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Fixed Point Theorems for Some Discontinuous Operators in Cone Metric Space

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Abstract. In this article, some fixed point theorems in cone metric spaces for operators belonging to the class $E(a,b,c)$ are proved.

1. Introduction

In [2] Derrick and Nova defined the following operator classes:

Let $(E, \|\cdot\|)$ be a Banach space, $K \subset E$ closed and $T : K \to K$ an arbitrary operator that satisfies the following condition for $a, b > 0$ and any $x, y \in K$:

(A) $\|(Tx - Ty) - b[(x - Tx) + (y - Ty)\| \leq a\|x - y\|$
(B) $\|(Tx - Ty) - b(x - Tx)\| \leq a\|x - y\| + b\|y - Ty\|$
(C) $\|(Tx - Ty) - a(x - y)\| \leq b[\|x - Tx\| + \|y - Ty\|]$
(D) $\|Tx - Ty\| \leq a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|]$

We shall say that T belongs or is of class $A(a,b)$ (respectively $B(a,b)$, $C(a,b)$, $D(a,b)$), when satisfies the condition (A) (respectively B, C, D).

Observe that, using the triangle inequality, that any map of class $A(a,b)$, $B(a,b)$ or $C(a,b)$ is of class $D(a,b)$.

Note that the condition (D) may hold even if the operator is discontinuous. In fact, any operator is in class $D(1,1)$. Since by triangle inequality:

$$\|Tx - Ty\| \leq \|Tx - x\| + \|x - y\| + \|y - Ty\|.$$

Recently, Khan and Samanipour [5] defined a new class of operators called $E(a,b,c)$ which includes operators of class $D(a,b)$.

Definition 1.1. Let $(E, \|\cdot\|)$ be a Banach space, $K \subset E$ closed and $T : K \to K$ an arbitrary operator that satisfies the following condition for $a, b, c > 0$ and any $x, y \in K$:

$$\|Tx - Ty\| \leq a\|x - y\| + b\big[\|x - Tx\| + \|y - Ty\|\big] + c\big[\|x - Ty\| + \|y - Tx\|\big].$$

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We shall say that T belongs or is of class $E(a,b,c)$.

Huang and Zhang [3] have replaced the real numbers by ordered Banach spaces and defined a cone metric space. They have proved some fixed point theorems of contractive mappings defined on these spaces. Further results on fixed point theorems in such spaces were obtained by several other mathematicians, see [1], [4] and [6].

The following concepts are borrowed from Huang and Zhang.

Let E be a real Banach space, and P a subset of E. Then P is called a cone if

(i) P is closed, nonempty, and $P \neq 0$,
(ii) $a, b \in R, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$; we shall write $x \ll y$ if $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P . The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \quad \text{implies} \quad \|x\| \leq K\|y\|.$$

Definition 1.2. Let X be a non-empty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(d1) $0 < d(x,y)$, for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
(d2) $d(x,y) = d(y,x)$, for all $x, y \in X$;
(d3) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X, and the pair (X,d) is called a cone metric space.

The sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if for every $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$, for every $n \geq n_0$, and is called a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) \ll c$, for every $m, n \geq n_0$. A cone metric space (X,d) is called a complete cone metric space if every Cauchy sequence in X is convergent to a point of X. A self-map T on X is said to be continuous if $\lim\limits_{n\to\infty} x_n = x$ implies that $\lim\limits_{n\to\infty} T(x_n) = T(x)$, for every sequence $\{x_n\}$ in X.

Lemma 1.3 ([3])**.** *Let (X,d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.*

Following results will be used in the sequel.

Lemma 1.4 ([3], Lemma 1)**.** *Let (X,d) be a cone metric space, and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ $(n \to \infty)$.*

Lemma 1.5 ([3], Lemma 3). *Let (X, d) be a cone metric space, and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.*

Lemma 1.6 ([3], Lemma 4). *Let (X, d) be a cone metric space, and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.*

The following example is a cone metric space, see [3].

Example 1.7. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E \mid x, y \geq 0\}$, $X = \mathbb{R}$, and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

In this paper, we have studied some fixed point theorems in cone metric spaces for operators T belonging to the class $E(a, b, c)$ and posses some special properties.

2. Main results

First, we introduce some new concepts.

Definition 2.1. Let (X, d) be a cone metric space, and P be a normal cone with normal constant K. Let $T : Y \to Y, Y \subset X$ and $x \in Y$. Then T is said to be c-asymptotically regular at x if for all natural numbers n, $T^n(x) \in Y$ and $\lim_{n\to\infty} d(T^n(x), T^{n+1}(x)) = 0$.

Definition 2.2. Let (X, d) be a cone metric space, and P be a normal cone with normal constant K. A sequence $\{x_n\}$ of elements of $Y \subset X$ is said to c-asymptotically T-regular if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Remark 2.3. It is obvious that T is c-asymptotically regular at $x \in Y$ if and only if for all natural numbers n, $T^n(x) \in Y$ and $\{T^n(x)\}$ is c-asymptotically T-regular.

Definition 2.4. Let (X, d) be a cone metric space, and P be a normal cone with normal constant K. We say that $T \in E(a, b, c)$ if the inequality

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)]$$

holds for all $x, y \in X$, $0 \leq a, b, c < 1$.

Now, we present the main results.

Theorem 2.5. *Let (X, d) be a cone metric space,and P be a normal cone with normal constant K , and $T : X \to X$. If $T \in E(a, b, c)$, $0 \leq a, b, c < 1$, $a + 2b + 2c < 1$. Then T is c-asymptotically regular at every point in X.*

Proof. Let x_0 be an arbitrary point in X. Define $x_n = T^n x_0$. Then we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq ad(x_{n-1}, x_n) + b[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &\qquad + c[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] \\ &= ad(x_{n-1}, x_n) + b[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\qquad + c[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\ &\leq ad(x_{n-1}, x_n) + b[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\qquad + c[d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \end{aligned}$$

so that

$$d(x_n, x_{n+1}) \leq \frac{a+b+c}{1-b-c} d(x_{n-1}, x_n).$$

Now we get

$$\|d(x_n, x_{n+1})\| \leq \left(\frac{a+b+c}{1-b-c}\right)^n K\|d(x_0, x_1)\|.$$

This implies

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(T^n x_0, T^{n+1} x_0) = 0.$$

Since x_0 is arbitrary, T is c-asymptotically regular at every point in X. This completes the proof. □

Theorem 2.6. *Let X be complete cone metric space,and P be a normal cone with normal constant K ,and $T : X \to X$ be a mapping in $E(a, b, c)$, $0 \leq a, b, c < 1$. Then a sequence $\{x_n\}$in X is c-asymptotically T-regular if and only if it converges to a fixed point of T.*

Proof. Suppose that $\lim_{n\to\infty} x_n = z$ and $z = Tz$. Then we have

$$d(x_n, Tx_n) \leq d(x_n, z) + d(z, Tx_n).$$

So by letting $n \to \infty$, we see that $d(x_n, Tx_n) \to 0$. Hence $\{x_n\}$ is c-asymptotically T-regular. Conversely,

$$\begin{aligned} d(Tx_n, Tx_m) &\leq ad(x_n, x_m) + b[d(x_n, Tx_n) + d(x_m, Tx_m)] \\ &\qquad + c[d(x_n, Tx_m) + d(x_m, Tx_n)] \\ &\leq a[d(x_n, Tx_n) + d(Tx_n, Tx_m) + d(Tx_m, x_m)] \\ &\qquad + b[d(x_n, Tx_n) + d(x_m, Tx_m)] \\ &\qquad + c[d(x_n, Tx_n) + d(Tx_n, Tx_m) \\ &\qquad + d(x_m, Tx_m) + d(Tx_m, Tx_n)] \end{aligned}$$

So that

$$d(Tx_n, Tx_m) \leq \frac{a+b+c}{1-a-2c}[d(x_n, Tx_n) + d(x_m, Tx_m)]$$

$$\|d(Tx_n, Tx_m)\| \leq \left(\frac{a+b+c}{1-a-2c}\right) K\|d(x_n, Tx_n) + d(x_m, Tx_m)\| \to 0.$$

Letting $m, n \to \infty$, we observe that $\{Tx_n\}$ is a Cauchy sequence. Since X is complete $\{Tx_n\}$ converges to, say, z in X. Since $\lim_{n\to\infty} d(x_n, Tx_n) \to 0$, $x_n \to z$ as $n \to \infty$. We assert that $z = Tz$. For if $z \neq Tz$, then

$$\begin{aligned} d(z, Tz) &\leq d(z, Tx_n) + d(Tx_n, Tz) \\ &\leq d(z, Tx_n) + ad(x_n, z) + b[d(x_n, Tx_n) + d(z, Tz)] \\ &\qquad + c[d(z, Tx_n) + d(x_n, Tz)] \\ &\leq d(z, x_n) + d(x_n, Tx_n) + ad(x_n, z) + b[d(x_n, Tx_n) + d(z, Tz)] \\ &\qquad + c[d(z, x_n) + d(x_n, Tx_n) + d(x_n, z) + d(z, Tz)]. \end{aligned}$$

From this we obtained

$$\|d(z, Tz)\| \leq \left(\frac{1+a+2c}{1-b-c}\right) K(\|d(x_n, Tx_n)\| + \|d(z, x_n)\|) \to 0.$$

Hence $z = Tz$. This completes the proof. □

Theorem 2.7. *Let X be complete cone metric space, and P be a normal cone with normal constant K, and $T : X \to X$ be a mapping in $E(a, b, c), a, b, c \geq 0,\ a + 2b + 2c < 1$. Then T has a unique fixed point in X.*

Proof. By Theorem 2.5 T is c-asymptotically regular at every point in X. Let x_0 be an arbitrary point in X. Define $x_n = T^n x_0$. Then the sequence $\{x_n\}$ is c-asymptotically T-regular (see Remark 2.3). Thus by Theorem 2.6 the sequence $\{x_n\}$ converges to a point z in X such that $z = Tz$. To show that z is unique, suppose z and z_1 are two fixed points of T. Then, we have

$$\begin{aligned} d(z, z_1) &= d(Tz, Tz_1) \\ &\leq ad(z, z_1) + b[d(z, Tz) + d(z_1, Tz_1)] + c[d(z, Tz_1) + d(z_1, Tz)] \\ &\leq ad(z, z_1) + c[d(z, z_1) + d(z_1, Tz_1) + d(z_1, z) + d(z, Tz)], \end{aligned}$$

so

$$d(z, z_1) \leq (a + 2c)d(z, z_1),$$

which implies that $d(z, z_1) = 0$. Hence $z = z_1$. This completes the proof. □

In Theorem 2.7, for establishing the existence of fixed points we have used the c-asymptotic regularity of T at one point only. Keeping this in mind we obtain an extension of the above theorem in which the condition $a + 2b + 2c < 1$ may be relaxed. Thus we have the following theorem. Note that $a + 2b + 2c$ may exceed 1 in this ease.

Theorem 2.8. *Let X be complete cone metric space, and P be a normal cone with normal constant K and $T : X \to X$ be a mapping in $E(a, b, c), a, b, c \geq 0,\ b, c < 1$. If T is c-asymptotically regular at some* *point in X, then T has a fixed point in X. Further, if $a + 2c < 1$, then the fixed point is unique.*

Proof. Let T be c-asymptotically regular at $x_0 \in X$. Define $x_n = T^n x_0$. Then the result immediately follows from Theorem 2.7. This completes the proof. □

REFERENCES

[1] M. Abbas and G. Jungck, *Common fixed point results for non commuting mappings without continuity in cone metric space*, J. Math. Anal. Appl., **341** (2008), 416-420.
[2] W.R. Derrick and L. Nova, *Interior properties and fixed points of certain discontinuous operators*, Elsevier Science, (1992), 239-245.
[3] Huang Long-Guang and Zhang Xian, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332** (2007), 1468-1476.
[4] D. Ilić and V. Rakočević, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl., **341** (2008), 876-882.
[5] M.S. Khan and M. Samanipour, *Fixed point theorems for operators of class $E(a, b, c)$*, (Submitted).
[6] P. Raja and S.M. Vaezpour, *Some extensions of Banach's contraction principle in complete cone metric spaces*, J. Fixed point theory. Appl., (To appear).

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