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Common Fixed Point Theorems for Subcompatible D-Maps

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ABSTRACT. The purpose of this paper is to establish a common fixed point theorem for two pairs of subcompatible single and set-valued D-maps in a metric space. This result improves, extends and generalizes the result of [1] and others.

1. INTRODUCTION

In the sequel $(\mathcal{X}, d)$ denotes a metric space and $B(\mathcal{X})$ is the set of all nonempty bounded subsets of $\mathcal{X}$. We define

$$\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}$$

for all A, B in $B(\mathcal{X})$. If $A = \{a\}$, we write $\delta(A, B) = \delta(a, B)$. Also, if $B = \{b\}$, we write $\delta(A, B) = d(a, b)$. From the definition of $\delta(A, B)$ it follows immediately that

$$\begin{aligned}
&\delta(A, B) \geq 0, \\
&\delta(A, B) = \delta(B, A), \\
&\delta(A, B) \leq \delta(A, C) + \delta(C, B), \\
&\delta(A, A) = \operatorname{diam} A, \\
&\delta(A, B) = 0 \quad \text{iff} \quad A = B = \{a\}
\end{aligned}$$

for all A, B, C in $B(\mathcal{X})$.

Definition 1.1 ([3]). *A sequence $\{A_n\}$ of nonempty subsets of $\mathcal{X}$ is said to be convergent to a subset A of $\mathcal{X}$ if:*

(i) each point a in A is the limit of a convergent sequence $\{a_n\}$, where a_n is in A_n for $n \in \mathbb{N}$,
(ii) for arbitrary $\varepsilon > 0$, there exists an integer m such that $A_n \subseteq A_\varepsilon$ for $n > m$, where A_ε denotes the set of all points x in $\mathcal{X}$ for which there

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exists a point a in A, depending on x, such that $d(x, a) < \epsilon$. A is then said to be the limit of the sequence $\{A_n\}$.

Lemma 1.1 ([3]). *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(\mathcal{X})$ converging to A and B in $B(\mathcal{X})$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Lemma 1.2 ([4]). *Let $\{A_n\}$ be a sequence in $B(\mathcal{X})$ and y be a point in $\mathcal{X}$ such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(\mathcal{X})$.*

To generalize commuting and weakly commuting maps, Jungck [5] introduced the concept of compatible maps. When f and g are self-maps of a metric space $(\mathcal{X}, d)$, he defines f and g to be compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0 \tag{1}$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

Further, Jungck et al. [7] gave another generalization of weakly commuting maps by introducing compatible maps of type (A). f and g above are compatible of type (A) if they satisfy instead of (1) the two equalities

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(gfx_n, f^2x_n) = 0.$$

Extending type (A) maps, Pathak and Khan [10] introduced the notion of compatible maps of type (B). f and g are compatible of type (B) if in lieu of (1) we have

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n)\right]$$

and

$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n)\right].$$

In their paper [9], Pathak et al. added another extension of compatible maps of type (A) by giving the concept of compatible maps of type (C). f and g above are compatible of type (C) if they satisfy the two inequalities

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leq \frac{1}{3}\Big[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n) + \lim_{n\to\infty} d(ft, g^2x_n)\Big]$$

and

$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leq \frac{1}{3}\Big[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n) + \lim_{n\to\infty} d(gt, f^2x_n)\Big].$$

In 1996, Jungck [6] gave a generalization of the above concepts by introducing the notion of weakly compatible maps. f and g are weakly compatible if they commute at their coincidence points, i.e., if $ft = gt$ for some $t \in \mathcal{X}$, then $fgt = gft$.

Afterwards, Jungck and Rhoades [8] extended the above notion to the setting of single and set-valued maps. $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to B(\mathcal{X})$ are subcompatible if

$$\{t \in \mathcal{X}/Ft = \{ft\}\} \subseteq \{t \in \mathcal{X}/Fft = fFt\}.$$

Recently, Djoudi and Khemis [2] introduced the concept of D-maps as follows: f and F above are D-maps if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$\lim_{n\to\infty} fx_n = t \quad \text{and} \quad \lim_{n\to\infty} Fx_n = \{t\}$$

for some $t \in \mathcal{X}$.

Example 1.1.

(1) Let $\mathcal{X} = [1, \infty)$ with the usual metric d. Define $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to B(\mathcal{X})$ as follows

$$fx = x \quad \text{and} \quad Fx = [1, x] \quad \text{for} \quad x \in \mathcal{X}.$$

Let $x_n = 1 + \dfrac{1}{n}$ for $n \in \mathbb{N}^* = \{1, 2, \dots\}$. Then,

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_n = 1 \quad \text{and} \quad \lim_{n\to\infty} Fx_n = \lim_{n\to\infty} [1, x_n] = \{1\}.$$

Therefore f and F are D-maps.

(2) Endow $\mathcal{X} = [1, \infty)$ with the usual metric d and define

$$fx = x + 3 \quad \text{and} \quad Fx = [1, x] \quad \text{for every} \quad x \in \mathcal{X}.$$

Suppose there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $fx_n \to t$ and $y_n \to t$ for some $t \in \mathcal{X}$, with $y_n \in Fx_n = [1, x_n]$. Then, $\lim_{n\to\infty} x_n = t - 3$ and $1 \le t \le t - 3$, which is impossible.

Let $\mathbb{R}_+$ be the set of all non-negative real numbers and $\mathcal{G}$ be the set of all continuous functions $G : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the conditions

(G_1) : G is nondecreasing in variables t_5 and t_6,

(G_2) : there exists $\theta \in (1, \infty)$, such that for every $u, v \ge 0$ with

  (G_a) : $G(u, v, u, v, u + v, 0) \ge 0$ or

  (G_b) : $G(u, v, v, u, 0, u + v) \ge 0$

  we have $u \ge \theta v$.

(G_3) : $G(u, u, 0, 0, u, u) < 0 \ \forall u > 0$.

In [1], Djoudi established and proved the next result.

Theorem 1.1. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ be maps from a complete metric space $\mathcal{X}$ into itself having the following conditions*

(*i*) *$\mathcal{A}$, $\mathcal{B}$ are surjective,*
(*ii*) *the pairs of maps $\mathcal{A}$, $\mathcal{S}$ as well as $\mathcal{B}$, $\mathcal{T}$ are weakly compatible,*
(*iii*) *the inequality*

$$G\big(d(\mathcal{A}x,\mathcal{B}y),d(\mathcal{S}x,\mathcal{T}y),d(\mathcal{A}x,\mathcal{S}x),\\ d(\mathcal{B}y,\mathcal{T}y),d(\mathcal{A}x,\mathcal{T}y),d(\mathcal{B}y,\mathcal{S}x)\big)\geq 0$$

for all $x,y\in\mathcal{X}$, where $G\in\mathcal{G}$. Then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Our aim here is to extend the above result to the setting of single and set-valued maps in a metric space by deleting some conditions required on G. Also, we give a generalization of our result.

2. IMPLICIT RELATIONS

Let $\mathbb{R}_+$ and let Φ be the set of all continuous functions $\varphi:\mathbb{R}_+^6\to\mathbb{R}$ satisfying the conditions

(φ_1) : for every $u,v\geq 0$ with
(φ_a) : $\varphi(u,v,u,v,u+v,0)\geq 0$ or
(φ_b) : $\varphi(u,v,v,u,0,u+v)\geq 0$ we have $u\geq v$.
(φ_2) : $\varphi(u,u,0,0,u,u)<0\ \forall u>0$.

Example 2.1.

$$\varphi(t_1,t_2,t_3,t_4,t_5,t_6)=t_1^p-t_2^p-\frac{\alpha t_5^{p-1}t_6+\beta t_5t_6^{p-1}}{1+\gamma t_3^p+\delta t_4^p},$$

where $\alpha,\beta>0,\gamma,\delta\geq 0$ and p is an integer such that $p\geq 2$.

(φ_1) : For $u\geq 0$ and $v\geq 0$ we have

$$\varphi(u,v,u,v,u+v,0)=\varphi(u,v,v,u,0,u+v)=u^p-v^p\geq 0,$$

which implies that $u\geq v$.

(φ_2) : $\varphi(u,u,0,0,u,u)=-(\alpha+\beta)u^p<0\ \forall u>0$.

Example 2.2.

$$\varphi(t_1,t_2,t_3,t_4,t_5,t_6)=t_1^p-at_2^p-bt_3^p-ct_4^p-dt_5^{p-1}t_6-et_5t_6^{p-1},$$

where $a\geq 1$, $0\leq b,c<1$, $a+b+c\geq 1$, $a+d+e>1$ and p is an integer such that $p\geq 2$.

(φ_1) : For $u\geq 0$ and $v\geq 0$ we have

$$\varphi(u,v,u,v,u+v,0)=u^p-av^p-bu^p-cv^p\geq 0$$

which implies that

$$u\geq\left(\frac{a+c}{1-b}\right)^{\frac{1}{p}}v\geq v.$$

Similarly, we have

$$\varphi(u,v,v,u,0,u+v) = u^p - av^p - bv^p - cu^p \geq 0$$

which implies that

$$u \geq \left(\frac{a+b}{1-c}\right)^{\frac{1}{p}} v \geq v.$$

(φ_2) : $\varphi(u,u,0,0,u,u) = u^p(1-a-d-e) < 0 \; \forall u > 0.$

Example 2.3.

$$\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = \min\{t_1,t_3,t_4\} - kt_1,$$

where $k > 1$.

(φ_1) : Let $u \geq 0$ and $v \geq 0$. Suppose that $u < v$. Then

$$\begin{aligned}\varphi(u,v,u,v,u+v,0) = \varphi(u,v,v,u,0,u+v) = \\ = \min\{u,v\} - ku = u - ku \geq 0\end{aligned}$$

which implies that $u \geq ku > u$ which is a contradiction. Then $u \geq v$.

(φ_2) : $\varphi(u,u,0,0,u,u) = \min\{u,0\} - ku = -ku < 0$, $\forall u > 0$.

Example 2.4.

$$\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = \min\left\{t_1^2, t_3t_4\right\} - \alpha t_5t_6 - \beta t_1^2,$$

where $\alpha \geq 0$ and $\beta > 1$.

(φ_1) : Let $u \geq 0$ and $v \geq 0$. Suppose that $u < v$. Then

$$\begin{aligned}\varphi(u,v,u,v,u+v,0) = \varphi(u,v,v,u,0,u+v) = \\ = \min\left\{u^2, uv\right\} - \beta u^2 = u^2 - \beta u^2 \geq 0\end{aligned}$$

which implies that $u^2 \geq \beta u^2 > u^2$, which is a contradiction. Then $u \geq v$.

(φ_2) : $\varphi(u,u,0,0,u,u) = \min\left\{u^2,0\right\} - \alpha u^2 - \beta u^2 = -(\alpha+\beta)u^2 < 0$, $\forall u > 0$.

3. Main results

Theorem 3.1. *Let f, g be self-maps of a metric space $(\mathcal{X}, d)$ and let $F, G : \mathcal{X} \to B(\mathcal{X})$ be two set-valued maps satisfying the conditions*

(1) *f and g are surjective,*
(2) $\varphi\big(d(fx,gy), \delta(Fx,Gy), \delta(fx,Fx), \delta(gy,Gy), \delta(fx,Gy), \delta(gy,Fx)\big) \geq 0$ *for all x, y in $\mathcal{X}$, where $\varphi \in \Phi$.*

If either

(3) *f and F are subcompatible D-maps; g and G are subcompatible, or*
(3′) *g and G are subcompatible D-maps; f and F are subcompatible,*

then f, g, F and G have a unique common fixed point $t \in \mathcal{X}$ such that

$$Ft = Gt = \{t\} = \{ft\} = \{gt\}.$$

Proof. Suppose that F and f are D-maps, then, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = t$ and $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in \mathcal{X}$. Since f and g are surjective, then, there exist two points u and v in $\mathcal{X}$ such that $t = fu = gv$. First, we show that $\{t\} = Gv$. Indeed, by inequality (2) we get

$$\varphi\big(d(fx_n, gv), \delta(Fx_n, Gv), \delta(fx_n, Fx_n), \delta(gv, Gv), \delta(fx_n, Gv), \delta(gv, Fx_n)\big) \geq 0.$$

Since φ is continuous, using Lemma 1.1 we obtain at infinity

$$\varphi(0, \delta(t, Gv), 0, \delta(t, Gv), \delta(t, Gv), 0) \geq 0,$$

thus, by (φ_a) we have $Gv = \{t\}$, i.e., $Gv = \{t\} = \{gv\}$. Since G and g are subcompatible, then $Ggv = gGv$ and hence $GGv = Ggv = gGv = \{ggv\}$. We claim that $Ggv = \{t\}$. Suppose not, then $\delta(t, Ggv) > 0$ and by (2) we get

$$\varphi\big(d(fx_n, g^2v), \delta(Fx_n, Ggv), \delta(fx_n, Fx_n), \delta(g^2v, Ggv), \delta(fx_n, Ggv), \delta(g^2v, Fx_n)) \geq 0.$$

Since φ is continuous, using lemma 1.1 we obtain at infinity

$$\begin{aligned} 0 &\leq \varphi(d(t, g^2v), \delta(t, Ggv), 0, 0, \delta(t, Ggv), \delta(g^2v, t)) \\ &= \varphi(\delta(t, Ggv), \delta(t, Ggv), 0, 0, \delta(t, Ggv), \delta(Ggv, t)) \end{aligned}$$

contradicts (φ_2), then $Ggv = \{t\} = \{gv\} = \{ggv\}$. , by inequality (2) we have

$$\begin{aligned} 0 &\leq \varphi(d(fu, gv), \delta(Fu, Gv), \delta(fu, Fu), \delta(gv, Gv), \delta(fu, Gv), \delta(gv, Fu)) \\ &= \varphi(0, \delta(Fu, t), \delta(t, Fu), 0, 0, \delta(t, Fu)) \end{aligned}$$

which by (φ_b) implies that $Fu = \{t\} = \{fu\}$. Since F and f are subcompatible, then $Ffu = fFu$ and hence $FFu = Ffu = fFu = \{ffu\}$. If $\delta(Ffu, t) > 0$, then by inequality (2) we have

$$\begin{aligned} 0 &\leq \varphi\big(d(f^2u, gv), \delta(Ffu, Gv), \delta(f^2u, Ffu), \\ &\qquad \delta(gv, Gv), \delta(f^2u, Gv), \delta(gv, Ffu)\big) = \\ &= \varphi(\delta(Ffu, t), \delta(Ffu, t), 0, 0, \delta(Ffu, t), \delta(t, Ffu)) \end{aligned}$$

contradicts (φ_2). Hence $Ffu = \{t\} = \{fu\} = \{ffu\}$. Therefore $t = fu = gv$ is a common fixed point of both f, g, F and G.

Similarly, we can obtain this conclusion by using $(3')$ in lieu of (3).

Now, suppose that f, g, F and G have two common fixed points t and t' such that $t' \neq t$. Then inequality (2) gives

$$\varphi(d(ft, gt'), \delta(Ft, Gt'), \delta(ft, Ft), \delta(gt', Gt'), \delta(ft, Gt'), \delta(gt', Ft)) = \\ = \varphi(d(t, t'), d(t, t'), 0, 0, d(t, t'), d(t', t)) \geq 0$$

contradicts (φ_2). Therefore $t' = t$. □

If we let in the above theorem, $F = G$ and $f = g$ then we get the following result.

Corollary 3.1. *Let $(\mathcal{X}, d)$ be a metric space and let $f : \mathcal{X} \to \mathcal{X}$, $F : \mathcal{X} \to B(\mathcal{X})$ be a single and a set-valued map, respectively such that*

(i) f is surjective,
(ii) $\varphi\big(d(fx, fy), \delta(Fx, Fy), \delta(fx, Fx), \delta(fy, Fy), \delta(fx, Fy), \delta(fy, Fx)\big) \geq 0$ for all x, y in $\mathcal{X}$, where $\varphi \in \Phi$. If f and F are subcompatible D-maps, then, f and F have a unique common fixed point $t \in \mathcal{X}$ such that

$$Ft = \{t\} = \{ft\}.$$

Now, if we put $f = g$ then we get the next corollary.

Corollary 3.2. *Let f be a self-map of a metric space $(\mathcal{X}, d)$ and let $F, G : \mathcal{X} \to B(\mathcal{X})$ be two set-valued maps satisfying the conditions*

(i) f is surjective,
(ii) $\varphi(d(fx, fy), \delta(Fx, Gy), \delta(fx, Fx), \delta(fy, Gy), \delta(fx, Gy), \delta(fy, Fx)) \geq 0$ for all x, y in $\mathcal{X}$, where $\varphi \in \Phi$.

If either

(iii) f and F are subcompatible D-maps; f and G are subcompatible, or
(iii)′ f and G are subcompatible D-maps; f and F are subcompatible.

Then, f, F and G have a unique common fixed point $t \in \mathcal{X}$ such that

$$Ft = Gt = \{ft\} = \{t\}.$$

Corollary 3.3. *If in Theorem 3.1 we have instead of (2) the inequality*

$$d^p(fx, gy) \geq \delta^p(Fx, Gy) + \\ + \frac{\alpha\delta^{p-1}(fx, Gy)\delta(gy, Fx) + \beta\delta(fx, Gy)\delta^{p-1}(gy, Fx)}{1 + \gamma\delta^p(fx, Fx) + \delta\delta^p(gy, Gy)}$$

for all x, y in $\mathcal{X}$, where $\alpha, \beta > 0$, $\gamma, \delta \geq 0$ and p is an integer such that $p \geq 2$. Then, f, g, F and G have a unique common fixed point $t \in \mathcal{X}$.

Proof. Take a function φ as in Example 2.1, then

$$\varphi\big(d(fx,gy),\delta(Fx,Gy),\delta(fx,Fx),\delta(gy,Gy),\delta(fx,Gy),\delta(gy,Fx)\big) = \\ = d^p(fx,gy)-\delta^p(Fx,Gy)- \\ -\frac{\alpha\delta^{p-1}(fx,Gy)\delta(gy,Fx)+\beta\delta(fx,Gy)\delta^{p-1}(gy,Fx)}{1+\gamma\delta^p(fx,Fx)+\delta\delta^p(gy,Gy)}\geq 0,$$

which implies that

$$d^p(fx,gy)\geq \delta^p(Fx,Gy)+ \\ \frac{\alpha\delta^{p-1}(fx,Gy)\delta(gy,Fx)+\beta\delta(fx,Gy)\delta^{p-1}(gy,Fx)}{1+\gamma\delta^p(fx,Fx)+\delta\delta^p(gy,Gy)}$$

for all x,y in $\mathcal{X}$, where $\alpha,\beta>0$, $\gamma,\delta\geq 0$ and p is an integer such that $p\geq 2$. Conclude by using Theorem 3.1. □

Remark. As in Corollary 3.3 we can get other corollaries using Examples 2.2-2.4.

Corollary 3.4. *Let f,g,F and G be maps satisfying* (1)*,* (3) *and* (3′) *of Theorem 3.1. Suppose that for all $x,y\in\mathcal{X}$ we have the inequality*

$$d^p(fx,gy)\geq \delta^p(Fx,Gy)+\delta^{p-1}(fx,Gy)\delta(gy,Fx)+\delta(fx,Gy)\delta^{p-1}(gy,Fx)$$

where p is an integer such that $p\geq 2$. Then, f,g,F and G have a unique common fixed point $t\in\mathcal{X}$.

Proof. Take a function φ as in Example 2.1 with $\alpha=\beta=1$ and $\gamma=\delta=0$. Observe by condition (2)

$$\varphi\big(d(fx,gy),\delta(Fx,Gy),\delta(fx,Fx),\delta(gy,Gy),\delta(fx,Gy),\delta(gy,Fx)\big) = \\ = d^p(fx,gy)-\delta^p(Fx,Gy)-\delta^{p-1}(fx,Gy)\delta(gy,Fx)- \\ \delta(fx,Gy)\delta^{p-1}(gy,Fx)\geq 0.$$

Conclude by using Theorem 3.1. □

Remark. We can get other results if we let in the corollaries $f=g$ and also $f=g$ and $F=G$.

Now, we give a generalization of Theorem 3.1.

Theorem 3.2. *Let f,g be self-maps of a metric space $(\mathcal{X},d)$ and $F_n:\mathcal{X}\to B(\mathcal{X})$, $n\in\mathbb{N}^*=\{1,2,\dots\}$ be set-valued maps with*

(*i*) *f and g are surjective,*

(*ii*) *the inequality*

$$\varphi(d(fx,gy),\delta(F_nx,F_{n+1}y),\delta(fx,F_nx),\delta(gy,F_{n+1}y), \\ \delta(fx,F_{n+1}y),\delta(gy,F_nx))\geq 0$$

holds for all x,y in $\mathcal{X}$, where $\varphi\in\Phi$. If either (*iii*) *f and $\{F_n\}_{n\in\mathbb{N}^*}$ are subcompatible D-maps; g and $\{F_{n+1}\}_{n\in\mathbb{N}^*}$ are subcompatible, or*

(*iv*) *g and $\{F_{n+1}\}_{n\in\mathbb{N}^*}$ are subcompatible D-maps; f and $\{F_n\}_{n\in\mathbb{N}^*}$ are subcompatible.*

Then, there is a unique common fixed point $t \in \mathcal{X}$ such that

$$F_n t = \{t\} = \{ft\} = \{gt\}, \qquad n \in \mathbb{N}^*.$$

Proof. Letting $n = 1$, we get the hypotheses of Theorem 3.1 for the maps f, g, F_1 and F_2 with the unique common fixed point t. Now, t is a unique common fixed point of f, g, F_1 and of f, g, F_2. Otherwise, if t' is a second distinct fixed point of f, g and F_1, then by inequality (ii), we get

$$\varphi\big(d(ft', gt), \delta(F_1t', F_2t), \delta(ft', F_1t'), \delta(gt, F_2t), \delta(ft', F_2t), \delta(gt, F_1t')\big) = \varphi\big(d(t', t), d(t', t), 0, 0, d(t', t), d(t, t')\big) \geq 0$$

which contradicts (φ_2) hence $t' = t$.

By the same method, we prove that t is the unique common fixed point of the maps f, g and F_2.

Now, by letting $n = 2$, we get the hypotheses of Theorem 3.1 for the maps f, g, F_2 and F_3 and consequently they have a unique common fixed point t'. Analogously, t' is the unique common fixed point of f, g, F_2 and of f, g, F_3. Thus $t' = t$. Continuing in this way, we clearly see that t is the required point. □

References

[1] A. Djoudi, *General fixed point theorems for weakly compatible maps*, Demonstratio Math, **38** (2005), No. **1**, 197-205.

[2] A. Djoudi, R. Khemis, *Fixed points for set and single valued maps without continuity*, Demonstratio Math, **38** (2005), No. **3**, 739-751.

[3] B. Fisher, *Common fixed points of mappings and set-valued mappings*, Rostock. Math. Kolloq, No. **18** (1981), 69-77.

[4] B. Fisher, S. Sessa, *Two common fixed point theorems for weakly commuting mappings*, Period. Math. Hungar, **20** (1989), No. **3**, 207-218.

[5] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. & Math. Sci., **9** (1986), No. **4**, 771-779.

[6] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci., **4** (1996), No. **2**, 199-215.

[7] G. Jungck, P.P. Murthy, Y.J. Cho, *Compatible mappings of type* (A) *and common fixed points*, Math. Japon., **38** (1993), No. **2**, 381-390.

[8] G. Jungck, B.E. Rhoades, *Fixed points for set valued functions without continuity*, Indian J. Pure Appl. Math., **29** (1998), No. **3**, 227-238.

[9] H.K. Pathak, Y.J. Cho, S.M. Kang, B. Madharia, *Compatible mappings of type (C) and common fixed point theorems of Greguš type*, Demonstratio Math., **31** (1998), No. **3**, 499-518.

[10] H.K. Pathak, M.S. Khan, *Compatible mappings of type (B) and common fixed point theorems of Greguš type*, Czechoslovak Math. J., **45(120)** (1995), No. **4**, 685-698.

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