functions. Firstly, we introduce a new classes of sets called e^*-open sets. The class of e^*-open sets generalize the classes of e-open sets, δ-semiopen sets and δ-preopen sets. Properties and the relationships of e^*-open sets are investigated. On the other hand, we introduce the notions of $(\mathcal{D}, \mathcal{S})$-sets, $(\mathcal{DS}, \mathcal{E})$-sets, $(\mathcal{D}, \mathcal{S})^*$-sets, $(\mathcal{DS}, \mathcal{E})^*$-sets, e^*-continuity, $(\mathcal{D}, \mathcal{S})^*$-continuity and $(\mathcal{DS}, \mathcal{E})^*$-continuity. Finally, we obtain some new decompositions of continuous functions via these new concepts.

In this paper (X, τ) and (Y, σ) represent topological spaces. For a subset A of a space X, $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively. A subset A of a space (X, τ) is called α-open [7] (resp. β-open [1], preopen [5], regular open [10], regular closed [10]) if:

$$A \subset int(cl(int(A))) \quad (\text{resp. } A \subset cl(int(cl(A))),$$

$$A \subset int(cl(A)), \quad A = int(cl(A)), \quad A = cl(int(A))).$$

A subset A of a space (X, τ) is called δ-open [11] if for each $x \in A$ there exists a regular open set V such that $x \in V \subset A$. A set A is said to be δ-closed if its complement is δ-open. A point $x \in X$ is called a δ-cluster points of A [11] if $A \cap int(cl(V)) \neq \varnothing$ for each open set V containing x. The

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set of all δ-cluster points of A is called the δ-closure of A and is denoted by δ-$cl(A)$. The δ-interior of A is the union of all regular open sets contained in A and is denoted by δ-$int(A)$. A subset A of a space (X,τ) is called δ-preopen [9] (resp. δ-semiopen [8], e-open [3], e-closed [3]) if:

$$A \subset int(\delta\text{-}cl(A)) \quad (\text{resp. } A \subset cl(\delta\text{-}int(A)),$$

$$A \subset cl(\delta\text{-}int(A)) \cup int(\delta\text{-}cl(A)), \quad cl(\delta\text{-}int(A)) \cap int(\delta\text{-}cl(A)) \subset A).$$

The complement of a δ-semiopen (resp. δ-preopen) set is called δ-semiclosed (resp. δ-preclosed). The intersection of all e-closed (resp. δ-semiclosed, δ-preclosed) sets, each containing a set A in a topological space X is called the e-closure [3] (resp. δ-semiclosure [8], δ-preclosure [9]) of A and it is denoted by e-$cl(A)$ (resp. δ-$scl(A)$, δ-$pcl(A)$). The union of all e-open (resp. δ-semiopen, δ-preopen) sets, each contained in a set A in a topological space X is called the e-interior [3] (resp. δ-semiinterior [8], δ-preinterior [9]) of A and it is denoted by e-$int(A)$ (resp. δ-$sint(A)$, δ-$pint(A)$).

Lemma 1.1 ([8]). *The following hold for a subset A of a space X:*

(1) δ-$sint(A) = A \cap cl(\delta\text{-}int(A))$ *and* δ-$scl(A) = A \cup int(\delta\text{-}cl(A))$;

(2) δ-$pcl(A) = A \cup cl(\delta\text{-}int(A))$;

(3) δ-$scl(\delta\text{-}sint(A)) = \delta\text{-}sint(A) \cup int(cl(\delta\text{-}int(A)))$ *and* δ-$sint(\delta\text{-}scl(A)) = \delta\text{-}scl(A) \cap cl(int(\delta\text{-}cl(A)))$;

(4) δ-$cl(\delta\text{-}sint(A)) = cl(\delta\text{-}int(A))$;

(5) δ-$scl(\delta\text{-}int(A)) = int(cl(\delta\text{-}int(A)))$.

Theorem 1.1. *([3]) Let A be a subset of a space X. Then:*

(1) e-$cl(A) = \delta\text{-}pcl(A) \cap \delta\text{-}scl(A)$;

(2) δ-$int(e\text{-}cl(A)) = int(cl(\delta\text{-}int(A)))$.

2. e^*-OPEN SETS, $(\mathcal{D},\mathcal{S})$-SETS AND $(\mathcal{DS},\mathcal{E})$-SETS

Definition 2.1. A subset A of a space X is called e^*-open if $A \subset cl(int(\delta\text{-}cl(A)))$.

Theorem 2.1. *The following are equivalent for a subset A of a space X:*

(1) *A is e^*-open,*

(2) *there exists a δ-preopen set U such that $U \subset \delta\text{-}cl(A) \subset \delta\text{-}cl(U)$,*

(3) *δ-$cl(A)$ is regular closed.*

Remark 2.1. (1) Let A be a subset of a space X. Then the following diagram holds:

$$\begin{tikzcd}
& e^*\text{-open} & \\
& e\text{-open} \arrow[u] & \\
\delta\text{-semiopen} \arrow[ur] & & \delta\text{-preopen} \arrow[ul] \\
\delta\text{-open} \arrow[u] \arrow[dr] & & \text{preopen} \arrow[u] \\
& \text{open} \arrow[ur] &
\end{tikzcd}$$

(2) None of these implications is reversible as shown in the following example and in [3].

(3) Every β-open set is e^*-open but the converse is not true in general as shown in the following example.

Example 2.1. Let $X = \{a, b, c, d\}$ and let $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the set $\{b, d\}$ is e^*-open but it is neither e-open nor β-open.

Definition 2.2. A subset A of a space X is called e^*-closed if $int(cl(\delta\text{-}int(A))) \subset A$.

Theorem 2.2. (1) *The union of any family of e^*-open sets is an e^*-open set;*

(2) *The intersection of any family of e^*-closed sets is an e^*-closed set.*

Definition 2.3. Let A be a subset of a space X.

(1) The intersection of all e^*-closed sets containing A is called the e^*-closure of A and is denoted by $e^*\text{-}cl(A)$;

(2) The e^*-interior of A, denoted by $e^*\text{-}int(A)$, is defined by the union of all e^*-open sets contained in A.

Lemma 2.1. *The following hold for a subset A of a space X:*

(1) *$e^*\text{-}cl(A)$ is e^*-closed,*

(2) *$X \backslash e^*\text{-}cl(A) = e^*\text{-}int(X \backslash A)$.*

Theorem 2.3. *The following hold for a subset A of a space X:*

(1) *A is e^*-open if and only if $A = A \cap cl(int(\delta\text{-}cl(A)))$;*

(2) *A is e^*-closed if and only if $A = A \cup int(cl(\delta\text{-}int(A)))$;*

(3) *$e^*\text{-}cl(A) = A \cup int(cl(\delta\text{-}int(A)))$;*

(4) *$e^*\text{-}int(A) = A \cap cl(int(\delta\text{-}cl(A)))$.*

Proof. (1) : Let A be e^*-open. Then $A \subset cl(int(\delta\text{-}cl(A)))$. We obtain $A \subset A \cap cl(int(\delta\text{-}cl(A)))$. Conversely, let $A = A \cap cl(int(\delta\text{-}cl(A)))$. We have $A = A \cap cl(int(\delta\text{-}cl(A))) \subset cl(int(\delta\text{-}cl(A)))$ and hence, A is e^*-open.

(3) : Since $e^*\text{-}cl(A)$ is e^*-closed, $int(cl(\delta\text{-}int(A))) \subset int(cl(\delta\text{-}int(e^*\text{-}cl(A))))$ $\subset e^*\text{-}cl(A)$. Hence, $A \cup int(cl(\delta\text{-}int(A))) \subset e^*\text{-}cl(A)$.

Conversely, since:

$$\begin{aligned}
&int(cl(\delta\text{-}int(A \cup int(cl(\delta\text{-}int(A)))))) = \\
&= int(cl(\delta\text{-}int(A \cup \delta\text{-}int(\delta\text{-}cl(\delta\text{-}int(A)))))) = \\
&= int(cl(\delta\text{-}int(A) \cup \delta\text{-}int(\delta\text{-}int(\delta\text{-}cl(\delta\text{-}int(A)))))) = \\
&= int(cl(\delta\text{-}int(A) \cup \delta\text{-}int(\delta\text{-}cl(\delta\text{-}int(A))))) = \\
&= int(cl(\delta\text{-}int(\delta\text{-}cl(\delta\text{-}int(A))))) = \\
&= int(cl(\delta\text{-}int(A))) \subset A \cup int(cl(\delta\text{-}int(A))),
\end{aligned}$$

then $A \cup int(cl(\delta\text{-}int(A)))$ is e^*-closed containing A and hence:

$$e^*\text{-}cl(A) \subset A \cup int(cl(\delta\text{-}int(A))).$$

Thus, we obtain $e^*\text{-}cl(A) = A \cup int(cl(\delta\text{-}int(A)))$.

(2) follows from (1) and (4) follows from (3). □

Theorem 2.4. *Let A be a subset of a space X. Then the following hold:*

(1) $e^*\text{-}cl(\delta\text{-}int(A)) = int(cl(\delta\text{-}int(A)))$;
(2) $\delta\text{-}int(e^*\text{-}cl(A)) = int(cl(\delta\text{-}int(A)))$;
(3) $e^*\text{-}int(\delta\text{-}cl(A)) = \delta\text{-}cl(e^*\text{-}int(A)) = cl(int(\delta\text{-}cl(A)))$;
(4) $e^*\text{-}int(e\text{-}cl(A)) = \delta\text{-}sint(\delta\text{-}scl(A)) \cap \delta\text{-}pcl(A)$;
(5) $e^*\text{-}cl(e\text{-}int(A)) = \delta\text{-}scl(\delta\text{-}sint(A)) \cup \delta\text{-}pint(A)$;
(6) $e\text{-}cl(e^*\text{-}int(A)) = \delta\text{-}sint(\delta\text{-}scl(A)) \cap \delta\text{-}pcl(A)$;
(7) $e\text{-}int(e^*\text{-}cl(A)) = \delta\text{-}scl(\delta\text{-}sint(A)) \cup \delta\text{-}pint(A)$.

Proof. Proof is similar to the proof of Theorem 2.15 in [3]. □

Theorem 2.5. *Let A be a subset of a space X. Then:*

$$\delta\text{-}scl(\delta\text{-}sint(A)) \subset \delta\text{-}sint(\delta\text{-}scl(A)).$$

Proof. Proof is similar to the proof of Theorem 3.2 in [3]. □

Definition 2.4. A subset A of a space X is said to be a $(\mathcal{D},\mathcal{S})$-set if $\delta\text{-}sint(\delta\text{-}scl(A)) = int(A)$.

Theorem 2.6. *Let A be a subset of a space X. Then if A is a $(\mathcal{D},\mathcal{S})$-set, it is δ-semiclosed.*

Proof. Since A is a $(\mathcal{D},\mathcal{S})$-set, we have:

$$\begin{aligned}
A \supset int(A) &= \delta\text{-}sint(\delta\text{-}scl(A)) = \delta\text{-}scl(A) \cap cl(int(\delta\text{-}cl(A))) \supset \\
&\supset int(\delta\text{-}cl(A)) \cap cl(int(\delta\text{-}cl(A))) \supset int(\delta\text{-}cl(A)).
\end{aligned}$$

This implies that A is δ-semiclosed. □

The following example shows that the implication in Theorem 2.6 is not reversible.

Example 2.2. Let $X = \{a, b, c, d\}$ and let $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the set $\{a, b, d\}$ is δ-semiclosed but it is not a $(\mathcal{D}, \mathcal{S})$-set.

Definition 2.5. A subset K of a space X is said to be a $(\mathcal{D}, \mathcal{S})^*$-set if there exist an open set A and a $(\mathcal{D}, \mathcal{S})$-set B such that $K = A \cap B$.

Remark 2.2. Every $(\mathcal{D}, \mathcal{S})$-set and every open set is a $(\mathcal{D}, \mathcal{S})^*$-set but not conversely.

Example 2.3. Let $X = \{a, b, c, d\}$ and let $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the set $\{d\}$ is a $(\mathcal{D}, \mathcal{S})^*$-set but it is not open. The set $\{a\}$ is a $(\mathcal{D}, \mathcal{S})^*$-set but it is not a $(\mathcal{D}, \mathcal{S})$-set.

Theorem 2.7. *Let X be a topological space and $A \subset X$. Then A is e^*-open if and only if $A \subset \delta\text{-}sint(\delta\text{-}scl(A))$.*

Proof. Since A is e^*-open, then $A \subset cl(int(\delta\text{-}cl(A)))$. By Lemma 1.1:

$$\begin{aligned} A &\subset cl(int(\delta\text{-}cl(A))) \cap A \subset \\ &\subset \delta\text{-}scl(A) \cap cl(int(\delta\text{-}cl(A))) = \\ &= \delta\text{-}sint(\delta\text{-}scl(A)). \end{aligned}$$

Conversely, since $A \subset \delta\text{-}sint(\delta\text{-}scl(A))$, by Lemma 1.1 we obtain:

$$\begin{aligned} A &\subset \delta\text{-}sint(\delta\text{-}scl(A)) = \\ &= \delta\text{-}scl(A) \cap cl(int(\delta\text{-}cl(A))) \subset \\ &\subset cl(int(\delta\text{-}cl(A))). \end{aligned}$$

Thus, A is e^*-open. □

Theorem 2.8. *Let K be a subset of a topological space X. Then the following are equivalent:*

(1) *K is open;*
(2) *K is α-open and a $(\mathcal{D}, \mathcal{S})^*$-set;*
(3) *K is preopen and a $(\mathcal{D}, \mathcal{S})^*$-set;*
(4) *K is δ-preopen and a $(\mathcal{D}, \mathcal{S})^*$-set;*
(5) *K is e-open and a $(\mathcal{D}, \mathcal{S})^*$-set;*
(6) *K is e^*-open and a $(\mathcal{D}, \mathcal{S})^*$-set.*

Proof. (6) $\Rightarrow$ (1) : Since K is e^*-open and a $(\mathcal{D},\mathcal{S})^*$-set, then there exist an open set A and a $(\mathcal{D},\mathcal{S})$-set B such that $K = A \cap B$. On the other hand:

$$\begin{aligned} K &\subset \delta\text{-}sint(\delta\text{-}scl(A \cap B)) \subset \\ &\subset \delta\text{-}sint(\delta\text{-}scl(A)) \cap \delta\text{-}sint(\delta\text{-}scl(B)) \subset \\ &\subset \delta\text{-}sint(\delta\text{-}cl(A)) \cap int(B) = \\ &= cl(int(\delta\text{-}cl(A))) \cap int(B) \subset \\ &\subset cl(cl(\delta\text{-}cl(A))) \cap int(B) = \\ &= \delta\text{-}cl(A) \cap int(B) \end{aligned}$$

by Lemma 1.1. We have $K \subset \delta\text{-}cl(A) \cap int(B) \cap A = int(B) \cap A$ and then $K = A \cap int(B)$. Thus, K is open.

The other implications are obvious. $\square$

Definition 2.6. A subset K of a space X is said to be a $(\mathcal{DS},\mathcal{E})$-set if $\delta\text{-}scl(\delta\text{-}sint(K)) = int(K)$.

Theorem 2.9. *Let K be a subset of a space X. Then if K is a $(\mathcal{DS},\mathcal{E})$-set, then it is e^*-closed.*

Proof. Let K be a $(\mathcal{DS},\mathcal{E})$-set. By Lemma 1.1:

$$\begin{aligned} K &\supset int(K) = \\ &= \delta\text{-}scl(\delta\text{-}sint(K))) = \\ &= \delta\text{-}sint(K) \cup int(cl(\delta\text{-}int(K))) \supset \\ &\supset int(cl(\delta\text{-}int(K))). \end{aligned}$$

This implies that K is e^*-closed. $\square$

The following example shows that this implication is not reversible.

Example 2.4. Let $X = \{a,b,c,d\}$ and let $\tau = \big\{\varnothing, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\big\}$. Then the set $\{a,c,d\}$ is e^*-closed but it is not a $(\mathcal{DS},\mathcal{E})$-set.

Definition 2.7. A subset K of a space X is said to be a $(\mathcal{DS},\mathcal{E})^*$-set if there exist an open set A and a $(\mathcal{DS},\mathcal{E})$-set B such that $K = A \cap B$.

Remark 2.3. Every open and every $(\mathcal{DS},\mathcal{E})$-set is a $(\mathcal{DS},\mathcal{E})^*$-set but not conversely.

Example 2.5. Let $X = \{a,b,c,d\}$ and let $\tau = \big\{\varnothing, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\big\}$. Then the set $\{c,d\}$ is a $(\mathcal{DS},\mathcal{E})^*$-set but it is not open. Then the set $\{a,b,d\}$ is a $(\mathcal{DS},\mathcal{E})^*$-set but it is not a $(\mathcal{DS},\mathcal{E})$-set.

Theorem 2.10. *Let X be a topological space and $K \subset X$. Then K is δ-semiopen if and only if $K \subset \delta\text{-}scl(\delta\text{-}sint(K))$.*

Proof. Proof is similar to the proof of Theorem 2.7. $\square$

Theorem 2.11. *Let K be a subset of a space X. Then if K is δ-semiopen and a $(\mathcal{DS},\mathcal{E})^*$-set, K is open.*

Proof. Since K is δ-semiopen and a $(\mathcal{DS},\mathcal{E})^*$-set, there exist an open set A and a $(\mathcal{DS},\mathcal{E})$-set B such that $K = A \cap B$ and:

$$\begin{aligned} K &\subset \delta\text{-}scl(\delta\text{-}sint(K)) = \\ &= \delta\text{-}scl(\delta\text{-}sint(A \cap B)) \subset \\ &\subset \delta\text{-}scl(\delta\text{-}sint(A)) \cap \delta\text{-}scl(\delta\text{-}sint(B)) = \\ &= \delta\text{-}scl(\delta\text{-}sint(A)) \cap int(B) \subset \\ &\subset \delta\text{-}cl(\delta\text{-}sint(A)) \cap int(B) = \\ &= cl(\delta\text{-}int(A)) \cap int(B) \subset \\ &\subset cl(A) \cap int(B). \end{aligned}$$

Since $K \subset cl(A) \cap int(B) \cap A = A \cap int(B)$, $K = A \cap int(B)$. Thus, K is open. □

Definition 2.8. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be e^*-continuous (resp. $(\mathcal{D},\mathcal{S})^*$-continuous, $(\mathcal{DS},\mathcal{E})^*$-continuous) if $f^{-1}(A)$ is e^*-open (resp. a $(\mathcal{D},\mathcal{S})^*$-set, a $(\mathcal{DS},\mathcal{E})^*$-set) in X for every $A \in \sigma$.

Definition 2.9. A function $f : (X,\tau) \to (Y,\sigma)$ is called e-continuous [3] (resp. δ-almost continuous [9], δ-semicontinuous [2], α-continuous [6], precontinuous [5]) if $f^{-1}(A)$ is e-open (resp. δ-preopen, δ-semiopen, α-open, preopen) for each $A \in \sigma$.

The following remark is immediate from Theorem 2.8.

Remark 2.4. For a function $f : X \to Y$, the following are equivalent:

(1) f is continuous;
(2) f is α-continuous and $(\mathcal{D},\mathcal{S})^*$-continuous;
(3) f is precontinuous and $(\mathcal{D},\mathcal{S})^*$-continuous;
(4) f is δ-almost continuous and $(\mathcal{D},\mathcal{S})^*$-continuous;
(5) f is e-continuous and $(\mathcal{D},\mathcal{S})^*$-continuous;
(6) f is e^*-continuous and $(\mathcal{D},\mathcal{S})^*$-continuous.

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ERDAL EKICI
DEPARTMENT OF MATHEMATICS
CANAKKALE ONSEKIZ MART UNIVERSITY
TERZIOGLU CAMPUS
17020 CANAKKALE
TURKEY
E-mail address: eekici@comu.edu.tr