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Power Semigroups that are 0-Archimedean*

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Abstract. In this paper we give a structural characterization for semigroups whose power semigroups are 0-Archimedean. We prove that these semigroups are exactly the nilpotent semigroups.

1. Introduction and preliminaries

Power semigroups of various semigroups were studied by a number of authors [1, 2, 3, 5, 6, 7] and [8]. In the present paper we are going to prove that the power semigroup of a semigroup S is 0-Archimedean if and only if S is a nilpotent semigroup.

By $\mathbf{Z}^+$ we denote the set of all positive integers. If X is a non-empty set, then with $P(X)$ we denote the *partitive set* of the set X, i.e. the set of all subsets of X. Let S be a semigroup. On the partitive set of a semigroup S we define a multiplication with:

$$AB = \{x \in S \mid (\exists a \in A)(\exists b \in B)\, x = ab\}, \qquad A, B \in P(S).$$

Then under this operation the set $P(S)$ is a semigroup which we call a *partitive semigroup* of a semigroup S. Definitions and notations which we use for multiplication of elements of a semigroup S, we will use for multiplication of elements of a semigroup $P(S)$, too. The set of all idempotents of a semigroup S we denote by $E(S)$. A subsemiogroup $\langle a \rangle$ of a semigroup S generated by one element subset $\{a\}$ of S we call a *monogenic* or a *cyclic* subsemigroup of S.

By S^0 (S^1) we denote a semigroup S with zero 0 (with identity 1).

The following lemmas are very helpful results for the further work.

Lemma 1 ([1]). *A semigroup S is a group if and only if S is the zero in $P(S)$.*

Lemma 2 ([5]). *Let A be an ideal of a semigroup S, then $P(A)$ is an ideal of $P(S)$.*

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A semigroup S is called a *homogroup* if it has a two-sided ideal which is a subgroup of S. This notion is introduced by G. Thierrin in [9].

Let a and b be elements of a semigroup S. Then:

$$a \mid b \Leftrightarrow b \in S^1 a S^1, \qquad a \longrightarrow b \Leftrightarrow (\exists n \in \mathbf{Z}^+)\ a \mid b^n.$$

A semigroup $S = S^0$ is *0-Archimedean* if $a \longrightarrow b$, for all $a, b \in S - \{0\}$. These semigroups are introduced and studied by S. Bogdanović and M. Ćirić in [4].

Let $S = S^0$. An element $a \in S$ is *nilpotent* if there is $n \in \mathbf{Z}^+$ such that $a^n = 0$. The set of all nilpotent elements from a semigroup S we denote by $\mathrm{Nil}(S)$. A semigroup S is *nil-semigroup* if $S = \mathrm{Nil}(S)$. An ideal extension S of a semigroup T is a *nil-extension* of T if S/T is a nil-semigroup, i.e. if $\sqrt{T} = S$. A semigroup $S = S^0$ is *nilpotent* if there is $n \in \mathbf{Z}^+$ such that $S^{n+1} = 0$. If $S^{n+1} = 0$, then we say that S is $(n+1)$-*nilpotent*. A semigroup S is *nilpotent*, the class of nilpotency $n + 1$, if S is $(n + 1)$-nilpotent and it is not n-nilpotent. An ideal extension S of a semigroup T by nilpotent semigroup we call *nilpotent extension* of T.

A semigroup $S = S^0$ is a *null semigroup*, if $S^2 = 0$, i.e. if $ab = 0$, for all $a, b \in S$. A semigroup $S = S^0$ is 0-*simple* if and only if

$$(\forall a, b \in S - \{0\})\ a \in SbS.$$

An element a of a semigroup S is *intra-π-regular* if there is $n \in \mathbf{Z}^+$ such that $a^n \in S a^{2n} S$, i.e. if some its power is intra-regular. A semigroup S is *intra-π-regular* if all its elements are intra-π-regular.

An element a of a semigroup S is *periodic* if there are $m, n \in \mathbf{Z}^+$, such that $a^m = a^{m+n}$. A semigroup S is *periodic* if every its element is periodic.

2. THE RESULTS

We start with the following theorem:

Theorem 1. *A semigroup S is a homogroup if and only if $P(S)$ has the zero.*

Proof. Let G be a group-ideal of a semigroup S. Then by Lemma 1 G is the zero in $P(G)$. Assume $A \in P(S)$, then

$$GA \subseteq GS \subseteq G; \qquad AG \subseteq SG \subseteq G.$$

So, $GA, AG \subseteq P(G)$. Whence $G \cdot GA = G$ and $AG \cdot G = G$. Thus $AG = G$ and $GA = G$. Therefore, G is the zero in $P(S)$.

Conversely, let G be the zero of $P(S)$, then G is the zero of $P(G)$ and by Lemma 1 we have that G is a group. Since $GS \cup SG \subseteq G$ we then have that G is an ideal of S, i.e. S is a homogroup. □

Lemma 3. *Let S be a 0-Archimedean semigroup and A is an ideal of S, then A is 0-Archimedean.*

Proof. It is clear that $0 \in A$. Assume $a, b \in A - \{0\}$ such that $a^n = xby$ for some $x, y \in S$ and $n \in \mathbf{Z}^+$. Then

$$a^{n+2} = (ax)b(ya) \in AbA.$$

Thus A is 0-Archimedean. □

Now we prove the main result of this paper:

Theorem 2. *Let S be a semigroup. Then $P(S)$ is 0-Archimedean if and only if S is nilpotent.*

Proof. Let $P(S)$ be a 0-Archimedean semigroup. Then we have two cases:

(1) $P(S)$ is a nil-semigroup. Then by Theorem 1 [1] S is nilpotent.

(2) $P(S)$ is a non-nil semigroup. By Theorem 1 S has a subgroup G which is an ideal of S. If e is the identity of G, then there exist $k \in \mathbf{Z}^+$ and $X, Y \in P(S)$ such that $\{e\}^k = XGY = G$, since G is the zero of $P(S)$. Thus $|G| = 1$, i.e. S has the zero 0.

For $a, b \in S - \{0\}$ we have that $\{a\}^n = X\{b\}Y$ for some $n \in \mathbf{Z}^+$ and $X, Y \in P(S)$, whence $a^n = xby$ for all $x \in X$, $y \in Y$ and for some $n \in \mathbf{Z}^+$. Hence, S is 0-Archimedean.

For $a \in S - \{0\}$ we have that $\{a\}, \langle a\rangle \in P(S) - \{0\}$, whence $\{a\}^n = B\langle a\rangle C$, for some $n \in \mathbf{Z}^+$ and $B, C \in P(S)$. Now we have that $a^n = ba^{2n}c$ for every $b \in B$ and $c \in C$. Thus S is intra-π-regular.

Since S is 0-Archimedean intra-π-regular, then by Theorem 3 [4] S is a nil-extension of a 0-simple semigroup K. For S and $a \in K - \{0\}$ there exist $n \in \mathbf{Z}^+$ and $X, Y \in P(S)$ such that

$$S^n = X\{a\}Y \subseteq XKY \subseteq K = K^2 = K^n \subseteq S^n.$$

Hence, S is a nilpotent extension of K. By Lemma 3 $P(K)$ is an ideal of $P(S)$ and by Lemma 1 $P(K)$ is 0-Archimedean. For $a \in K - \{0\}$ there exist $B, C \in P(K)$ such that $\{a\}^n = BKC$, for some $n \in \mathbf{Z}^+$. Since K is 0-simple, then we have that $\{a\}^n K = B(KCK) = BK$ and $K\{a\}^n = (KBK)C = KC$.

Now $\{a\}^n = BKC = \{a\}^n KC = \{a\}^n K\{a\}^n$, whence $a^n = a^n \cdot a \cdot a^n$, $a^n = a^n \cdot a^2 \cdot a^n$. So $a^n = a^{n+1}$, i.e. S is periodic. Assume $e \in E(K - \{0\})$, then there exist $k \in \mathbf{Z}^+$ and $B, C \in P(K)$ such that $\{e\}^k = B\{0, e\}C$, whence $0 \in \{e\}$, i.e. $e = 0$, which is not possible.

The converse follows by Theorem 1 [1]. □

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