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General Convexity, General Concavity, Fixed Points, Geometry, and Min-Max Points

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Abstract. This paper continues the study of general convexity and general concavity which are described in an abstract form on arbitrary sets. The main feature is the systematic use of a very versatile technique introduced in this paper via ATM-maps and MTM-maps. In this sense we give simplest applications of ATM-maps and MTM-maps to fixed point theory, geometry, variational inequalities, and minimax theory.

1. General convex topological spaces

Fundamental elements of general convexity. In this section we shall consider general convexity introduced in 1997 by Tasković [42] which is described in an abstract form on sets and topological spaces. Also, we formulate a fixed point theorem for general nonexpansive mappings in the general convex topological spaces.

In this sense, let X be a nonempty set or topological space and let $A : X \times X \to \mathbb{R}_+^0 := [0, +\infty)$ be a function. In this part we considered a topological space X (or a nonempty set X) with a **general convex structure** denoted by $G(x, y, \lambda)$ iff there exist a mapping $G : X \times X \times I \to X$ (for the closed unit interval $I := [0, 1]$) and a bisection function $g : (\mathbb{R}_+^0)^2 \to \mathbb{R}_+^0$ satisfying the following inequality

$$\text{(S)} \qquad A\Big(z, G(x, y, \lambda)\Big) \leq \max\Big\{A(z, x), A(z, y), g\Big(A(z, x), A(z, y)\Big)\Big\}$$

for all $x, y, z \in X$ and arbitrary $\lambda \in [0, 1]$. Similarly, a topological space (or a nonempty set) X is with a **general affine structure** iff there exists a mapping $G : X \times X \times I \to X$ such that (S) for all $x, y, z \in X$ and arbitrary $\lambda \in \mathbb{R}$.

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The topological space X with a general convex structure is called a **general convex topological space**. A subset K of X is **general convex** iff $G(x, y, \lambda) \in K$ for all $x, y \in K$ and arbitrary $\lambda \in [0, 1]$.

On the other hand, a nonempty set X with a general convex structure is called **general convex space**. A subset K of X is **general convex** iff $G(x, y, \lambda) \in K$ for all $x, y \in K$ and arbitrary $\lambda \in [0, 1]$.

There are many examples of general convex (topological) spaces. We give some examples here. Clearly a Banach space, or any convex subset of it, is a general convex metric space with $G(x, y, \lambda) := \lambda x + (1 - \lambda)y$. Also, more generally, if X is a linear space with a translation invariant matric ρ (de facto, for $A(x, y) := \rho(x, y)$ and for $g(s, t) := \lambda s + (1 - \lambda)t$) satisfying

$$\rho\Big[0, \lambda x + (1 - \lambda)y\Big] \leq \max\Big\{\rho[0, x], \rho[0, y], \lambda\rho[0, x] + (1 - \lambda)\rho[0, y]\Big\}$$

for all $x, y \in X$ and arbitrary $\lambda \in [0, 1]$, then X is a general convex metric space. If (X, ρ) is a metric space and if there exist a mapping $G : X \times X \times I \to X$ and a mapping $g : (\mathbb{R}^0_+)^2 \to \mathbb{R}^0_+$ such that

$$\rho\Big[z, G(x, y, \lambda)\Big] \leq \max\Big\{\rho[z, x], \rho[z, y], g(\rho[z, x], \rho[z, y])\Big\}$$

for all $x, y, z \in X$ and arbitrary $\lambda \in [0, 1]$, then X is a general convex metric space, i.e., (X, A) is an example of a general convex topological space for $A(x, y) := \rho[x, y]$ with topology of metric space.

We notice, T a k a h a s h i introduced in 1970 the notion of metric space (X, ρ) with a convex structure such that there exists a mapping $W : X \times X \times I \to X$ which satisfies the following inequality

$$\rho\Big[z, W(x, y, \lambda)\Big] \leq \lambda\rho[z, x] + (1 - \lambda)\rho[z, y]$$

for all $x, y, z \in X$ and arbitrary $\lambda \in [0, 1]$. This is also an example of general convex (topological) space.

Obviously, the class of our general convex topological spaces includes as a paradigmatic example the preceding class of convex metric spaces of T a k a h a s h i [40]. There are many other examples, but we consider these examples as paradigmatic.

We notice, the topological space X with a general affine structure is called a **general affine topological space**. A subset M of X is **general affine** (or *general upper affine*) if $G(x, y, \lambda) \in M$ for all $x, y \in M$ and arbitrary $\lambda \in \mathbb{R}$.

Similarly, the nonempty set X with a general affine structure is called a **general affine space**. A subset M of X is **general affine** (or *general upper affine*) if $G(x, y, \lambda) \in M$ for all $x, y \in M$ and arbitrary $\lambda \in \mathbb{R}$.

Theorem 1. *If X is a general convex topological space and if $\{K_\gamma\}_{\gamma \in \mathfrak{G}}$ is any family of general convex (general affine) sets, then*

$$K := \bigcap\nolimits_{\gamma \in \mathfrak{G}} K_\gamma$$

is a general convex (general affine) subset of X. If in addition $\{K_\gamma\}_{\gamma\in\mathfrak{G}}$ is a chain (meaning for $\alpha,\beta\in\mathfrak{G}$, either $K_\alpha\subset K_\beta$ or $K_\beta\subset K_\alpha$), then $M:=\cup_{\gamma\in\mathfrak{G}}K_\gamma$ is general convex (general affine).

Proof. For $x,y\in K$ we obtain $x,y\in K_\gamma$ for every $\gamma\in\mathfrak{G}$. Thus $G(x,y,\lambda)\in K_\gamma$ for every $\gamma\in\mathfrak{G}$, because $\{K_\gamma\}_{\gamma\in\mathfrak{G}}$ is a general convex family. This means that $G(x,y,\lambda)\in K$ for all $x,y\in K$ and arbitrary $\lambda\in[0,1]$, i.e., K is a general convex subset of X. Similarly we have the proof for other part of statement.

In the theory of metric spaces (also, for general convex topological space), it is extremely convenient to use a geometrical language inspired by classical geometry.

Given a general convex topological space (or only nonempty set) X, with the function A, a point $a\in X$, and a real number $r>0$, the **open ball** (respective, **closed ball**, **sphere**) of center a and radius r is the set

$$K(a,r):=\big\{x\in X \mid A(a,x)<r\big\}$$

(respective, $B(a,r):=\big\{x\in X\mid A(a,x)\le r\big\}$, $S(a,r):=\big\{x\in X\mid A(a,x)=r\big\}$). Open and closed balls of center a always contain the point a, but a sphere of center a may be empty. □

We will, in further, denote by $\mathfrak{G}(P)$ the set of all upper bisection function $g:P^2\to P$ which are increasing satisfying $g(t,t)\le t$ for every $t\in P$.

Theorem 2. *Let X be a nonempty set with a general convex structure and with a bisection function $g\in\mathfrak{G}(\mathbb{R}^0_+)$. Then open and closed balls in X are general convex subsets of X.*

Proof. For $x,y\in B(a,r)$ and for arbitrary $\lambda\in[0,1]$, there exists a general convex structure $G(x,y,\lambda)\in X$. Since X is a nonempty set with general convex structure, we obtain

$$A\big(a,G(x,y,\lambda)\big)\le\max\big\{A(a,x),A(a,y),g\big(A(a,x),A(a,y)\big)\big\}\le\cdots$$
$$\cdots\le\max\Big\{A(a,x),A(a,y),\max\big\{A(a,x),A(a,y)\big\}\Big\}\le\max\Big\{r,r,\max\big\{r,r\big\}\Big\}\le r.$$

Thus we have $G(x,y,\lambda)\in B(a,r)$ for arbitrary $\lambda\in[0,1]$ and for all $x,y\in B(a,r)$. Similarly, $K(a,r)$ is general convex subsets of X. The proof is complete. □

Theorem 3. *Let X be a nonempty set with a general convex structure and with a bisection function $g\in\mathfrak{G}(\mathbb{R}^0_+)$. If $A:X\times X\to\mathbb{R}^0_+$ is a symmetric space function satisfying the inequality $A(a,b)\le\max\{A(a,c),A(c,b)\}$ for all $a,b,c\in X$ and if $A(a,a)=0$, then*

$$A(x,y)=\max\Big\{A(x,G(x,y,\lambda)),A(G(x,y,\lambda),y)\Big\}\tag{J}$$

for all $x,y\in X$ and arbitrary $\lambda\in[0,1]$. If X is a nonempty set with a general affine structure, then equality (J) holds for all $x,y\in X$ and for arbitrary $\lambda\in\mathbb{R}$.

Proof. Since X is a nonempty set with the general convex structure for the functions A and G, we obtain

$$A(x,y) \le \max\Big\{A(x, G(x,y,\lambda)), A(G(x,y,\lambda), y)\Big\} \le \cdots$$
$$\cdots \le \max\Big\{A(x,x), A(x,y), A(x,y), A(y,y), \max\{A(x,x), A(y,y), A(x,y), A(x,y)\}\Big\} =$$
$$= A(x,y)$$

for all $x, y \in X$ and for arbitrary $\lambda \in [0,1]$. Thus (J) holds. Similarly in case when X is with a general affine structure. The proof is complete. □

Further facts. In connection with the preceding facts, we notice that the preceding notation of general convex structure can be improved in the following sense.

Let X be a nonempty set, let $P := (P, \preccurlyeq)$ be a partially ordered set with order $\preccurlyeq$, and let $A : X \times X \to P$ be a function. A mapping $G : X \times X \times I \to X$ is said to be **ordered general convex structure** on X if there exists a bisection function $g : P^2 \to P$ such that

$$\text{(1)} \qquad A\Big(z, G(x,y,\lambda)\Big) \preccurlyeq \sup\Big\{A(z,x), A(z,y), g\big(A(z,x), A(z,y)\big)\Big\}$$

for all $x, y, z \in X$ and for all arbitrary $\lambda \in [0,1]$. A nonempty subset K of X is said to **order general convex** if $G(x,y,\lambda) \in K$ for all $x, y \in K$ and for arbitrary $\lambda \in [0,1]$.

Similarly to the other facts we also have the definitions of an order general affine structure and of an order general affine set.

The analogous with the preceding statements and facts we directly have the following results.

Theorem 4. *If X is a nonempty set with an order general convex (order general affine) structure and if $\{K_\gamma\}_{\gamma\in\mathfrak{G}}$ is any family order general convex (order general affine) sets, then*

$$K := \bigcap\nolimits_{\gamma\in\mathfrak{G}} K_\gamma$$

is an order general convex (order general affine) subset of X.

The proof of this statement is totally analogous to the proof of Theorem 1. Further, the **open ball** (respective, **closed ball**, **sphere**) of a center a and radius r is the set

$$K(a,r) := \big\{x \in X \mid A(a,x) \prec r\big\}$$

(respective, $B(a,r) := \big\{x \in X \mid A(a,x) \preccurlyeq r\big\}$, $S(a,r) := \big\{x \in X \mid A(a,x) = r\big\}$). Open and closed balls of center a always contain the point a, but a sphere of center a may be empty.

Theorem 5. *Let X be a nonempty set with an order general convex structure and with a bisection function $g \in \mathfrak{G}(P)$. Then open and closed balls in X are order general convex subsets of X.*

A brief proof of this statement based on the preceding proof of Theorem 2 may be found in Tasković [44].

Theorem 6. *Let X be a nonempty set with an order general convex structure. If $A : X \times X \to P$ is a symmetric function which satisfying the following inequalities*

$$\sup\Big\{A(a,a), A(b,b)\Big\} \preccurlyeq A(a,b) \preccurlyeq \sup\Big\{A(a,c), A(c,b)\Big\}$$

for all $a, b, c \in X$, then $A(x,y) = \sup\big\{A\big(x, G(x,y,\lambda)\big), A\big(G(x,y,\lambda), y\big)\big\}$ holds for all $x, y \in X$ and for arbitrary $\lambda \in [0,1]$. (This is analogous to the proof of Theorem 3).

A nonempty subset K of X is said to **ordered general convex** if $G(x,y,\lambda) \in K$ for all $x, y \in K$ and for arbitrary $\lambda \in [0,1]$. Similarly, K is **ordered general affine** if $G(x,y,\lambda) \in K$ for all $x, y \in K$ and for every $\lambda \in \mathbb{R}$.

In this sense, a nonempty set X with an ordered general convex structure is a called **ordered general convex space**.

We say that G is **ordered general affine structure** if (1) holds for all $x, y, z \in X$ and for every $\lambda \in \mathbb{R}$. A nonempty set X with an ordered general affine structure is called **ordered general affine space**. For the further facts see: T a s k o v i ć [44].

A theorem of fixed point. Takahashi [40] introduced a notion of convex metric spaces and studied the fixed point theory for nonexpansive mappings in such setting.

For the convex metric spaces Kirk [27] and Goebel and Kirk [19] use the term "hyperbolic type space". They studied the iteration processes for nonexpansive mappings in the abstract framework and generalize and unify some known results.

In this part we give a statement of fixed point in general convex topological spaces. In this sense let X be a general convex topological space for a continuous function $A : X \times X \to \mathbb{R}^0_+$.

For $S \subset X$ we denote the **diameter** of S by $\delta(S) := \sup\{A(x,y) \mid x, y \in S\}$. A point $x \in S$ is a **diametral point** of S provided

$$\sup\Big\{A(x,y) : y \in S\Big\} = \delta(S).$$

Lemma 1. *Let M be a nonempty compact subset of general convex topological space X with a bisection function $g \in \mathfrak{G}(\mathbb{R}^0_+)$, and let K be the least closed general convex set containing M. If the diameter $\delta(M)$ is positive, then there exists an element $u \in K$ such that*

$$\sup\Big\{A(x,u) : x \in M\Big\} < \delta(M).$$

A brief proof of this statement based on compactness may be found in Tasković [42], and [43]. Also, for these facts see Tasković [44].

We notice that this statement gives us the following definition. A general convex topological space is said to have **normal structure** if for each closed bounded general convex subset S of X which contains at least two points, there exists $x \in S$ which is not a diametral point of S.

Compact convex sets, possesses normal structure obtained by Brodskij and Milman [2]. It is obvious that a compact convex metric space has normal structure obtained by Takahashi [40]. Every bounded closed general convex subset of uniformly convex Banach space has normal structure, too.

A general convex topological space X will be said to have **Šmulian property** if every bounded decreasing net of nonempty closed general convex subsets of X has a nonempty intersection.

Šmulian [39] proved that a necessary and sufficient condition that a Banach space X is reflexive is that Šmulian property holds.

From the Šmulian property and Theorem 1 and 2, we obtain the following fundamental fact over the method of Kirk [28].

Lemma 2. *Let F be a subset of a general convex topological space X with the Šmulian property and with a bisection function $g \in \mathfrak{G}(\mathbb{R}_+^0)$. Then the* **centre** *of F, i.e.,*

$$F_c := \Big\{ x \in F \mid R_x(F) := \sup \big\{ A(x,y) \mid y \in F \big\} = \inf_{x \in F} R_x(F) := R(F) \Big\}$$

is a nonempty, closed and general convex set.

Proof. Let $F(x,n) := \{y \in F \mid A(x,y) \leq R(F) + 1/n\}$ for every $n \in \mathbb{N}$ and $x \in X$. It is easily seen that the sets $C_n = \cap_{x \in F} F(x,n)$ form a decreasing sequence on nonempty closed general convex sets, and hence, from the Šmulian property and Theorem 1 and 2, $F_c = \cap_{n \in \mathbb{N}} C_n$ is a nonempty, closed and general convex set. The proof is complete. □

The existence of fixed points of a nonexpansive mapping on Hilbert space was established in 1965 by Browder, and Browder [3], Göhde [18] and Kirk [28] for such mappings on uniformly convex Banach spaces, independently of each other.

On the other hand, a general fixed point result for isometries was obtained by Brodskij and Milman [2]. But, Kirk [28] first proved that if K is a nonempty weakly compact convex subset of a Banach space and if K has a normal structure, then every nonexpansive self-map T of K has a fixed point.

Later, Takahashi [40] introduced a notion of convex metric spaces and studied the fixed point statements for nonexpansive mappings in such setting. For all the preceding results the concept of normal structure, due to Brodskij-Milman [2], play a key role. For further facts of this see: Talman [41], and Machado [32].

Let X be a topological space (or only a nonempty set), K be a subset of X, and $A : X \times X \to \mathbb{R}_+^0$ be a continuous function. A mapping T of K into K is said to be **general nonexpansive** if the following inequality holds in the form as:

$$\text{(D)} \qquad A(Tx,Ty) \leq \sup\Big\{A(x,y) : x,y \in E\Big\}$$

for all $x, y \in E$ and for every closed general convex subset E of K with at least two points such that $T(E) \subset E$. In this section we will prove the following result for general nonexpansive mappings in general convex topological spaces.

Theorem 7. *Let X be a general convex topological space with the Šmulian property, and with a bisection function $g \in \mathfrak{G}(\mathbb{R}^0_+)$, and K be a nonempty bounded closed general convex subset of X with normal structure. If T is a general nonexpansive mapping of K into itself, then T has a fixed point in K.*

Some remarks. Let X be a Banach space. A mapping T of a subset K of X into K is called **half-diametral contraction** on K if the following inequality holds in the form as:

$$\text{(C)} \qquad \|Tx - Ty\| \leq \sup\Big\{\|x-y\| : y \in E\Big\}$$

for all $x, y \in E$ and for every closed general convex subset E of K with at least two points such that $T(E) \subset E$.

In a former paper Tasković [42] investigated a nonempty bounded closed and general convex subset K of a reflexive Banach space X, and showed that if K has normal structure and if T is a half-diametral contractive mapping of K into itself, then T has a fixed point in K.

We notice, by interchanging x and y in (C) we see that a nature condition of fixed point appears above which may be written

$$\text{(G)} \qquad \|Tx - Ty\| \leq \max\Big\{\sup\Big(\|x-z\| : z \in K\Big), \sup\Big(\|y-z\| : z \in K\Big)\Big\}$$

for all $x, y \in K$. But, the inequality (C) can not be written in some form of type (G). Also, we notice that condition (C) is not equivalent to condition (G).

More than that, we notice that the fact: *that (D) holds for every closed convex subset E of K with at least two points such that $T(E) \subset E$, can not to omit in Theorem 7.*

Proof of Theorem 7. Let $\mathfrak{F}$ denote the collection of all nonempty closed and general convex subsets of K, each of which is mapped into itself by T. By Šmulian property and Zorn's lemma $\mathfrak{F}$ has a minimal element which we denote by F. We show that F consists of a single point. Let $x \in F_c$. Then

$$A(Tx,Ty) \leq \sup\big\{A(x,y) : x,y \in F\big\} = R_x(F) \quad \text{for every} \quad y \in F,$$

and hence $T(F)$ is contained in the spherical ball $B(Tx, R(F))$ centered at Tx with radius $R(F)$. Since $T(F \cap B) \subset F \cap B$, the minimality of F implies $F \subset B$. Hence $R_{T(x)}(F) \leq R(F)$. Since $R(F) \leq R_x(F)$ for all $x \in F$, thus we obtain $R_{T(x)}(F) = R(F)$. Hence $Tx \in F_c$ and F_c is mapped into itself by T. By Lemma 2 we have $F_c \in \mathfrak{F}$. If $z, w \in F_c$, then we obtain

$A(z, w) \leq R_z(F) = R(F)$. Hence, by normal structure (i.e., by Lemma 1), $\delta(F_c) \leq R(F) < \delta(F)$. Since this contradicts the minimality of F, $\delta(F) = 0$ and F consists of a single point. The proof is complete.

Further, let X be a linear space with general convex structure $G : X \times X \times I \to X$. If $\lambda_i \in \mathbb{R}$ $(i = 1, \ldots, n)$, $\lambda \in I$, and $\lambda_1 + \cdots + \lambda_n = 1$, then

$$x = \lambda_1 G(x_1, x_1, \lambda) + \cdots + \lambda_n G(x_n, x_n, \lambda)$$

is called **general affine combination** of $x_1, \ldots, x_n$, the latter being elements of a linear space X. If in addition $\lambda_i \geqslant 0$ $(i = 1, \ldots, n)$, then x is called a **general convex combination** of $x_1, \ldots, x_n \in X$. We have the following characterization of general convex sets.

Theorem 8. *A set K of a linear space X is general convex (general affine) if and only if every general convex (general affine) combination of points of K lies in K.*

Proof. Since a set that contains all general convex combinations of its points is obviously general convex, we only need to consider a general convex set K and show that it contains any general convex combination of its points. Our proof is by induction of the number of points of K occurring in a general convex combination, the conclusion following from the definition for $n = 2$. Assuming the result true for any general convex combination with n or fewer points, we consider one with $n + 1$ points,

$$x = \lambda_1 G(x_1, x_1, \lambda) + \cdots + \lambda_{n+1} G(x_{n+1}, x_{n+1}, \lambda).$$

Not all the λ_i's can be as great as one, so we relabel if necessary so that $\lambda_{n+1} < 1$. Then we obtain the following equalities in the following form as

$$x = (1 - \lambda_{n+1}) \sum_{k=1}^{n} \frac{\lambda_k}{1 - \lambda_{n+1}} G(x_k, x_k, \lambda) + \lambda_{n+1} G(x_{n+1}, x_{n+1}, \lambda) =$$
$$= (1 - \lambda_{n+1}) y + \lambda_{n+1} G(x_{n+1}, x_{n+1}, \lambda).$$

Now $y \in K$ by assumption, and thus so is x, being a general convex combination of two points of K. The proof in the general affine case follows exactly the same pattern.

We call the intersection of all general convex sets containing a given set K is the **general convex hull** of K, denoted by $G(\operatorname{conv}(K))$, $g.\operatorname{conv}(K)$, or $GH(K)$. Similarly, the intersection of all general affine sets containing K is called **general affine hull** of K. By Theorem 1, the general convex hull is a general convex set; the general affine hull is a general affine set.

This means that the terminology "general convex hull" is reasonable. It is, however, not true that the general convex hull of a closed set is necessarily closed. For this we can consider the set of the form as $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = (1 + x^4)^{-1}\}$. □

Theorem 9. *For any K of a linear space X, the general convex (general affine) hull of K consists precisely of all general convex (general affine) combinations of elements of K.*

Proof. We prove the statement for general convex hulls, leaving the one for general affine hulls to the reader. Let $GH(K)$ denote the general convex hull of K and $GC(K)$ the set of general convex combinations of elements of K. Now $K \subset GH(K)$, and since $GH(K)$ is general convex, Theorem 8 shows that $GC(K) \subset GH(K)$. Conversely, if $x = \lambda_1 G(x_1, x_1, \lambda) + \ldots + \lambda_n G(x_n, x_n, \lambda)$ and $y = \eta_1 G(y_1, y_1, \lambda) + \ldots + \eta_m G(y_m, y_m, \lambda)$ are two elements of $GC(K)$, then

$$z = \lambda x + (1-\lambda)y = \sum_{k=1}^{n} \lambda\lambda_k G(x_k, x_k, \lambda) + \sum_{j=1}^{m} (1-\lambda)\eta_j G(y_j, y_j, \lambda)$$

is another element of $GC(K)$ since $\sum_{k=1}^{n} \lambda\lambda_k + \sum_{j=1}^{m}(1-\lambda)\eta_j = \lambda + (1-\lambda) = 1$.

Thus $GC(K)$ is a convex set, i.e., $GC(K)$ is a general convex set with $G(x, y, \lambda) := \lambda x + (1-\lambda)y$, containing K. Therefore $GH(K) \subset GC(K)$. The proof is complete. □

We notice that this statement can be improved if $X = \mathbb{R}^n$. In this case, $GH(K)$ consists of all general convex combinations of $n+1$ or fewer elements of K.

Also, from the preceding proof of Theorem 9, directly, we have that the set of all general convex combinations is a general convex set.

Before proving a slightly more general version of this statement, let us introduce the concept of **dimension** for a general convex set. First if K is general affine, we define its dimension to be that of the subspace of which it is a translate. More generally, if K is general convex, its **dimension** is the dimension of the general affine hull of K. We are now ready to prove an extension on general convex sets of Carathéodory's theorem for convex sets.

Theorem 10. *If a set K of a linear space X and its general convex hull $GH(K)$ has dimension m, then for each $x \in GH(K)$, there exist $m+1$ points $x_0, x_1, \ldots, x_m$ of K such that x is a general convex combination of these points.*

Proof. Let $x \in GH(K)$. Then $x = \sum_{i=0}^{n} \lambda_i G(x_i, x_i, \lambda)$ where $x_i \in K$, $\lambda_i > 0$, and $\sum_{i=0}^{n} \lambda_i = 1$. Now suppose $n+1$, the number of terms in the general convex combination, is greater than $m+1$ and let $B = \{G(x_0, x_0, \lambda), \ldots, G(x_n, x_n, \lambda)\}$. Then

$$\dim(\text{g. affine hull}(B)) \leq \dim(\text{g. affine hull}(K)) = m \leq n-1$$

and therefore $\{G(x_1, x_1, \lambda) - G(x_0, x_0, \lambda), \ldots, G(x_n, x_n, \lambda) - G(x_0, x_0, \lambda)\}$ is a linearly dependent set (from: Some remarks and examples - 2). Thus there are constants $\alpha_1, \ldots, \alpha_n$, not all 0, such that $\sum_{i=1}^{n} \alpha_i(G(x_i, x_i, \lambda) - G(x_0, x_0, \lambda)) = 0$. Let $\alpha_0 = -\sum_{i=1}^{n} \alpha_i = 0$. Then $\sum_{i=0}^{n} \alpha_i G(x_i, x_i, \lambda) = 0$ and $\sum_{i=0}^{n} \alpha_i = 0$. Since all the λ_i's are positive, we may choose a positive number t so that $\gamma_i = \lambda_i - t\alpha_i \geq 0$ for $i = 0, 1, \ldots, n$ and so that $\gamma_k = 0$ for some k. Then

$$x = \sum_{i=0}^{n} \lambda_i G(x_i, x_i, \lambda) = \sum_{i=0}^{n} (\gamma_i + t\alpha_i) G(x_i, x_i, \lambda) = \sum_{i \neq k} \gamma_i G(x_i, x_i, \lambda)$$

with $\sum_{i \neq k} \gamma_i = \sum_{i=0}^{n} \gamma_i = \sum_{i=0}^{n} (\lambda_i - t\alpha_i) = \sum_{i=0}^{n} \lambda_i = 1$. Thus x is a general convex combination of the n points $x_0, \ldots x_{k-1}, x_{k+1}, \ldots, x_n$. Now either $n = m + 1$ and we are argument can be repeated. Eventually we are able to represent x as a general convex combination of $m+1$ or fewer points of K. The proof is complete. □

Some remarks and examples. 1. We notice that each of the following are general convex: 1) Any interval in $\mathbb{R}$. 2) $\{(r, t) \in \mathbb{R}^2$: $|r| + |t| \leq 4$. 3) $\{(r, t) \in \mathbb{R}^2$: $t \leq \log r$, $r > 0\}$. 4) The set of $n \times n$ matrices with nonnegative entries.

2. Let K be a subset of a linear space L. If $K =$ g. affin hull($\{G(x_0, x_0, \lambda), \ldots, G(x_n, x_n, \lambda)\}$) of a finite set, then it is the translate of the subspace g. affin hull($\{0, G(x_1, x_1, \lambda) - G(x_0, x_0, \lambda), \ldots, G(x_n, x_n, \lambda) - G(x_0, x_0, \lambda)\}$). In this case K has dimension n if and only if $\{G(x_1, x_1, \lambda) - G(x_0, x_0, \lambda), \ldots, G(x_n, x_n, \lambda) - G(x_0, x_0, \lambda)\}$ is a linearly independent set.

3. For any sets K and R in linear space L, let $K + R := \{x + y : x \in K,\ y \in R\}$, and let $\alpha K := \{\alpha x : x \in K\}$.If K and R are general convex, then: 1) $K + R$ is general convex, 2) αK is general convex for all $\alpha \in \mathbb{R}$, 3) $K = qK + (1 - q)K$ for $q \in [0, 1]$, $(\alpha_1 + \alpha_2)K = \alpha_1 K + \alpha_2 K$ for $\alpha_1, \alpha_2 \geq 0$.

4. Let $K_1, K_2, \ldots$ be a sequence of general convex sets. If $K_j \subset K_{j+1}$ (for $j \in \mathbb{N}$), then $\cup_{j=1}^{\infty} K_j$ is general convex. Also, $\liminf_{j \to \infty} K_j$ is a general convex set.

5. The simplest general convex sets are those which are general convex hulls of a finite set of points, that is, sets of the form $K = GH(\{x_0, \ldots, x_n\})$. Such a set is called a **general polytope**. If $G(x_1, x_1, \lambda) - G(x_0, x_0, \lambda), \ldots, G(x_n, x_n, \lambda) - G(x_0, x_0, \lambda)$ are linearly independent (so g.dim$(K) = n$), then K is called a **general n-simplex** with vertices $x_0, \ldots, x_n$. A point x in a general n-simplex can be vritten in a unique way as a general convex combination $x = \sum_{i=0}^{n} \lambda_i G(x_i, x_i, \lambda)$ of the vertices. The numbers $\lambda_0, \lambda_1, \ldots, \lambda_n$ are called **general barycentric coordinates** of x.

In the theory of metric spaces (also, for general convex topological spaces), it is extremely convenient to use a geometrical language inspired by classical geometry.

On the other hand, from the notation of general convex combination, we obtain similarly to Theorem 8 the following statement.

Theorem 11. *A set K of a linear space X is order general convex (order general affine) if and only if every general convex (general affine) combination of points of K lies in K.*

This is totally analogous to the former proof of Theorem 8. In connection with this, we notice, from Theorems 8 and 11 as an immediate consequence we obtain the main fact for the preceding notations, i.e., we have that the notation of general convex set is equivalent to the notation of order general convex set in a linear space! We are now in a position to formulate the following statement.

Theorem 12. *Let K be a subset of a linear space X, then the following statements are equivalent*:
(a) *K is a general convex set,*
(b) *K is an order general convex set,*
(c) *every general convex combination of points of K lies in K.*

Proof of Lemma 1. Since M is nonempty and compact, we may find $x_1, x_2 \in M$ such that $A(x_1, x_2) = \delta(M)$. Let $M_0 \subset M$ be maximal so that $M_0 \supset \{x_1, x_2\}$ and $A(x, y) = 0$ or $\delta(M)$ for all $x, y \in M_0$. Since M is compact and we are assuming $\delta(M) > 0$, M_0 must be finite. Let us assume $M_0 = \{x_1, x_2, \ldots, x_n\}$. Since X is a general convex topological space, we can define

$$y_1 = G\left(x_1, x_2, \frac{1}{2}\right), \quad y_2 = G\left(x_3, y_1, \frac{1}{3}\right), \ldots,$$
$$y_{n-2} = G\left(x_{n-1}, y_{n-3}, \frac{1}{3} - 1\right), \quad y_{n-1} = G\left(x_n, y_{n-2}, \frac{1}{n}\right) := u.$$

Since M is a compact set, we can find $y_0 \in M$ such that $A(y_0, u) = \sup\{A(x, u) : x \in M\}$. Now, since X is a general convex topological space, from (S), we obtain the following inequalities

$$A(y_0, u) = A\left(y_0, G\left(x_n, y_{n-2}, \frac{1}{n}\right)\right) \leq$$
$$\leq \max\Big\{A(y_0, x_n), A(y_0, y_{n-2}), g\Big(A(y_0, x_n), A(y_0, y_{n-2})\Big)\Big\} \leq \cdots$$
$$\cdots \leq \max\Big\{A(y_0, x_n), A(y_0, y_{n-2}), \max\Big[A(y_0, x_{n-1}), A(y_0, y_{n-3})\Big]\Big\} \leq \cdots$$
$$\cdots \leq \max\Big\{A(y_0, x_n), A(y_0, x_{n-1}), \ldots, A(y_0, x_1)\Big\} \leq \delta(M).$$

Thus, if $A(y_0, u) = \delta(M)$, then we must have $A(y_0, x_k) = \delta(M) > 0$ for all $k \in \{1, 2, \ldots, n\}$, which means that $y_0 \in M_0$ by definition of M_0. But then we must have $y_0 = x_k$ for some $k = 1, 2, \ldots, n$. This is a contradiction. Therefore, $\sup\{A(x, u) : x \in M\} = A(y_0, u) < \delta(M)$, i.e., $\sup\{A(x, u) : x \in M\} < \delta(M)$ holds. The proof of Lemma 1 is complete.

Let X be a topological space (or only a nonempty set), K be a subset of X, and $A : X \times X \to \mathbb{R}^0_+$ be a continuous function. A mapping T of K into K is said to be **strictly general nonexpansive** if

$$\text{(D)} \qquad A(Tx, Ty) \leq \sup\Big\{A(x, y) : x, y \in K\Big\}$$

for all $x, y \in K$. In this section we will prove the following result for strictly general nonexpansive mappings in general convex topological spaces.

Let X be a Banach space. A mapping T of a subset K of X into itself is called **diametral contraction** on K if

$$\|Tx - Ty\| \leq \sup\Big\{\|x - y\| : x, y \in E\Big\}$$

for all $x, y \in E$ and for every closed general convex subset E of K with at least two points such that $T(E) \subset E$.

Families of strictly general nonexpansive mappings. Fixed-point theorems for families of continuous linear or affine transformations have been obtained by Markoff [33], Kakutani [23], Day [5] and Göhde [18]. Fixed-point theorems for families of nonexpansive mappings established by De Marr [8] (for Banach spaces), F. Browder [3] (for uniformly convex spaces), and T. Lim [31] (for reflexive Banach spaces).

Also, Belluce and Kirk [1] extend a theorem by showing that any finite family of commuting nonexpansive self-mappings of such a set K always has a common fixed point.

In 1980 Tasković investigated a nonempty commutative family $\mathfrak{F}$ of diametral contractive mappings of a nonempty compact convex subset K into itself, and showed that the family $\mathfrak{F}$ has a common fixed point in K.

In this part we extend this result and a result of Tasković [42] for general nonexpansive mappings in general convex topological spaces.

Let X be a compact general convex topological space. A family $\mathfrak{F}$ of general nonexpansive mappings T of X into itself is said to have **invariant property** in X if for any compact general convex subset K of X such that $T(K) \subset K$ for each $T \in \mathfrak{F}$ there exists a compact subset $M \subset K$ such that $T(M) = M$ for each $T \in \mathfrak{F}$.

Theorem 13. *Let X be a compact general convex topological space with a bisection function $g \in \mathfrak{G}(\mathbb{R}^0_+)$. If $\mathfrak{F}$ is a family of strictly general nonexpansive mappings with invariant property in X, then the family $\mathfrak{F}$ has a common fixed point.*

Proof. By using Zorn's lemma, we can find a minimal nonempty general convex compact set $K \subset X$ such that K is an invariant under each $T \in \mathfrak{F}$. If K consists of a single point, then the invariant property implies that T has a fixed point. Also, by hypothesis, there exists a compact subset M of K such that $M = \{T(x) : x \in M\}$ for each $T \in \mathfrak{F}$. If M contains more than one point by Lemma 1 there exists an element u in the least general convex set containing M such that the condition of nondiametral point holds. Let us define

$$K_0 := \cap_{x\in M}\Big\{y \in K : A(x, y) \leq \sup\Big(A(u, x) : x \in M\Big)\Big\},$$

then K_0 is the nonempty closed general convex proper subset of K invariant under each $T \in \mathfrak{F}$. This is a contradiction to the minimality of K. The proof is complete. $\square$

Strictly general convex spaces. Let X be a nonempty set, let $A : X \times X \to \mathbb{R}^0_+$ a continuous function, and let $I := [0,1]$ be the closed unit interval. Let $G : X \times X \times I \to X$ be a general convex structure on a topological space (or only on a nonempty set) X. We say that G is a **strict general convex structure** if it has the property that whenever $w \in X$ and there is $(x, y, \lambda) \in X \times X \times I$ and there is $g : (\mathbb{R}^0_+)^2 \to \mathbb{R}^0_+$ such that

$$A(z,w) \leq \max\Big\{A(z,x), A(z,y), g\big(A(z,x), A(z,y)\big)\Big\} \quad \text{for every} \quad z \in X,$$

then $w = G(x, y, \lambda)$. If G is a strict general convex structure on the topological space (or on the nonempty set) X we call X is a **strict general convex topological space** (or a **strict general convex space**).

We give a preliminary example here. For example, the plane equipped with the norm in the following form as $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$ is strictly general convex space in our sense, but not in the former sense.

We say that G is a **strict general affine structure** if it has the property that whenever $w \in X$ and there is $(x, y, \lambda) \in X \times X \times \mathbb{R}$ and there is $g : (\mathbb{R}^0_+)^2 \to \mathbb{R}^0_+$ such that

$$A(z,w) \leq \max\Big\{A(z,x), A(z,y), g\big(A(z,x), A(z,y)\big)\Big\} \quad \text{for every} \quad z \in X,$$

then $w = G(x, y, \lambda)$. If G is a strict general affine structure on the topological space (or on the nonempty set) X, we call X is a **strictly general affine topological space** (or a **strictly general affine space**).

We notice, strict convex structures on a metric space (X, ρ) were introduced by T a k a h a s h i [40], and have been studied by M a c h a d o [32] and T a l m a n [41] as well.

Theorem 14. *Let X be a strict general convex space with a bisection function $g \in \mathfrak{G}(\mathbb{R}^0_+)$. Then the following equality holds in the following form as*

$$G\Big(G(x,y,\lambda), y, \gamma\Big) = G(x,y,\beta) \tag{2}$$

for all $x, y \in X$ and for all $\lambda, \gamma, \beta \in [0,1]$. If X is a strict general affine space with a bisection function $g \in \mathfrak{G}(\mathbb{R}^0_+)$, then (2) holds for all $x, y \in X$ and for all $\lambda, \gamma, \beta \in \mathbb{R}$.

Proof. Let $z \in X$ be an arbitrary point. Then we have the following inequalities in the following adequate form as

$$A\Big(z, G\Big(G(x,y,\lambda), y, \gamma\Big)\Big) \leq \max\Big\{A\Big(z, G(x,y,\lambda)\Big), A(z,y)\Big\} \leq$$
$$\leq \max\Big\{\max\Big[A(z,x), A(z,y)\Big], A(z,y), g\big(A(z,x), A(z,y)\big)\Big\} \leq \max\Big\{A(z,x), A(z,y)\Big\},$$

whence, by strictness of general convex space, we obtain $G(G(x,y,\lambda), y, \gamma) = G(x,y,\beta)$ for all $x, y \in X$ and for all $\lambda, \gamma, \beta \in I$. The proof is complete. $\square$

We notice, it does not appear that even a strict general convex structure is necessarily continuous as a function from $X \times X \times I$ to X. But we have the following fact.

Theorem 15. *Let G be a general convex structure on a topological space X with a bisection function $g \in \mathfrak{G}(\mathbb{R}_+^0)$. If $A(a, b) = 0$ iff $a = b$ for all $a, b \in X$, then G is a continuous function at each point (x, x, λ) of $X \times X \times I$.*

Proof. Let $\{(x_n, y_n, \lambda_n)\}_{n\in\mathbb{N}}$ be a sequence in $X \times X \times I$ which converges to (x, x, λ). But this is immediate, since the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ both converge to x, and (S) yields

$$A\Big(x, G(x_n, y_n, \lambda_n)\Big) \le \max\Big\{A(x, x_n), A(x, y_n), g\big(A(x, x_n), A(x, y_n)\big)\Big\},$$

i.e., $A(x, G(x_n, y_n, \lambda) \le \cdots \le \max\{A(x, x_n), A(x, y_n)\}$ for each $n \in \mathbb{N}$. The proof is complete. □

Theorem 16. *Let G be a strict general convex structure on a compact Hausdorff topological space X with a bisection function $g \in \mathfrak{G}(\mathbb{R}_+^0)$. Then G is a continuous function as a mapping from $X \times X \times I$ to X.*

Proof. Let $\{(x_n, y_n, \lambda_n)\}_{n\in\mathbb{N}}$ be a sequence in $X \times X \times I$ which converges to (x, y, λ), and let w be a limit point of the sequence $\{G(x_n, y_n, \lambda_n)\}_{n\in\mathbb{N}}$. Select a subsequence $\{G(x_{n(k)}, y_{n(k)}, \lambda_{n(k)})\}_{n\in\mathbb{N}}$ which converges to w. Then for any $z \in X$ we have

$$A\Big(z, G(x_{n(k)}, y_{n(k)}, \lambda_{n(k)})\Big) \le$$
$$\le \max\Big\{A(z, x_{n(k)}), A(z, y_{n(k)}), g\big(A(z, x_{n(k)}), A(z, y_{n(k)})\big)\Big\},$$

i.e., $\le \max\{A(z, x_{n(k)}), A(z, y_{n(k)})\}$ for all $k \in \mathbb{N}$. By continuity of the function A, we conclude that $A(z, w) \le \max\Big\{A(z, x), A(z, y)\Big\}$. Strictness now guarantees that $w = G(x, y, \lambda)$. It follows that $G(x, y, \lambda)$ is the only limit point of the sequence $\{G(x_n, y_n, \lambda_n)\}_{n\in\mathbb{N}}$. Since X is compact, the sequence $\{G(x_n, y_n, \lambda_n)\}_{n\in\mathbb{N}}$ must converge to $G(x, y, \lambda)$. The proof is complete. □

Some remarks. In connection with the preceding notations we have the following definition. Let X be a nonempty set, let $A : X \times X \to \mathbb{R}_+^0$ a continuous function, and let $I := [0, 1]$ be the closed unit interval. Let X be a topological space (or a nonempty set) and let

$$I^n = \Big\{(\lambda_1, \dots, \lambda_n) \in I \times \cdots \times I : \lambda_1 + \cdots \lambda_n = 1\Big\}.$$

In further, a **strong convex structure** on X is a (continuous) function $K : X^n \times I^n \to X$ such that the following inequality holds in the form:

$$A\Big(z, K(x_1, \dots, x_n, \lambda_1, \dots, \lambda_n)\Big) \le \max\Big\{A(z, x_1), \dots, A(z, x_n)\Big\}$$

for all $z, x_1, \dots, x_n \in X$ and for all $\lambda_1, \dots, \lambda_n \in I$ with property $\lambda_1 + \cdots + \lambda_n = 1$.

A topological space (or a nonempty set) with a strong general convex structure will be called **strongly general convex topological space** (or **strongly general convex space**).

On the other hand, the strong general convex structure on X is a special form of the ordered general convex structure on X.

Let X be a nonempty set, let $P := (P, \preccurlyeq)$ be a partially ordered set with order $\preccurlyeq$, and let $A : X \times X \to P$ be a function.

A mapping $K : X^n \times I^n \to X$ is said to be n**-ordered general convex structure** on X if there is a function $g : P^n \to P$ such that

$$
\begin{aligned}
&A\Big(z, K(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n)\Big) \preccurlyeq \\
&\preccurlyeq \sup\Big\{A(z, x_1), \ldots, A(z, x_n), g\Big(A(z, x_1), \ldots, A(z, x_n)\Big)\Big\}
\end{aligned} \tag{3}
$$

for all $z, x_1, \ldots, x_n \in X$ and for all $\lambda_1, \ldots, \lambda_n \in I$ with property $\lambda_1 + \cdots + \lambda_n = 1$.

We say that K is n**-ordered general affine structure** if (3) holds for all points $z, x_1, \ldots, x_n \in X$ and for all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with property $\lambda_1 + \cdots + \lambda_n = 1$.

For results on topological space with n-ordered general convex structure see: T a s k o v i ć [42]. We notice, strong convex structures for metric spaces were introduced by T a l m a n [41] and studied the fixed point statements for condensing multifunctions in such setting.

2. ATM-mappings

This paragraph is primarily devoted to illustring the use of the results established in this paper. Let E be a vector space. The set of all subsets of E is denoted by 2^E and g.conv(A) will denote the **general convex hull** of any $A \in 2^E$. A subset $A \subset E$ is called **finitely closed** if its intersection with each finite dimensional flat $L \subset E$ is closed in the Eucliedan topology of L. We notice that a set closed in any topology making E a topological vector space is necessarily finitely closed. Recall that a family $\{A_\lambda : \lambda \in \mathfrak{L}\}$ of sets is said to have the finite intersection property if the intersection of each finite subfamily is not empty. A **flat** (or *linear variety*) in E is a subset of form $L + a$, where L is a linear subspace.

Let E be a vector space and $X \subset E$ an arbitrary subset. A function $G : X \to 2^E$ is called simply an **ATM-mapping** provided

$$\text{g.conv}\{x_1, \ldots, x_r\} \subset \cup_{i=1}^{r} G(x_i)$$

for each finite subset $\{x_1, \ldots, x_r\} \subset X$. In this sense, the essential property of ATM-maps is given in the following statement.

Theorem 17. (ATM-maps principle). *Let E be a vector space, X an arbitrary subset of E, and $G : X \to 2^E$ an ATM-map such that each $G(x)$ is finitely closed. Then the family $\{G(x) : x \in X\}$ of sets has the finite intersection property.*

Proof. (*By contradiction*). We assume that $\cap_{i=1}^{n} G(x_i) =$ the empty set $(= \varnothing)$. Working in the finite-dimensional flat L spanned by $\{x_1, \ldots, x_n\}$, let ρ be the Euclidean metric in L and $C = \text{g.conv}\{x_1, \ldots, x_n\} \subset L$. (We notice that because each $L \cap G(x_i)$ is closed in L, we have $\rho(x, L \cap G(x_i) = 0$ if and only if $x \in L \cap G(x_i)$. Since $\cap_{i=1}^{n} L \cap G(x_i) = \varnothing$ by assumption, the function $\lambda : C \to \mathbb{R}$ given by $c \mapsto \sum_{i=1}^{n} \rho(c, L \cap G(x_i))$ is not zero for any $c \in C$ and we can define a continuous function $f : C \to C$ by setting

$$f(c) = \frac{1}{\lambda(c)} \sum_{i=1}^{n} \rho(c, L \cap G(x_i)) G(x_i, x_i, \lambda). \tag{4}$$

Application of Brouwer's theorem, in the preceding case (4), we obtain that f has a fixed point $c_0 \in C$. Let $I = \{i : \rho(c_0, L \cap G(x_i)) \neq 0\}$. Then the fixed point c_0 cannot belong to $\cup\{G(x_i) : i \in I\}$. However,

$$c_0 = f(c_0) \in \text{g.conv}\{x_i : i \in I\} \subset \{G(x_i) : i \in I\}$$

and, with this contradiction, the proof is complete. (This proof is give by an essential idea of Ky Fan [13]). □

Annotations. In the special case when X is the set of vertices of a simplex in $\mathbb{R}^n$ the preceding statement was discovered by Knaster-Kuratowski-Mazurkiewicz in 1929; and their method of proof was based on a combinatorial result of Sperner in 1928. In the special case when $\text{g.conv}\{x_1, \ldots, x_n\} = \text{conv}\{x_1, \ldots, x_n\}$ we obtain directly well known result as KKM-maps principle by Ky Fan in 1961.

As an immediate consequence of the preceding ATM-maps principle directly we obtain the following result which is an extension of a theorem (for KKM-maps) by Ky Fan [13].

Corrollary 1. *Let E be a topological vector space, $X \subset E$ an arbitrary subset, and $G : C \to 2^E$ an ATM-mapping. If all the sets $G(x)$ are closed in E, and if one is compact, then $\cap\{G(x) : x \in X\}$ is a nonempty set.*

We now observe that the conclusion $\cap\{G(x) : x \in X\} \neq \varnothing$ can be reached in another way, which avoids placing any compactness restriction on the sets $G(x)$. It involves using an auxiliary family of sets and a suitable topology on E.

Corrollary 2. *Let E be a vector space, $X \subset E$ an arbitrary subset, and $G : X \to 2^E$ an ATM-mapping. Assume there is a set-valued map $R : X \to 2^E$ such that $G(x) \subset R(x)$ for each $x \in X$, and for which*

$$\cap\{\mathfrak{R}(x) : x \in X\} = \cap\{G(x) : x \in X\},$$

and if there is some topology on E such that each $\mathfrak{R}(x)$ is a compact set, then $\cap\{G(x) : x \in X\}$ is a nonempty set.

The proof of this consequence is elementary from Theorem 17. The corresponding statement for KKM-maps may be found in Ky Fan [14]. Also see: Tasković [44]. In the next we give the simplest applications of ATM-maps to fixed point theory.

Proposition 1. *Let C be a general convex compact subset of a normed space E, and let $f : C \to E$ be a continuous mapping. Then there exists at least one $y_0 \in C$ such that*

$$\|y_0 - f(y_0)\| = \inf_{x \in C} \|x - f(y_0)\|. \tag{5}$$

Proof. (*Application of ATM-mappings*). Define the mapping $G : C \to 2^E$ by the following equality in the form as

$$G(x) = \{y \in C : \|y - f(y)\| \leq \|x - f(y)\|\};$$

because f is a continuous mapping, the sets $G(x)$ are closed in C, therefore compact. We verife that G is an ATM-map. Let $y \in \text{g.conv}\{x_1, \ldots, x_n\} \subset C$. If $y \notin \cup_{i=1}^n G(x_i)$ then $\|y - f(y)\| > \|x_i - f(y)\|$ for $i = 1, \ldots, n$. This shows that the points x_i all lie in an open ball of radius $\|y - f(y)\|$ centered at $f(y)$, therefore so also does their general convex hull and, in particular, y. Thus, $\|y - f(y)\| > \|y - f(y)\|$, which is a contradiction.

On the other hand, by compactness of the $G(x)$ we find a point y_0 such that $y_0 \in \cap\{G(x) : x \in C\}$ and hence $\|y_0 - f(y_0)\| \leq \|x - f(y_0)\|$ for all $x \in C$. This clearly implies (5) and the proof is complete. □

Proposition 2. *Let C be a general convex compact set in a normed space E, and let $f : C \to E$ be a continuous mapping and such that, for each $c \in C$ $(c \neq f(c))$, the line segment $[c, f(c)]$ contains at least two points of C. Then f has a fixed point.*

Proof. (*Application of Proposition 1*). By the preceding statement of Proposition 1 there exists a $y_0 \in C$ such that the following equality holds in the form as (5). We show that y_0 is a fixed point. The segment $[y_0, f(y_0)]$ must contain a point of C other than y_0, say that $x = \lambda y_0 + (1-\lambda) f(y_0)$ for some $0 < \lambda < 1$. Then $\|y_0 - f(y_0)\| \leq \lambda \|y_0 - f(y_0)\|$ and, thus we must have $\|y_0 - f(y_0)\| = 0$. The proof is complete. □

Annotations. We notice that in special case for convex set in a normed space from Propositions 1 and 2 directly we obtain the corresponding results of Ky Fan [13]. On the other hand, this statements just proved implies that any continuous self-map of a compact general convex set in a normed space has a fixed point. We extend this result to arbitrary locally general convex spaces. The following result is an extension of Schauder-Tychonoff theorem in 1935.

Locally general convex spaces. A linear topological space is **locally general convex** iff every neighborhood of zero contains a general convex neighborhood of zero. Every locally general convex topology on a vector space is determined by some family $\{p_\alpha : \alpha \in \mathfrak{A}\}$ of semi-norms having the property that $p_\alpha(x) = 0$ for all $\alpha \in \mathfrak{A}$ if and only if $x = 0$. In the topology determined by this family of seminorms a set V is **open** if and only if for each $v \in V$ there exists some $\varepsilon > 0$ and finitely many $\alpha_1, \ldots, \alpha_n \in \mathfrak{A}$ such that $\cap_{i=1}^n \{x : p_{\alpha_i}(x - v) < \varepsilon\} \subset V$.

Let A be a subset of a locally general convex space E. The **general convex closure** of A in notation g.Conv(A) is the smallest closed general convex subset containing A.

Theorem 18. *Let C be a compact general convex set in a locally general convex topological space E. Then every continuous mapping $f : C \to C$ has a fixed point.*

Proof. (*Application of ATM-maps*). Let $\{p_i\}_{i\in I}$ be the family of all continuous seminorms in E. For each index $i \in I$ set

$$A_i = \{y \in C : p_i(y - f(y)) = 0\}.$$

In further, a point $y_0 \in C$ is a fixed point for f if and only if $y_0 \in \cap_{i\in I} A_i$. By compactness of C we need to show only that each finite intersection $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}$ is a nonempty set. Given $\{i_1, i_2, \ldots, i_n\}$ define the mapping $G : C \to 2^E$ by

$$G(x) = \left\{ y \in C : \sum_{j=1}^{n} p_{i_j}(y - f(y)) \le \sum_{j=1}^{n} p_{i_j}(x - f(y)) \right\};$$

and, as in the proof of Proposition 1, we verify that G is an ATM-map, so there is a point $y_0 \in C$ such that the following inequality holds in the form as

$$\sum_{j=1}^{n} p_{i_j}(y_0 - f(y_0)) \le \sum_{j=1}^{n} p_{i_j}(x - f(y_0)) \quad \text{for all} \quad x \in C;$$

and we have therefore $p_{i_j}(y_0 - f(y_0)) = 0$ for $0 \le j \le n$, i.e., this means that $y_0 \in A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}$. The proof is complete. □

In what follows we frequently denote the fixed-point set of mapping f by Fix(f). As an immediate consequence of the preceding result we obtain the following extension of well-known Markoff-Kakutani theorem by Markoff in 1936.

Theorem 19. *Let C be a compact general convex set in a locally general convex linear space E, and let $\mathfrak{D}$ be a commuting family of continuous affine maps of C into itself. Then $\mathfrak{D}$ has a common fixed point.*

Proof. (*Application of Theorem 18*). By Theorem 18 the set Fix(f) is nonempty for each $f \in \mathfrak{D}$. Moreover, the set Fix(f) is compact, being closed in the compact set C, and Fix(f) is general convex because f is an affine mapping. We must prove that $\cap\{\text{Fix}(f) : f \in \mathfrak{D}\}$ is nonempty. Because each set Fix(f) is compact, it is sufficient to show that each finite intersection Fix$(f_1, \ldots, f_n) \equiv \cap_{i=1}^{n} \text{Fix}(f_i)$ is nonempty. We proceed by induction on the number n of f_i, the result being true for $n = 1$. Assume that Fix$(f_1, \ldots, f_i)$

is nonempty whenever $i < n$, and consider any n members $f_1, \dots, f_n \in \mathfrak{D}$. Because $\mathfrak{D}$ is commuting, we find that

$$f_n(\mathrm{Fix}(f_1, \dots, f_{n-1})) \subset \mathrm{Fix}(f_1, \dots, f_{n-1}),$$

for if $x \in \mathrm{Fix}(f_1, \dots, f_{n-1})$, then $f_i(f_n(x)) = f_n(f_i(x)) = f_n(x)$ for each $i < n$ so $f_n(x) \in \mathrm{Fix}(f_1, \dots, f_{n-1})$. Since $\mathrm{Fix}(f_1, \dots, f_{n-1})$ is a nonempty compact general convex set, we conclude from Theorem 18 that $\mathrm{Fix}(f_1, \dots, f_n)$ is nonempty. The proof is complete. □

Annotations. Let A be a subset of a locally general convex space E. A point $x \in A$ is an **extreme point** of A if it is not contained in the interior of any line segment (or of any general convex structure) having its endpoints in A.

The the following facts hold: 1) *If A is a compact general convex subset of a locally general convex space E, then A is the general convex closure of its set of extreme points.* 2) *If B is any compact subset of E, and if g.Conv(B) is also compact, then all the extreme points of g.Conv(B) belongs to B.*

We notice that this statement via 1) and 2) is an extension of *Krein-Milman theorem* for convex sets, see: T a s k o v i ć [44].

With regard to this, the weak topology in a locally general convex linear space E is the smallest topology for E with which all the continuous linear functionals $f : E \to \mathbb{R}$ remain continuous. This topology, which is in general smaller than the original topology, also makes E into a locally general convex linear space. We speak of weakly open sets, weak compactness, and further when referring to the weak topology. In this sense we have the following result: *A general convex set $C \subset E$ is weakly closed if and only if it is closed.* This result is an extension of **Mazur's theorem** for convex sets. For this see: T a s k o v i ć [44].

3. A new variational inequality

In the next we apply the ATM-maps to get a fairly general version of a new variational inequality which is a generalization of Hartman-Stampacchia variational inequality.

Let $H := (H, (,))$ be a Hilbert space and C be any subset of H. We recall that a mapping $f : C \to H$ is called **monotone** on C iff $(f(x) - f(y), x - y) \geq 0$ for all $x, y \in C$. We say that $f : C \to H$ is **hemi-continuous** iff $f \mid L \cap C$ is continuous for each one-dimensional flat $L \subset H$.

Theorem 20. *Let H be a Hilbert space, C a closed bounded general convex (with general convex structure $G(x, y, \lambda)$) subset of H, and $f : C \to H$ monotone and hemi-continuous. Then there exists a point $y_0 \in C$ such that*

$$\Big(f(G(y_0, y_0, \lambda)), G(y_0, y_0, \lambda) - G(x, x, \lambda)\Big) \leq 0 \quad \text{for all} \quad x \in C.$$

Proof. (*Application of ATM-maps and Corollary 2*). In this sense define for every $x \in C$ the following set in the form as

$$G(x) = \Big\{y \in C : (f(G(y, y, \lambda)), G(y, y, \lambda) - G(x, x, \lambda)) \leq 0\Big\};$$

and the statement will be proved by showing that $\cap\{G(x) : x \in C\}$ is a nonempty set. First we establish that $G : C \to 2^H$ is an ATM-mapping. Indeed, let $y_0 \in \text{g.conv}\{x_1, \ldots, x_n\}$. If $y_0 \notin \cup\{G(x_i)\colon i = 1, \ldots, n\}$, we would have $(f(G(y_0, y_0, \lambda)),\ G(y_0, y_0, \lambda) - G(x_i, x_i, \lambda)) > 0$ for each $i = 1, \ldots, n$. Since all the x_i would therefore lie in the half-space $\{x \in H : (f(G(y_0, y_0, \lambda)), G(y_0, y_0, \lambda)) > (f(G(y_0, y_0, \lambda)), G(x, x, \lambda))\}$ we also would g.conv$\{x_1, \ldots, x_n\}$, and we have the contradiction $(f(G(y_0, y_0, \lambda)),\ G(y_0,$ $y_0,\ \lambda)) > (f(G(y_0, y_0, \lambda)), G(y_0, y_0, \lambda))$. Thus G is an ATM-map. Consider in the next the map $\mathfrak{R} : C \to 2^H$ given by

$$\mathfrak{R}(x) = \Big\{y \in C : \Big(f(G(x, x, \lambda)), G(y, y, \lambda) - G(x, x, \lambda)\Big) \le 0\Big\};$$

and we show that $\mathfrak{R}(x)$ satisfies the requirements of Corollary 2. Thus $G(x) \subset \mathfrak{R}(x)$ for each $x \in C$. Because of this, it is enough to show

$$\cap\{\mathfrak{R}(x) : x \in C\} \subset \cap\{G(x) : x \in C\}.$$

Assume $y_0 \in \cap\{\mathfrak{R}(x) : x \in C\}$. Choose any $x \in C$ and let $z_t = tG(x, x, \lambda) + (1 - t)G(y_0, y_0, \lambda)$, because C is general convex we have $z_t \in C$ for each $0 \le t \le 1$. Since $y_0 \in \mathfrak{R}(z_i)$ for each $t \in [0, 1]$, we find that $(f(G(z_t, z_t, \lambda)), G(y_0, y_0, \lambda) - G(z_t, z_t, \lambda)) \le 0$ for all $t \in [0, 1]$. This says that $t(f(G(z_t, z_t, \lambda)), G(y_0, y_0, \lambda) - G(x, x, \lambda)) \le 0$ for all $t \in [0, 1]$, and in particular, that $(f(G(z_t, z_t, \lambda)), G(y_0, y_0, \lambda) - G(x, x, \lambda)) \le 0$ for $0 < t \le 1$. Now, let $t \to 0$. The continuity of f on the ray joining y_0 and x gives $f(G(z_t, z_t, \lambda)) \to f(G(y_0, y_0, \lambda))$ and therefore that $(f(G(z_0, z_0, \lambda)), G(y_0, y_0, \lambda) - G(x, x, \lambda)) \le 0$. Thus, $y_0 \in G(x)$ for each $x \in C$ and $\cap\{\mathfrak{R}(x) : x \in C\} = \cap\{G(x) : x \in C\}$.

We now equip H with the weak topology. Then C, as a closed bounded general convex set in a Hilbert space, it weakly compact. Therefore each $\mathfrak{R}(x)$, being the intersection of the closed half-space $\{y \in H : (f(G(x, x, \lambda)), G(y,$ $y,\ \lambda)) \le (f(G(x, x, \lambda),\ G(x, x, \lambda))$ with C is, for the same reason, also weakly compact. Thus, all the requirements in Corollary 2 are satisfied, so $\cap\{G(x) : x \in C\}$ is nonempty and, as we have observed, the proof is complete. □

4. Min-max points

The concept of an ATM-map can be used to establish first a general coincidence statement for set-valued maps which has numerous applications.

Theorem 21. (Coincidence statement). *Let $X \subset E$ and $Y \subset F$ be nonempty compact general convex (with general convex structure $G(x, y, \lambda)$) sets in the linear topological spaces E and F. Let $A, B : X \to 2^Y$ be two set-valued mappings such that: (i) $A(x)$ is open and $B(x)$ is a nonempty general convex set for each $x \in X$. (ii) $B^{-1}(y)$ is open and $A^{-1}(y)$ is a nonempty general convex set for each $y \in Y$. Then there is an $x_0 \in X$ such that the set $A(x_0) \cap B(x_0)$ is nonempty.*

Proof. (*Application of ATM-maps*). Let $Z = X \times Y$ and define $G : X \times Y \to 2^{E\times F}$ by $(x,y) \mapsto Z \backslash (B^{-1}(y) \times A(x))$. Each $G(x,y)$ is a nonempty set closed in $X \times Y$, therefore compact. We now observe that $Z = \cup\{B^{-1}(y) \times A(x) : (x,y) \in Z\}$ in the sense, for, given any $(x_0, y_0) \in Z$ choose an (x,y) in the nonempty set $A^{-1}(y_0) \times B(x_0)$; then $(x_0,y_0) \in B^{-1}(y) \times A(x)$. Thus $\cap\{G(z) : z \in Z\}$ is empty set and G cannot be an ATM-map. Therefore there are elements $z_1, \dots, z_n \in Z$ such that g.conv$\{z_1,\dots,z_n\}$ is not contained in $\cup_{i=1}^n G(z_i)$; so that some general convex combination $w = \sum_{i=1}^n \lambda_i G(z_i, z_i, \lambda) \notin \sum_{i=1}^n G(z_i)$. Because Z is general convex, the point w belongs to Z, so $w \in Z \backslash \cup_{i=1}^n G(z_i) = \cap_{i=1}^n B^{-1}(y_i) \times A(x_i)$. Writing $w = (\sum_{i=1}^n \lambda_i G(x_i, x_i, \lambda), \sum_{i=1}^n \lambda_i G(y_i, y_i, \lambda))$ we have $\sum_{i=1}^n \lambda_i G(x_i, x_i, \lambda) \in B^{-1}(y_i)$ for each $i = 1, \dots, n$ and $\sum_{i=1}^n \lambda_i G(y_i, y_i, \lambda) \in A(x_i)$ for each $i = 1, \dots, n$. The first inclusion shows that each $y_i \in B(\sum_{i=1}^n \lambda_i G(y_i, y_i, \lambda))$ and therefore that

$$\sum_{i=1}^n \lambda_i G(y_i, y_i, \lambda) \in B\left(\sum_{i=1}^n \lambda_i G(x_i, x_i, \lambda)\right).$$

The second inclusion shows that each $x_i \in A^{-1}(\sum_{i=1}^n \lambda_i G(y_i, y_i, \lambda))$, therefore $\sum_{i=1}^n \lambda_i\, G(x_i, x_i, \lambda) \in A^{-1}(\sum_{i=1}^n \lambda_i G(y_i, y_i, \lambda))$ consequently $\sum_{i=1}^n \lambda_i\, G(y_i, y_i, \lambda) \in A(\sum_{i=1}^n \lambda_i\, G(x_i, x_i, \lambda))$. Thus $A(\sum_{i=1}^n \lambda_i G(x_i, x_i, \lambda)) \cap B(\sum_{i=1}^n \lambda_i\, G(x_i, x_i, \lambda))$ is a nonempty set, and the proof is complete. □

In further we give an immediate application to game theory by establishing a general version of the minimax principle as an extension of the von Neumann minimax principle.

Recall that a real valued function $f : X \to \mathbb{R}$ on a topological space is *lower* (respectively *upper*) *semicontinuous* iff $\{x : f(x) > r\}$ (respectively $\{x : f(x) < r\}$) is open for each $r \in \mathbb{R}$. If X is a general convex set in a linear space, then f is **general quasi-concave** (respectively **general quasi-convex**) iff $\{x \in X : f(x) > r\}$ (respectively $\{x \in X : f(x) < r\}$) is general convex set for each $r \in \mathbb{R}$.

Theorem 22. (Min-max principle). *Let $X \subset E$ and $Y \subset F$ be two nonempty compact general convex sets in the linear topological spaces E and F. Let $f : X \times Y \to \mathbb{R}$ satisfy: (a) $y \mapsto f(x,y)$ is lower semicontinuous and general quasi-convex for each fixed $x \in X$. (b) $x \mapsto f(x,y)$ is upper semicontinuous and general quasi-concave for each fixed $y \in Y$. Then the following equality holds in the form as*

$$\max_{x\in X} \min_{y \in Y} f(x,y) = \min_{y\in Y} \max_{x \in X} f(x,y).$$

Proof. (*Application of Coincidence statement*). Because of upper semicontinuity, $\max_{x\in X} f(x,y)$ exists for each $y \in Y$ and is a lower semicontinuous function of $y \in Y$, so $\min_{y\in Y} \max_{x\in X} f(x,y)$ exists. Similarly,

$\max_{x\in X}\min_{y\in Y} f(x,y)$ exists. Since $f(x,y) \leq \max_{x\in X} f(x,y)$ we have $\min_{y\in Y} f(x,y) \leq \min_{y\in Y}\max_{x\in X} f(x,y)$; therefore

$$\max_{x\in X}\min_{y\in Y} f(x,y) \leq \min_{y\in Y}\max_{x\in X} f(x,y).$$

In further, we shall show that the preceding inequality cannot hold. For, assume it did, then there would be some r with the following inequalities in the form as

$$\max_{x\in X}\min_{y\in Y} f(x,y) < r < \min_{y\in Y}\max_{x\in X} f(x,y).$$

Define $A, B : X \to 2^Y$ by $A(x) = \{y : f(x,y) > r\}$ and $B(x) = \{y : f(x,y) < r\}$. These set-valued maps would then satisfy the conditions of Theorem 21. Indeed, each $A(x)$ is open by lower semicontinuity of $y \mapsto f(x,y)$, each $B(x)$ is general convex by the general quasi-convexity of $y \mapsto f(x,y)$, and is nonempty because $\max_{x\in X}\min_{y\in Y} f(x,y) < r$. Since $A^{-1}(y) = \{x : f(x,y) > r\}$ and $B^{-1}(y) = \{x : f(x,y) < r\}$, we find in the same way that each $A^{-1}(y)$ is nonempty and general convex and each $B^{-1}(y)$ is open. Then by Theorem 21 there would be some (x_0, y_0) with $y_0 \in A(x_0) \cap B(x_0)$, which gives the contradiction $r < f(x_0, y_0) < r$. Thus the inequality cannot hold, and the proof is complete. □

5. GEOMETRY OF GENERAL CONVEXITY

This paragraph is primarily devoted to illustring the use of the results and facts on general convexity in this paper. In this sense first we extend a result of Radon in 1916 on division a finite set.

Theorem 23. *Let $K = \{x_1, \dots, x_m\}$ be a finite set of points in the space $\mathbb{R}^n$. If $n+2 \leq m$, then K can be division on two disjoint subsets A and B such that* g.conv$(A) \cap$ g.conv(B) *is a nonempty set.*

Proof. The points of the set K are affine dependent because $n+2 \leq m$, thus there exist numbers $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ (at least one of this numbers is $\neq 0$) such that

$$\lambda_1 G(x_1, x_1, \lambda) + \cdots + \lambda_m G(x_m, x_m, \lambda) = 0 \quad \text{for} \quad \lambda_1 + \cdots + \lambda_m = 0.$$

Let $r \in \{1, 2, \dots, m-1\}$ be an index such that the following inequalities hold in the form as $\lambda_1, \dots, \lambda_r \geq 0$ and $\lambda_{r+1}, \dots, \lambda_m < 0$. Thus we obtain

$$\gamma := \lambda_1 + \cdots + \lambda_r = -(\lambda_{r+1} + \cdots + \lambda_m) > 0, \tag{6}$$

and, in the context of the preceding facts, we obtain the following equality as a consequence of the preceding equalities in the following form as

$$\frac{\lambda_1}{\gamma} G(x_1, x_1, \lambda) + \cdots + \frac{\lambda_r}{\gamma} G(x_r, x_r, \lambda) =$$
$$= \left(-\frac{\lambda_{r+1}}{\gamma}\right) G(x_{r+1}, x_{r+1}, \lambda) + \cdots + \left(-\frac{\lambda_m}{\gamma}\right) G(x_m, x_m, \lambda),$$

where
$$\frac{\lambda_1}{\gamma}+\cdots+\frac{\lambda_r}{\gamma}=\left(-\frac{\lambda_{r+1}}{\gamma}\right)+\cdots+\left(-\frac{\lambda_m}{\gamma}\right)=1$$
for nonnegative numbers $\lambda_1/\gamma,\ldots,\lambda_r/\gamma,-\lambda_{r+1}/\gamma,\ldots,-\lambda_m/\gamma$. If we choice that is $A=\{x_1,\ldots,x_r\}$ and that is $B=\{x_{r+1},\ldots,x_m\}$, then the following fact holds in the form as
$$\frac{\lambda_1}{\gamma}G(x_1,x_1,\lambda)+\cdots+\frac{\lambda_r}{\gamma}G(x_r,x_r,\lambda)\in \text{g.conv}(A)\cap\text{g.conv}(B),$$
which means that the set $\text{g.conv}(A)\cap\text{g.conv}(B)$ is nonempty. The proof is complete. □

Theorem 24. (Statement of division). *Let $K:=\{K_1,\ldots,K_m\}$ be a finite family of general convex sets in $\mathbb{R}^n$ with $m\geq n+1$. If every subfamily with $n+1$ elements of the family K has a nonempty intersection, then $K_1\cap K_2\cap\cdots\cdots\cap K_m$ is a nonempty set.*

Proof. (*Application of Theorem 23*). If $m=n+1$ then this statement holds. Assume that this statement holds for all form sets with $m-1$ sets. Let $\{K_1,\ldots,K_m\}$ be a collection with $m\geq n+2$ general convex sets in $\mathbb{R}^n$. On the other hand, from inductive assumed, for every index $j\in\{1,2,\ldots,m\}$ there exists a point $x_j\in\cap_{i=1}^m K_i$. Applying Theorem 23 with $m\geq n+2$, we obtain that there exist two disjoint sets A and B such that
$$A\cup B=\{x_1,\ldots,x_m\},\text{g.conv}(A)\cap\text{g.conv}(B)\neq\varnothing.$$

This means that there exists an index $r\in\{1,2,\ldots,m-1\}$ such that $A=\{x_1,\ldots,x_r\}$ and $B=\{x_{r+1},\ldots,x_m\}$. Since $x_j\in\cap_{i=r+1}^m K_i$ for $j\in\{1,\ldots,r\}$, we obtain $\text{g.conv}\{x_1,\ldots,x_r\}\subset\cap_{i=r+1}^m K_i$, which further means that the following fact holds as
$$\xi:=\frac{\lambda_1}{\gamma}G(x_1,x_1,\lambda)+\cdots+\frac{\lambda_r}{\gamma}G(x_r,x_r,\lambda)\in\bigcap_{i=r+1}^m K_i,$$
where γ is given as in (6). To repeat this procedure, with same reason as in the preceding cases we obtain the following fact $\text{g.conv}(\{x_{r+1},\ldots,x_m\})\subset\cap_{i=1}^r K_i$; which further means
$$\xi=\left(-\frac{\lambda_{r+1}}{\gamma}\right)G(x_{r+1},x_{r+1},\lambda)+\cdots+\left(-\frac{\lambda_m}{\gamma}\right)G(x_m,x_m,\lambda)\in\bigcap_{i=1}^r K_i.$$

From all the preceding facts we obtain that the following fact holds as $\xi\in\cap_{i=1}^m K_i$. Thus, from the induction principle, this statement of division holds. The proof is complete. □

An annotation. In connection with the preceding, let $\mathfrak{R}$ be a family of closed general convex sets in $\mathbb{R}^n$ with more or equally of $n+1$ elements. If at least one member of family $\mathfrak{R}$ is a compact set and if all $n+1$ members of family $\mathfrak{R}$ have a nonempty intersection, then the family $\mathfrak{R}$ has a nonempty intersection.

Proof. For given family $\mathfrak{R}$ set that is $\mathfrak{R} = \{K \cap R : K \in \mathfrak{R}\}$ where $R \in \mathfrak{R}$ a given compact set. The elements of family $\mathfrak{R}$ are compact sets (as general convex closed sets contained in the compact R). Application of Theorem 24 for the case of finite family of general convex sets, we obtain that the intersection of *centered family* $\mathfrak{R}$ is a nonempty set. The proof is complete. $\square$

Remarks. We notice that Theorem 24 is an extension of Helly theorem in 1923 on finite division. For this see: H e l l y [21] and K i r c h b e r g e r [26]. On the other hand for some extensions of Helly theorem see: K l e e and G r ü n b a u m for compact and convex figure on $\mathbb{R}^n$ (Also, for this see: D a n z e r - G r ü n b a u m-K l e e [6]).

Otherwise, the preceding statement of finite division has a version on finite intersection in the following form as.

Indeed, let $\{K_1, \ldots, K_m\}$ be a finite family of compact general convex sets K_i $(i = 1, \ldots, m)$ in $\mathbb{R}^n$. Then the intersection of all this sets is nonempty if and only if the intersection of mostly $n+1$ $(\leq m)$ sets K_i is nonempty.

An extension of Krasnoselskij's theorem. In this sense, the set $Z \subset \mathbb{R}^n$ with the respect on to point $p \in Z$ is called **general starred** iff for every point $x \in Z$ the following fact holds as

$$\mathfrak{D}(p, x) := \{G(x, p, \lambda) : \lambda \in [0, 1]\} \subset Z,$$

where $G(x, p, \lambda)$ is a general convex structure on Z. The set of all points of Z for which it is general starred is called *general kernel* of Z. Otherwise, if for two points a and b in Z is $G(a, b, \lambda) \subset Z$, then we recall that *from the point b to see the point a over the set Z*. We have the following result.

Theorem 25. *Let $K \subset \mathbb{R}^n$ be a compact set with least of all $n+1$ points. If for every subset of $n+1$ points of the set K there exists a point in K from which to see all $n+1$ points over K, then the set K is general starred.*

Proof. (*Application of Theorem 24*). For the point $x \in K$ to make the following set of points in the following form as

$$V_x := \{y \in K : G(x, y, \lambda) \subset K\},$$

such that this set of all points from who is to see the point x. Applying Theorem 24 we obtain that there exists a point

$$y \in \bigcap_{x \in K} \text{g.conv}(V_x),$$

because, from supposition, every $n+1$ elements of the family $\{V_x : x \in K\}$, thus and the family $\{\text{g.conv}(V_x) : x \in K\}$, has a nonempty intersection. We prove that $y \in \cap_{x \in K} V_x$.

Assume contrary, i.e., for $x \in K$ there exists a point $u \in G(x, y, \lambda) \backslash K$. Set $r \in G(x, u, \lambda)$ with the property $G(r, u, \lambda) \cap K$ is an empty set.

Let the point $w \in G(r, u, \lambda)$ with the property $d(w, r) = 2^{-1} d(u, K)$, where d is a metric on the space $\mathbb{R}^n$. Then $G(w, u, \lambda) \cap K$ is an empty set, which means (because of compactness of K) that there exist points $v \in G(w, u, \lambda)$ and $z \in K$ such that

$$d(v, z) = d(G(w, u, \lambda), K) > 0.$$

This means that the point $z \in K$ which is nearest to the point v, i.e., the set K not to cut an open ball (is denoted B) with center v and radius $d(v, z)$. Thus it follows that V_x is to include in closed halfspace $\mathfrak{D}$ which is determined with hiperplane H which to pass through the point z, normal is on $G(v, z, \lambda)$ and not contained the point v.

Since $y \in \text{g.conv}(V_x) \subset \mathfrak{D}$, thus directly it follows that angle $(y, z, v) \geq \pi/2$ and that angle $(z, v, y) < \pi/2$. On the other hand, from the following inequalities

$$d(v, K) \leq d(w, K) < d(u, K)$$

it follows that $v \neq u$. This means that, from the preceding two inequalities for angles on $G(v, u, \lambda)$ there exists a point which is nearer the point z of the point v, contrary to the preceding choice of the point v. The proof is complete. □

Annotations. We notice that in the case $G(x, y, \lambda) = [x, y] := \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$ directly we obtain well-known result for starred sets in 1946 of Krasnoselskij [30].

In connection with the preceding, two nonempty general convex sets C and D in an arbitrary linear space X are **general complementary** if they form a partition of X, that is, $C \cap D = \varnothing$ and $C \cup D = X$. In this sense we have the following result.

Theorem 26. *Let A and B be disjoint general convex subsets of an arbitrary linear space X. Then there exist general complementary general convex sets C and D in X such that $A \subset C$ and $B \subset D$.*

Proof. Let $\mathfrak{D}$ be the class of all general convex sets in X disjoint from B and containing A; certainly $A \in \mathfrak{D}$. After partially ordering $\mathfrak{D}$ by inclusion, we apply Zorn's lemma to obtain a maximal element $C \in \mathfrak{D}$. It now suffices to put $D \equiv X \backslash C$ and prove that D is general convex. If D were not general convex, there would be $c, z \in D$ and $y \in G(x, z, \lambda) \cap C$. Because C is a maximal element of $\mathfrak{D}$, there must be points $p, q \in C$ such that both $G(p, x, \lambda)$ and $G(q, z, \lambda)$ intersect B, say at points u and v, respectively. Now, however, we find that $G(u, v, \lambda) \cap \text{g.conv}\{p, q, y\} \neq \varnothing$, which contradicts the disjointness of B and C. The proof is complete. □

Annotations. We come now to the celebrated result of Stone in 1946. Two nonempty convex sets C and D in an arbitrary linear space X are *complementary* if they form a partition of X that is, $C \cap D = \varnothing$ and $C \cup D = X$. From the preceding statement directly we have:

Theorem 27. (Stone's lemma). *Let A and B be disjoint convex subsets of an arbitrary linear space X. Then there exist complementary convex sets C and D in X such that $A \subset C$ and $B \subset D$.*

We notice that Stone's lemma is a crucial result for separation statements in geometric functional analysis and further.

6. General concave topological spaces

Fundamental elements of general concavity. In this section we shall consider general concavity by Tasković [44] which is described in an abstract form on sets and topological spaces. Also, we formulate a fixed point theorem for general expansive mappings in the general concave topological spaces.

In this sense, let X be a nonempty set or topological space and let $A : X \times X \to \mathbb{R}^0_+ := [0, +\infty)$ be a function. In this part we considered a topological space X (or a nonempty set X) with a **general concave structure** denoted by $G(x, y, \lambda)$ iff there exist a mapping $G : X \times X \times I \to X$ (for the closed unit interval $I := [0, 1]$) and a function $d : (\mathbb{R}^0_+)^2 \to \mathbb{R}^0_+$ satisfying the following inequality

$$\text{(R)} \qquad A\Big(z, G(x, y, \lambda)\Big) \geq \min\Big\{A(z, x), A(z, y), d\Big(A(z, x), A(z, y)\Big)\Big\}$$

for all $x, y, z \in X$ and arbitrary $\lambda \in [0, 1]$. Similarly, a topological space (or a nonempty set) X is with a **general lower affine structure** if there exists a mapping $G : X \times X \times I \to X$ such that (R) for all $x, y, z \in X$ and arbitrary $\lambda \in \mathbb{R}$.

The topological space X with a general concave structure is called a **general concave topological space**. A subset K of X is **general concave** iff $G(x, y, \lambda) \in K$ for all $x, y \in K$ and arbitrary $\lambda \in [0, 1]$.

On the other hand, a nonempty set X with a general concave structure is called **general concave space**. A subset K of X is **general concave** iff $G(x, y, \lambda) \in K$ for all $x, y \in K$ and arbitrary $\lambda \in [0, 1]$.

There are many examples of general concave (topological) spaces. We give some examples here. Clearly a Banach space is a general concave metric space with $G(x, y, \lambda) := \lambda x + (1 - \lambda)y$. Also, more generally, if X is a linear space with a translation invariant matric ρ (de facto, for $A(x, y) := \rho(x, y)$ and for $d(s, t) := \lambda s + (1 - \lambda)t$) satisfying

$$\rho\Big[0, \lambda x + (1 - \lambda)y\Big] \geq \min\Big\{\rho[0, x], \rho[0, y], \lambda\rho[0, x] + (1 - \lambda)\rho[0, y]\Big\}$$

for all $x, y \in X$ and arbitrary $\lambda \in [0, 1]$, then X is a general concave metric space. If (X, ρ) is a metric space and if there exist a mapping $G : X \times X \times I \to X$ and a mapping $d : (\mathbb{R}^0_+)^2 \to \mathbb{R}^0_+$ such that

$$\rho\Big[z, G(x, y, \lambda)\Big] \geq \min\Big\{\rho[z, x], \rho[z, y], d\big(\rho[z, x], \rho[z, y]\big)\Big\}$$

for all $x, y, z \in X$ and arbitrary $\lambda \in [0,1]$, then X is a general concave metric space, i.e., (X, A) is an example of a general concave topological space for $A(x,y) := \rho[x,y]$ with topology of metric space.

We notice, a metric space (X, ρ) is with a concave structure iff there exists a mapping $R : X \times X \times I \to X$ which satisfies the following inequality

$$\rho\Big[z, R(x,y,\lambda)\Big] \geq \lambda\rho[z,x] + (1-\lambda)\rho[z,y]$$

for all $x, y, z \in X$ and arbitrary $\lambda \in [0,1]$. This is also an example of general concave (topological) space.

Obviously, the class of our general concave topological spaces includes as a paradigmatic example the preceding class of concave metric spaces. There are many other examples, but we consider these examples as paradigmatic.

We notice, the topological space X with a general lower affine structure is called a **general lower affine topological space**. A subset M of X is **general lower affine** if $G(x,y,\lambda) \in M$ for all $x, y \in M$ and arbitrary $\lambda \in \mathbb{R}$.

Similarly, the nonempty set X with a general lower affine structure is called a **general lower affine space**. A subset M of X is **general lower affine** if $G(x,y,\lambda) \in M$ for all $x, y \in M$ and arbitrary $\lambda \in \mathbb{R}$.

Theorem 28. *If X is a general concave topological space and if $\{K_\gamma\}_{\gamma\in\mathfrak{G}}$ is any family of general concave (general lower affine) sets, then*

$$K := \bigcap\nolimits_{\gamma\in\mathfrak{G}} K_\gamma$$

is a general concave (general lower affine) subset of X. If in addition $\{K_\gamma\}_{\gamma\in\mathfrak{G}}$ is a chain (meaning for $\alpha, \beta \in \mathfrak{G}$, either $K_\alpha \subset K_\beta$ or $K_\beta \subset K_\alpha$), then $M := \cup_{\gamma\in\mathfrak{G}} K_\gamma$ is general concave (general lower affine).

Proof. For $x, y \in K$ we obtain $x, y \in K_\gamma$ for every $\gamma \in \mathfrak{G}$. Thus $G(x,y,\lambda) \in K_\gamma$ for every $\gamma \in \mathfrak{G}$, because $\{K_\gamma\}_{\gamma\in\mathfrak{G}}$ is a general concave family. This means that $G(x,y,\lambda) \in K$ for all $x, y \in K$ and arbitrary $\lambda \in [0,1]$, i.e., K is a general concave subset of X. Similarly we have the proof for other part of statement. □

In the theory of metric spaces (also, for general concave topological space), it is extremely convenient to use a geometrical language inspired by classical geometry.

Given a general concave topological space (or only nonempty set) X, with the function A, a point $a \in X$, and a real number $r > 0$, the **open ball** (respective, **closed ball**, **sphere**) of center a and radius r is the set

$$K(a,r) := \big\{x \in X \mid A(a,x) > r\big\}$$

(respective, $B(a,r) := \big\{x \in X \mid A(a,x) \geq r\big\}$, $S(a,r) := \big\{x \in X \mid A(a,x) = r\big\}$). Open and closed balls of center a always contain the point a, but a sphere of center a may be empty.

We will, in further, denote by $\mathfrak{D}(P)$ the set of all lower bisection functions $d : P^2 \to P$ which are increasing satisfying $d(t,t) \geq t$ for every $t \in P$.

Theorem 29. *Let X be a nonempty set with a general concave structure and with a bisection function $d \in \mathfrak{D}(\mathbb{R}^0_+)$. Then open and closed balls in X are general concave subsets of X.*

Proof. For $x, y \in B(a,r)$ and for arbitrary $\lambda \in [0,1]$, there exists a general concave structure $G(x,y,\lambda) \in X$. Since X is a nonempty set with general concave structure, we obtain

$$A\big(a, G(x,y,\lambda)\big) \geq \min\Big\{A(a,x), A(a,y), d\big(A(a,x), A(a,y)\big)\Big\} \geq \cdots$$

$$\cdots \geq \min\Big\{A(a,x), A(a,y), \min\big\{A(a,x), A(a,y)\big\}\Big\} \geq \min\Big\{r, r, \min\{r,r\}\Big\} \geq r.$$

Thus we have $G(x,y,\lambda) \in B(a,r)$ for arbitrary $\lambda \in [0,1]$ and for all $x, y \in B(a,r)$. Similarly, $K(a,r)$ is general concave subsets of X. The proof is complete. □

Theorem 30. *Let X be a nonempty set with a general concave structure and with a bisection function $\in \mathfrak{D}(\mathbb{R}^0_+)$. If $A : X \times X \to \mathbb{R}^0_+$ is a symmetric space function satisfying the inequality $A(a,b) \geq \min\{A(a,c), A(c,b)\}$ for all $a, b, c \in X$ and if $A(a,a) \geq A(a,b)$, then*

$$\text{(Jm)} \qquad A(x,y) = \min\Big\{A(x, G(x,y,\lambda)), A(G(x,y,\lambda), y)\Big\}$$

for all $x, y \in X$ and arbitrary $\lambda \in [0,1]$. If X is a nonempty set with a general lower affine structure, then equality (Jm) holds for all $x, y \in X$ and for arbitrary $\lambda \in \mathbb{R}$.

Proof. Since X is a nonempty set with the general concave structure for the functions A and G, we obtain

$$A(x,y) \geq \min\Big\{A(x, G(x,y,\lambda)), A(G(x,y,\lambda), y)\Big\} \geq \cdots$$

$$\cdots \geq \min\Big\{A(x,x), A(x,y), A(x,y), A(y,y), \min\big\{A(x,x), A(y,y), A(x,y), A(x,y)\big\}\Big\} =$$

$$= A(x,y)$$

for all $x, y \in X$ and for arbitrary $\lambda \in [0,1]$. Thus (Jm) holds. Similarly in case when X is with a general lower affine structure. The proof is complete. □

Further facts. In connection with the preceding facts, we notice that the preceding notation of general concave structure can be improved in the following sense.

Let X be a nonempty set, let $P := (P, \preccurlyeq)$ be a partially ordered set with order $\preccurlyeq$, and let $A : X \times X \to P$ be a function. A mapping $G : X \times X \times I \to X$ is said to be **ordered general concave structure** on X if there exists a bisection function $d : P^2 \to P$ such that

$$\text{(7)} \qquad A\Big(z, G(x,y,\lambda)\Big) \succcurlyeq \inf\Big\{A(z,x), A(z,y), d\Big(A(z,x), A(z,y)\Big)\Big\}$$

for all $x, y, z \in X$ and for all arbitrary $\lambda \in [0, 1]$. A nonempty subset K of X is said to **order general concave** if $G(x, y, \lambda) \in K$ for all $x, y \in K$ and for arbitrary $\lambda \in [0, 1]$.

Similarly to the other facts we also have the definitions of an order general lower affine structure and of an order general lower affine set.

The analogous with the preceding statements and facts we directly have the following results.

Theorem 31. *If X is a nonempty set with an order general concave (order general lower affine) structure and if $\{K_\gamma\}_{\gamma\in\mathfrak{G}}$ is any family order general concave (order general lower affine) sets, then*

$$K := \bigcap\nolimits_{\gamma\in\mathfrak{G}} K_\gamma$$

is an order general concave (order general lower affine) subset of X.

The proof of this statement is totally analogous to the proof of Theorem 28. Further, the **open ball** (respective, **closed ball**, **sphere**) of a center a and radius r is the set

$$K(a, r) := \big\{x \in X \mid A(a, x) \succ r\big\}$$

(respective, $B(a, r) := \big\{x \in X \mid A(a, x) \succcurlyeq r\big\}$, $S(a, r) := \big\{x \in X \mid A(a, x) = r\big\}$). Open and closed balls of center a always contain the point a, but a sphere of center a may be empty.

Theorem 32. *Let X be a nonempty set with an order general concave structure and with a bisection function $d \in \mathfrak{D}(P)$. Then open and closed balls in X are order general concave subsets of X.*

A brief proof of this statement based on the preceding proof of Theorem 29 may be found in Tasković [42], and [44].

Theorem 33. *Let X be a nonempty set with an order general concave structure and with a bisection function $d \in \mathfrak{D}(P)$. If $A : X \times X \to P$ is a symmetric function which satisfying the following inequalities*

$$\inf\Big\{A(a, a), A(b, b)\Big\} \succcurlyeq A(a, b) \succcurlyeq \inf\Big\{A(a, c), A(c, b)\Big\}$$

for all $a, b, c \in X$, then $A(x, y) = \sup\big\{A\big(x, G(x, y, \lambda)\big), A\big(G(x, y, \lambda), y\big)\big\}$ holds for all $x, y \in X$ and for arbitrary $\lambda \in [0, 1]$. (This is analogous to the proof of Theorem 30).

A nonempty subset K of X is said to **general concave** if $G(x, y, \lambda) \in K$ for all $x, y \in K$ and for arbitrary $\lambda \in [0, 1]$. Similarly, K is **general lower affine** if $G(x, y, \lambda) \in K$ for all $x, y \in K$ and for every $\lambda \in \mathbb{R}$.

In this sence, a nonempty set X with a general concave structure is a called **general concave space**.

We say that G is **general lower affine structure** if (7) holds for all $x, y, z \in X$ and for every $\lambda \in \mathbb{R}$. A nonempty set X with a general lower affine structure is called **general lower affine space**. For the further facts see: T a s k o v i ć [42], and [44].

A theorem of fixed point. In this part we give a statement of fixed point in general concave topological spaces. In this sense let X be a general concave topological space for a continuous function $A : X \times X \to \mathbb{R}^0_+$.

For $S \subset X$ we denote the **diameter** of S by $\delta(S) := \inf\{A(x,y) \mid x, y \in S\}$. A point $x \in S$ is a **diametral point** (or *lower diametral point*) of S provided

$$\inf\Big\{A(x,y) : y \in S\Big\} = \delta(S).$$

Lemma 3. *Let M be a nonempty compact subset of general concave topological space X with a bisection function $d \in \mathfrak{D}(\mathbb{R}^0_+)$ and let K be the least closed general concave set containing M. If the diameter $\delta(M)$ is positive, then there exists an element $u \in K$ such that*

$$\inf\Big\{A(x,u) : x \in M\Big\} > \delta(M).$$

A brief proof of this statement based on compactness may be found in Tasković [43]. Also, for these facts see Tasković [44].

We notice that this statement gives us the following definition. A general concave topological space is said to have **normal structure** if for each closed bounded general concave subset S of X which contains at least two points, there exists $x \in S$ which is not a diametral point of S.

A general concave topological space X will be said to have **Šmulian property** if every bounded decreasing net of nonempty closed general concave subsets of X has a nonempty intersection.

Lemma 4. *Let F be a subset of a general concave topological space X with the Šmulian property and with a bisection function $d \in \mathfrak{D}(\mathbb{R}^0_+)$. Then the* **centre** *of F, i.e.,*

$$F_c := \Big\{x \in F \mid R_x(F) := \inf\big\{A(x,y) \mid y \in F\big\} = \sup_{x \in F} R_x(F) := R(F)\Big\}$$

is a nonempty, closed and general concave set.

Proof. Let $F(x,n) := \{y \in F \mid A(x,y) \geq R(F) + n\}$ for every $n \in \mathbb{N}$ and $x \in X$. It is easily seen that the sets $C_n = \cap_{x \in F} F(x,n)$ form a decreasing sequence on nonempty closed general concave sets, and hence, from the Šmulian property and Theorem 28 and 29, $F_c = \cap_{n \in \mathbb{N}} C_n$ is a nonempty, closed and general concave set. The proof is complete. □

Let X be a topological space (or only a nonempty set), K be a subset of X, and $A : X \times X \to \mathbb{R}^0_+$ be a continuous function. A mapping T of K into K is said to be **general expansive** if the following inequality holds in the form as:

$$\text{(Di)} \qquad A(Tx,Ty) \geq \inf\Big\{A(x,y) : x, y \in E\Big\}$$

for all $x, y \in E$ and for every closed general concave subset E of K with at least two points such that $T(E) \subset E$. In this section we will prove the following result for general expansive mappings in general concave topological spaces.

Theorem 34. *Let X be a general concave topological space with the Šmulian property, and with a bisection function $d \in \mathfrak{D}(\mathbb{R}^0_+)$, and K be a nonempty bounded closed general concave subset of X with normal structure. If T is a general expansive mapping of K into itself, then T has a fixed point in K.*

Some remarks. Let X be a Banach space. A mapping T of a subset K of X into K is called **half-diametral lower contraction** on K if the following inequality holds in the form as:

$$\text{(Ci)} \qquad \|Tx - Ty\| \geq \inf \Big\{ \|x - y\| : y \in E \Big\}$$

for all $x, y \in E$ and for every closed general concave subset E of K with at least two points such that $T(E) \subset E$.

We notice, by interchanging x and y in (Ci) we see that a nature condition of fixed point appears above which may be written

$$\text{(Gi)} \qquad \|Tx - Ty\| \geq \min \Big\{ \inf \Big(\|x - z\| : z \in K \Big), \inf \Big(\|y - z\| : z \in K \Big) \Big\}$$

for all $x, y \in K$. But, the inequality (Ci) can not be written in some form of type (Gi). Also, we notice that condition (Ci) is not equivalent to condition (Gi).

Proof of Theorem 34. Let $\mathfrak{F}$ denote the collection of all nonempty closed and general concave subsets of K, each of which is mapped into itself by T. By Šmulian property and Zorn's lemma $\mathfrak{F}$ has a minimal element which we denote by F. We show that F consists of a single point. Let $x \in F_c$. Then

$$A(Tx, Ty) \geq \inf \big\{ A(x, y) : x, y \in F \big\} = R_x(F) \quad \text{for every} \quad y \in F,$$

and hence $T(F)$ is contained in the spherical ball $B(Tx, R(F))$ centered at Tx with radius $R(F)$. Since $T(F \cap B) \subset F \cap B$, the minimality of F implies $F \subset B$. Hence $R_{T(x)}(F) \geq R(F)$. Since $R(F) \geq R_x(F)$ for all $x \in F$, thus we obtain $R_{T(x)}(F) = R(F)$. Hence $Tx \in F_c$ and F_c is mapped into itself by T. By Lemma 4 we have $F_c \in \mathfrak{F}$. If $z, w \in F_c$, then we obtain $A(z, w) \geq R_z(F) = R(F)$. Hence, by normal structure (i.e., by Lemma 3), $\delta(F_c) \geq R(F) > \delta(F)$. Since this contradicts the minimality of F, $\delta(F) = 0$ and F consists of a single point. The proof is complete.

Further, let X be a linear space with general concave structure $G : X \times X \times I \to X$. If $\lambda_i \in \mathbb{R}$ $(i = 1, \ldots, n)$, $\lambda \in I$, and $\lambda_1 + \cdots + \lambda_n = 1$, then

$$x = \lambda_1 G(x_1, x_1, \lambda) + \cdots + \lambda_n G(x_n, x_n, \lambda)$$

is called **general lower affine combination** of $x_1, \ldots, x_n$, the latter being elements of a linear space X. If in addition $\lambda_i \geqslant 0$ $(i = 1, \ldots, n)$, then x is called a **general concave combination** of $x_1, \ldots, x_n \in X$. We have the following characterization of general concave sets.

Theorem 35. *A set K of a linear space X is general concave (general lower affine) if and only if every general concave (general lower affine) combination of points of K lies in K.*

Proof. Since a set that contains all general concave combinations of its points is obviously general concave, we only need to consider a general concave set K and show that it contains any general concave combination of its points. Our proof is by induction of the number of points of K occurring in a general concave combination, the conclusion following from the definition for $n = 2$. Assuming the result true for any general concave combination with n or fewer points, we consider one with $n + 1$ points,

$$x = \lambda_1 G(x_1, x_1, \lambda) + \cdots + \lambda_{n+1} G(x_{n+1}, x_{n+1}, \lambda).$$

Not all the λ_i's can be as great as one, so we relabel if necessary so that $\lambda_{n+1} < 1$. Then we obtain the following equalities in the following form as

$$x = (1 - \lambda_{n+1}) \sum_{k=1}^{n} \frac{\lambda_k}{1 - \lambda_{n+1}} G(x_k, x_k, \lambda) + \lambda_{n+1} G(x_{n+1}, x_{n+1}, \lambda) =$$
$$= (1 - \lambda_{n+1}) y + \lambda_{n+1} G(x_{n+1}, x_{n+1}, \lambda).$$

Now $y \in K$ by assumption, and thus so is x, being a general concave combination of two points of K. The proof in the general lower affine case follows exactly the same pattern. □

We call the intersection of all general concave sets containing a given set K is the **general concave hull** of K, denoted by $G(\mathrm{conc}(K))$, $g.\,\mathrm{conc}(K)$, or $GC(K)$. Similarly, the intersection of all general lower affine sets containing K is called **general lower affine hull** of K. By Theorem 28, the general concave hull is a general concave set; the general lower affine hull is a general lower affine set.

Theorem 36. *For any K of a linear space X, the general concave (general lower affine) hull of K consists precisely of all general concave (general lower affine) combinations of elements of K.*

The proof of this statement is a totally analogous with the former proof of Theorem 9. Thus the proof of this statement we omit.

We notice that this statement can be improved if $X = \mathbb{R}^n$. In this case, $GC(K)$ consists of all general concave combinations of $n + 1$ or fewer elements of K.

Also, from the preceding proof of Theorem 36, directly, we have that the set of all general concave combinations is a general concave set.

In the theory of metric spaces (also, for general concave topological spaces), it is extremely convenient to use a geometrical language inspired by classical geometry.

Before proving a slightly more general version of this statement, let us introduce the concept of **dimension** for a general concave set. First if K is general lower affine, we define its dimension to be that of the subspace of which it is a translate. More generally, if K is general concave, its **dimension** is the dimension of the general lower affine hull of K.

Theorem 37. *If a set K of a linear space X and its general concave hull $GC(K)$ has dimension m, then for each $x \in GC(K)$, there exist $m+1$ points $x_0, x_1, \ldots, x_m$ in K such that x is a general concave combination of these points.*

The proof of this statement is a totally analogous with the former proof of Theorem 10. Thus we omit the proof of this statement!

On the other hand, from the notation of general concave combination, we obtain similarly to Theorem 35 the following statement.

Theorem 38. *A set K of a linear space X is order general concave (order general lower affine) if and only if every general concave (general lower affine) combination of points of K lies in K.*

This is totally analogous to the former proof of Theorem 35. In connection with this, we notice, from Theorems 35 and 37 as an immediate consequence we obtain the main fact for the preceding notations, i.e., we have that the notation of general concave set is equivalent to the notation of order general concave set in a linear space! We are now in a position to formulate the following statement.

Theorem 39. *Let K be a subset of a linear space X, then the following statements are equivalent*:
(a) *K is a general concave set,*
(b) *K is an order general concave set,*
(c) *every general concave combination of points of K lies in K.*

Proof of Lemma 3. Since M is nonempty and compact, we may find $x_1, x_2 \in M$ such that $A(x_1, x_2) = \delta(M)$. Let $M_0 \subset M$ be maximal so that $M_0 \supset \{x_1, x_2\}$ and $A(x, y) = 0$ or $\delta(M)$ for all $x, y \in M_0$. Since M is compact and we are assuming $\delta(M) > 0$, M_0 must be finite. Let us assume $M_0 = \{x_1, x_2, \ldots, x_n\}$. Since X is a general concave topological space, we can define

$$y_1 = G\left(x_1, x_2, \frac{1}{2}\right), \quad y_2 = G\left(x_3, y_1, \frac{1}{3}\right), \ldots,$$
$$y_{n-2} = G\left(x_{n-1}, y_{n-3}, \frac{1}{3} - 1\right), \quad y_{n-1} = G\left(x_n, y_{n-2}, \frac{1}{n}\right) := u.$$

Since M is a compact set, we can find $y_0 \in M$ such that $A(y_0, u) = \inf\{A(x, u) : x \in M\}$. Now, since X is a general concave topological space, from (R), we obtain the following inequalities

$$A(y_0, u) = A\left(y_0, G\left(x_n, y_{n-2}, \frac{1}{n}\right)\right) \geq$$
$$\geq \min\Big\{A(y_0, x_n), A(y_0, y_{n-2}), d(A\big(y_0, x_n), A(y_0, y_{n-2})\big)\Big\} \geq \cdots$$
$$\cdots \geq \min\Big\{A(y_0, x_n), A)(y_0, y_{n-2}), \min\Big[A(y_0, x_{n-1}), A(y_0, y_{n-3})\Big]\Big\} \geq \cdots$$
$$\cdots \geq \min\Big\{A(y_0, x_n), A(y_0, x_{n-1}), \ldots, A(y_0, x_1)\Big\} \geq \delta(M).$$

Thus, if $A(y_0, u) = \delta(M)$, then we must have $A(y_0, x_k) = \delta(M) > 0$ for all $k \in \{1, 2, \ldots, n\}$, which means that $y_0 \in M_0$ by definition of M_0. But then we must have $y_0 = x_k$ for some $k = 1, 2, \ldots, n$. This is a contradiction. Therefore, $\inf \Big\{ A(x, u) : x \in M \Big\} = A(y_0, u) > \delta(M)$, i.e., $\inf \{ A(x, u) : x \in M \} > \delta(M)$ holds. The proof of Lemma 3 is complete.

Let X be a topological space (or only a nonempty set), K be a subset of X, and $A : X \times X \to \mathbb{R}^0_+$ be a continuous function. A mapping T of K into K is said to be **strictly general expansive** if

$$\text{(D)} \qquad A(Tx, Ty) \geq \inf \Big\{ A(x, y) : x, y \in K \Big\}$$

for all $x, y \in K$. In this section we will prove the following result for strictly general expansive mappings in general concave topological spaces.

Let X be a Banach space. A mapping T of a subset K of X into itself is called **diametral lower contraction** on K if

$$\|Tx - Ty\| \geq \inf \Big\{ \|x - y\| : x, y \in E \Big\}$$

for all $x, y \in E$ and for every closed general concave subset E of K with at least two points such that $T(E) \subset E$.

In this part we extend this result and a result of T a s k o v i ć [44] for general expansive mappings in general concave topological spaces.

Let X be a compact general concave topological space. A family $\mathfrak{F}$ of general expansive mappings T of X into itself is said to have **invariant property** in X if for any compact general concave subset K of X such that $T(K) \subset K$ for each $T \in \mathfrak{F}$ there exists a compact subset $M \subset K$ such that $T(M) = M$ for each $T \in \mathfrak{F}$.

Theorem 40. *Let X be a compact general concave topological space with a bisection function $d \in \mathfrak{D}(\mathbb{R}^0_+)$. If $\mathfrak{F}$ is a family of strictly general expansive mappings with invariant property in X, then the family $\mathfrak{F}$ has a common fixed point.*

Proof. By using Zorn's lemma, we can find a minimal nonempty general concave compact set $K \subset X$ such that K is an invariant under each $T \in \mathfrak{F}$. If K consists of a single point, then the invariant property implies that T has a fixed point. Also, by hypothesis, there exists a compact subset M of K such that $M = \{T(x) : x \in M\}$ for each $T \in \mathfrak{F}$. If M contains more than one point by Lemma 3 there exists an element u in the least general concave set containing M such that the condition of nondiametral point holds. Let us define

$$K_0 := \cap_{x \in M} \Big\{ y \in K : A(x, y) \geq \inf \Big(A(u, x) : x \in M \Big) \Big\},$$

then K_0 is the nonempty closed general concave proper subset of K invariant under each $T \in \mathfrak{F}$. This is a contradiction to the minimality of K. The proof is complete. □

Strictly general concave spaces. Let X be a nonempty set, let $A : X \times X \to \mathbb{R}^0_+$ a continuous function, and let $I := [0,1]$ be the closed unit interval. Let $G : X \times X \times I \to X$ be a general concave structure on a topological space (or only on a nonempty set) X. We say that G is a **strict general concave structure** if it has the property that whenever $w \in X$ and there is $(x,y,\lambda) \in X \times X \times I$ and there is $d : (\mathbb{R}^0_+)^2 \to \mathbb{R}^0_+$ such that

$$A(z,w) \geq \min\Big\{A(z,x), A(z,y), d\big(A(z,x), A(z,y)\big)\Big\} \quad \text{for every} \quad z \in X,$$

then $w = G(x,y,\lambda)$. If G is a strict general concave structure on the topological space (or on the nonempty set) X we call X is a **strict general concave topological space** (or a **strict general concave space**).

We give a preliminary example here. For example, the plane equipped with the norm in the following form as $\|(x_1,x_2)\| = \min\{|x_1|,|x_2|\}$ is strictly general concave space in our sense, but not in the former sense.

We say that G is a **strict general lower affine structure** if it has the property that whenever $w \in X$ and there is $(x,y,\lambda) \in X \times X \times \mathbb{R}$ and there is $d : (\mathbb{R}^0_+)^2 \to \mathbb{R}^0_+$ such that

$$A(z,w) \geq \min\Big\{A(z,x), A(z,y), d\big(A(z,x), A(z,y)\big)\Big\} \quad \text{for every} \quad z \in X,$$

then $w = G(x,y,\lambda)$. If G is a strict general lower affine structure on the topological space (or on the nonempty set) X, we call X is a **strictly general lower affine topological space** (or a **strictly general lower affine space**).

Theorem 41. *Let X be a strict general concave space with a bisection function $d \in \mathfrak{D}(\mathbb{R}^0_+)$. Then the following equality holds in the following form as*

$$G\Big(G(x,y,\lambda), y, \gamma\Big) = G(x,y,\beta) \tag{8}$$

for all $x,y \in X$ and for all $\lambda,\gamma,\beta \in [0,1]$. If X is a strict general lower affine space with a bisection function $d \in \mathfrak{D}(\mathbb{R}^0_+)$, then (8) holds for all $x,y \in X$ and for all $\lambda,\gamma,\beta \in \mathbb{R}$.

Proof. Let $z \in X$ be an arbitrary point. Then we have the following inequalities in the following adequate form as

$$A\Big(z, G\Big(G(x,y,\lambda), y, \gamma\Big)\Big) \geq \min\Big\{A\Big(z, G(x,y,\lambda)\Big), A(z,y)\Big\} \geq$$

$$\geq \min\Big\{\min\Big[A(z,x), A(z,y), d\big(A(z,x), A(z,y)\big)\Big], A(z,y)\Big\} \geq \cdots \min\Big\{A(z,x), A(z,y)\Big\},$$

whence, by strictness of general concave space, we obtain $G(G(x,y,\lambda),y,\gamma) = G(x,y,\beta)$ for all $x,y \in X$ and for all $\lambda,\gamma,\beta \in I$. The proof is complete. □

We notice, it does not appear that even a strict general concave structure is necessarily continuous as a function from $X \times X \times I$ to X. But we have the following fact.

Theorem 42. *Let G be a general concave structure on a topological space X with a bisection function $d \in \mathfrak{D}(\mathbb{R}^0_+)$. If $A(a,b)=0$ iff $a=b$ for all $a,b \in X$, then G is a continuous function at each point (x,x,λ) of $X \times X \times I$.*

Proof. Let $\{(x_n,y_n,\lambda_n)\}_{n\in\mathbb{N}}$ be a sequence in $X \times X \times I$ which converges to (x,x,λ). But this is immediate, since the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ both converge to x, and (R) yields

$$A\Big(x, G(x_n,y_n,\lambda_n)\Big) \geq \min\Big\{A(x,x_n), A(x,y_n), d\big(A(x,x_n), A(x,y_n)\big)\Big\},$$

i.e., $A(x,G(x_n,y_n,\lambda)) \geq \cdots \geq \min\{A(x,x_n), A(x,y_n)\}$ for each $n \in \mathbb{N}$. The proof is complete. □

Theorem 43. *Let G be a strict general concave structure on a compact Hausdorff topological space X with a bisection function $d \in \mathfrak{D}(\mathbb{R}^0_+)$. Then G is a continuous function as a mapping from $X \times X \times I$ to X.*

Proof. Let $\{(x_n,y_n,\lambda_n)\}_{n\in\mathbb{N}}$ be a sequence in $X \times X \times I$ which converges to (x,y,λ), and let w be a limit point of the sequence $\{G(x_n,y_n,\lambda_n)\}_{n\in\mathbb{N}}$. Select a subsequence $\{G(x_{n(k)},y_{n(k)},\lambda_{n(k)})\}_{n\in\mathbb{N}}$ which converges to w. Then for any $z \in X$ we have

$$A\Big(z, G(x_{n(k)},y_{n(k)},\lambda_{n(k)})\Big) \geq$$
$$\geq \min\Big\{A(z,x_{n(k)}), A(z,y_{n(k)}), d\big(A(z,x_{n(k)}), A(z,y_{n(k)})\big)\Big\},$$

i.e., $\min\{A(z,x_{n(k)}), A(z,y_{n(k)})\}$ for all $k \in \mathbb{N}$. By continuity of the function A, we conclude that $A(z,w) \geq \min\Big\{A(z,x), A(z,y)\Big\}$. Strictness now guarantees that $w = G(x,y,\lambda)$. It follows that $G(x,y,\lambda)$ is the only limit point of the sequence $\{G(x_n,y_n,\lambda_n)\}_{n\in\mathbb{N}}$. Since X is compact, the sequence $\{G(x_n,y_n,\lambda_n)\}_{n\in\mathbb{N}}$ must converge to $G(x,y,\lambda)$. The proof is complete. □

Some remarks. In connection with the preceding notations we have the following definition. Let X be a nonempty set, let $A : X \times X \to \mathbb{R}^0_+$ a continuous function, and let $I := [0,1]$ be the closed unit interval. Let X be a topological space (or a nonempty set) and let

$$I^n = \Big\{(\lambda_1,\ldots,\lambda_n) \in I \times \cdots \times I : \lambda_1 + \cdots \lambda_n = 1\Big\}.$$

In further, a **strong concave structure** on X is a (continuous) function $K : X^n \times I^n \to X$ such that the following inequality holds in the form:

$$A\Big(z, K(x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_n)\Big) \geq \min\Big\{A(z,x_1),\ldots,A(z,x_n)\Big\}$$

for all $z, x_1, \ldots, x_n \in X$ and for all $\lambda_1,\ldots,\lambda_n \in I$ with property $\lambda_1 + \cdots + \lambda_n = 1$.

A topological space (or a nonempty set) with a strong general concave structure will be called **strongly general concave topological space** (or **strongly general concave space**).

On the other hand, the strong general concave structure on X is a special form of the ordered general concave structure on X.

Let X be a nonempty set, let $P := (P, \preccurlyeq)$ be a partially ordered set with order $\preccurlyeq$, and let $A : X \times X \to P$ be a function.

A mapping $K : X^n \times I^n \to X$ is said to be **n-ordered general concave structure** on X if there is a function $g : P^n \to P$ such that

$$\begin{aligned} &A\Big(z, K(x_1, \dots, x_n, \lambda_1, \dots, \lambda_n)\Big) \succcurlyeq \\ &\succcurlyeq \inf\Big\{A(z, x_1), \dots, A(z, x_n), g\Big(A(z, x_1), \dots, A(z, x_n)\Big)\Big\} \end{aligned} \tag{9}$$

for all $z, x_1, \dots, x_n \in X$ and for all $\lambda_1, \dots, \lambda_n \in I$ with property $\lambda_1 + \cdots + \lambda_n = 1$.

We say that K is **n-ordered general lower affine structure** if (9) holds for all $z, x_1, \dots, x_n \in X$ and for all $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ with property $\lambda_1 + \cdots + \lambda_n = 1$.

For results on topological space with n-ordered general concave structure see: Tasković [44].

7. MTM-mappings

This paragraph is primarily devoted to illustring the use of the results established in this paper. Let E be a vector space. The set of all subsets of E is denoted by 2^E and g.conc(A) will denote the **general concave hull** of any $A \in 2^E$. A subset $A \subset E$ is called **finitely closed** if its intersection with each finite dimensional flat $L \subset E$ is closed in the Eucliedan topology of L. We notice that a set closed in any topology making E a topological vector space is necessarily finitely closed.

Let E be a vector space and $X \subset E$ an arbitrary subset. A function $G : X \to 2^E$ is called simply an **MTM-mapping** provided

$$\text{g.conc}\{x_1, \dots, x_r\} \subset \cup_{i=1}^r G(x_i)$$

for each finite subset $\{x_1, \dots, x_r\} \subset X$. In this sense, the essential property of MTM-maps is given in the following statement.

Theorem 44. (MTM-maps principle). *Let E be a vector space, X an arbitrary subset of E, and $G : X \to 2^E$ a MTM-map such that each $G(x)$ is finitely closed. Then the family $\{G(x) : x \in X\}$ of sets has the finite intersection property.*

The proof of this statement is a totally analogous with the former proof of Theorem 17. Thus we omit the proof of this statement.

Corrollary 3. *Let E be a topological vector space, $X \subset E$ an arbitrary subset, and $G : C \to 2^E$ a MTM-mapping. If all the sets $G(x)$ are closed in E, and if one is compact, then $\cap\{G(x) : x \in X\}$ is a nonempty set.*

We now observe that the conclusion $\cap\{G(x) : x \in X\} \neq \varnothing$ can be reached in another way, which avoids placing any compactness restriction on the sets $G(x)$. It involves using an auxiliary family of sets and a suitable topology on E.

Corrollary 4. *Let E be a vector space, $X \subset E$ an arbitrary subset, and $G : X \to 2^E$ a MTM-mapping. Assume there is a set-valued map $R : X \to 2^E$ such that $G(x) \subset R(x)$ for each $x \in X$, and for which*

$$\cap\{\Re(x) : x \in X\} = \cap\{G(x) : x \in X\},$$

and if there is some topology on E such that each $\Re(x)$ is a compact set, then $\cap\{G(x) : x \in X\}$ is a nonempty set.

The proof of this consequence is elementary from Theorem 44. In the next we give the simplest applications of MTM-maps to fixed point theory.

Proposition 3. *Let C be a general concave compact subset of a normed space E, and let $f : C \to E$ be a continuous mapping. Then there exists at least one $y_0 \in C$ such that*

$$\|y_0 - f(y_0)\| = \sup_{x \in C} \|x - f(y_0)\|. \tag{10}$$

Proof. (*Application of MTM-mappings*). Define the mapping $G : C \to 2^E$ by the following equality in the form as

$$G(x) = \{y \in C : \|y - f(y)\| \geq \|x - f(y)\|\};$$

because f is a continuous mapping, the sets $G(x)$ are closed in C, therefore compact. We verife that G is an MTM-map. Let $y \in \text{g.conc}\{x_1, \ldots, x_n\} \subset C$. If $y \notin \cup_{i=1}^n G(x_i)$ then $\|y - f(y)\| < \|x_i - f(y)\|$ for $i = 1, \ldots, n$. This shows that the points x_i all lie in an open ball of radius $\|y - f(y)\|$ centered at $f(y)$, therefore so also does their general concave hull and, in particular, y. Thus, $\|y - f(y)\| < \|y - f(y)\|$, which is a contradiction.

On the other hand, by compactness of the $G(x)$ we find a point y_0 such that $y_0 \in \cap\{G(x) : x \in C\}$ and hence $\|y_0 - f(y_0)\| \geq \|x - f(y_0)\|$ for all $x \in C$. This clearly implies (10) and the proof is complete. □

Proposition 4. *Let C be a general concave compact set in a normed space E, and let $f : C \to E$ be a continuous mapping and such that, for each $c \in C$ $(c \neq f(c))$, the line segment $[c, f(c)]$ contains at least two points of C. Then f has a fixed point.*

For the proof of this statement we can be application of Proposition 3. A proof of this statement may be found in: T a s k o v i ć [44].

Annotations. This proposition just proved implies that any continuous self-map of a compact general concave set in a normed space has a fixed point. We extend this result to arbitrary locally general concave spaces.

Locally general concave spaces. A linear topological space is **locally general concave** iff every neighborhood of zero contains a general concave neighborhood of zero. Every locally general concave topology on a vector space is determined by some family $\{p_\alpha : \alpha \in \mathfrak{A}\}$ of semi-norms having the property that $p_\alpha(x) = 0$ for all $\alpha \in \mathfrak{A}$ if and only if $x = 0$. In the topology determined by this family of seminorms a set V is **open** if and only if for each $v \in V$ there exists some $\varepsilon > 0$ and finitely many $\alpha_1, \dots, \alpha_n \in \mathfrak{A}$ such that $\cap_{i=1}^{n} \{x : p_{\alpha_i}(x - v) > \varepsilon\} \subset V$.

Let A be a subset of a locally general concave space E. The **general concave closure** of A in notation g.Conc(A) is the smallest closed general concave subset containing A.

Theorem 45. *Let C be a compact general concave set in a locally general concave topological space E. Then every continuous mapping $f : C \to C$ has a fixed point.*

Proof. (*Application of MTM-maps*). Let $\{p_i\}_{i \in I}$ be the family of all continuous seminorms in E. For each index $i \in I$ set

$$A_i = \{y \in C : p_i(y - f(y)) = 0\}.$$

In further, a point $y_0 \in C$ is a fixed point for f if and only if $y_0 \in \cap_{i \in I} A_i$. By compactness of C we need to show only that each finite intersection $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}$ is a nonempty set. Given $\{i_1, i_2, \dots, i_n\}$ define the mapping $G : C \to 2^E$ by

$$G(x) = \Big\{ y \in C : \sum_{j=1}^{n} p_{i_j}(y - f(y)) \geq \sum_{j=1}^{n} p_{i_j}(x - f(y)) \Big\};$$

and, as in the proof of Proposition 3, we verify that G is an MTM-map, so there is a point $y_0 \in C$ such that the following inequality holds in the form as

$$\sum_{j=1}^{n} p_{i_j}(y_0 - f(y_0)) \geq \sum_{j=1}^{n} p_{i_j}(x - f(y_0)) \quad \text{for all} \quad x \in C;$$

and we have therefore $p_{i_j}(y_0 - f(y_0)) = 0$ for $0 \leq j \leq n$, i.e., this means that $y_0 \in A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}$. The proof is complete. □

In what follows we frequently denote the fixed-point set of mapping f by Fix(f). As an immediate consequence of the preceding result we obtain the following extension.

Theorem 46. *Let C be a compact general concave set in a locally general concave linear space E, and let $\mathfrak{D}$ be a commuting family of continuous affine maps of C into itself. Then $\mathfrak{D}$ has a common fixed point.*

For the proof of this statement we can be application of Theorem 45. A proof of this statement may be found in: T a s k o v i ć [44].

Annotations. Let A be a subset of a locally general concave space E. A point $x \in A$ is an **extreme point** of A if it is not contained in the interior of any line segment (or of any general concave structure) having its endpoints in A.

The the following facts hold: 1) *If A is a compact general concave subset of a locally general concave space E, then A is the general concave closure of its set of extreme points.* a) *If B is any compact subset of E, and if g.Conc(B) is also compact, then all the extreme points of g.Conc(B) belongs to B.*

With regard to this, the weak topology in a locally general concave linear space E is the smallest topology for E with which all the continuous linear functionals $f : E \to \mathbb{R}$ remain continuous. This topology, which is in general smaller than the original topology, also makes E into a locally general concave linear space. We speak of weakly open sets, weak compactness, and further when referring to the weak topology. In this sense we have the following result: *A general concave set $S \subset E$ is weakly closed if and only if it is closed.* For this see: T a s k o v i ć [44].

8. A new variational inequality

In the next we apply the MTM-maps to get a fairly general version of a new variational inequality which is an extension of Hartman-Stampacchia variational inequality.

Theorem 47. *Let H be a Hilbert space, C a closed bounded general concave (with general concave structure $G(x, y, \lambda)$) subset of H, and $f : C \to H$ monotone and hemi-continuous. Then there exists a point $y_0 \in C$ such that*

$$\Big(f(G(y_0, y_0, \lambda)), G(y_0, y_0, \lambda) - G(x, x, \lambda)\Big) \leq 0 \quad \textit{for all} \quad x \in C.$$

Proof. (*Application of MTM-maps and Corollary 4*). In this sense define for every $x \in C$ the following set in the form as

$$G(x) = \Big\{y \in C : (f(G(y, y, \lambda)), G(y, y, \lambda) - G(x, x, \lambda)) \leq 0\Big\};$$

and the statement will be proved by showing that $\cap\{G(x) : x \in C\}$ is a nonempty set. First we establish that $G : C \to 2^E$ is an MTM-mapping. Indeed, let $y_0 \in$ g.conc$\{x_1, \ldots, x_n\}$. If $y_0 \notin \cup\{G(x_i)\colon\ i = 1, \ldots, n\}$, we would have $(f(G(y_0, y_0, \lambda)),\ G(y_0, y_0, \lambda) - G(x_i, x_i, \lambda)) > 0$ for each $i = 1, \ldots, n$. Since all the x_i would therefore lie in the half-space $\{x \in H : (f(G(y_0, y_0, \lambda)), G(y_0, y_0, \lambda)) > (f(G(y_0, y_0, \lambda)), G(x, x, \lambda))\}$ we also would g.conc$\{x_1, \ldots, x_n\}$, and we have the contradiction $(f(G(y_0, y_0, \lambda)),\ G(y_0,$ $y_0,\ \lambda)) > (f(G(y_0, y_0, \lambda)), G(y_0, y_0, \lambda))$. Thus G is an MTM-map. Consider in the next the map $\mathfrak{R} : C \to 2^H$ given by

$$\mathfrak{R}(x) = \Big\{y \in C : \Big(f(G(x, x, \lambda)), G(y, y, \lambda) - G(x, x, \lambda)\Big) \leq 0\Big\};$$

and we show that $\mathfrak{R}(x)$ satisfies the requirements of Corollary 4. Thus $G(x) \subset \mathfrak{R}(x)$ for each $x \in C$. Because of this, it is enough to show

$$\cap\{\mathfrak{R}(x) : x \in C\} \subset \cap\{G(x) : x \in C\}.$$

Further proof is a totally analogy with the former proof of Theorem 20. In finish, we now equip H with the weak topology. Then C, as a closed bounded general concave set in a Hilbert space, it weakly compact. Therefore each $\mathfrak{R}(x)$, being the intersection of the closed half-space $\{y \in H : (f(G(x, x, \lambda)), G(y, y, \lambda)) \leq (f(G(x, x, \lambda),\ G(x, x, \lambda))$ with C is, for the same reason, also weakly compact. Thus, all the requirements in Corollary

4 are satisfied, so $\cap\{G(x) : x \in C\}$ is nonempty and, as we have observed, the proof is complete. □

9. Min-max points

The concept of an MTM-map can be used to establish first a general coincidence statement for set-valued maps which has numerous applications.

Theorem 48. (Coincidence statement). *Let $X \subset E$ and $Y \subset F$ be nonempty compact general concave (with general concave structure $G(x, y, \lambda)$) sets in the linear topological spaces E and F. Let $A, B : X \to 2^Y$ be two set-valued mappings such that: (i) $A(x)$ is open and $B(x)$ is a nonempty general concave set for each $x \in X$. (ii) $B^{-1}(y)$ is open and $A^{-1}(y)$ is a nonempty general concave set for each $y \in Y$. Then there is an $x_0 \in X$ such that the set $A(x_0) \cap B(x_0)$ is nonempty.*

For the proof of this statement we can give an application of MTM-maps. It is analogous to the proof of Theorem 21. Thus we omit the proof of this statement.

In further we give an immediate application to game theory by establishing a general version of the minimax principle as an extension of the von Neumann minimax principle.

Recall that a real valued function $f : X \to \mathbb{R}$ on a topological space is *lower* (respectively *upper*) *semicontinuous* iff $\{x : f(x) > r\}$ (respectively $\{x : f(x) < r\}$) is open for each $r \in \mathbb{R}$. If X is a general concave set in a linear space, then f is **general quasi-convex** (respectively **general quasi-concave**) iff $\{x \in X : f(x) > r\}$ (respectively $\{x \in X : f(x) < r\}$) is general concave set for each $r \in \mathbb{R}$.

Theorem 49. (Min-max principle). *Let $X \subset E$ and $Y \subset F$ be two nonempty compact general concave sets in the linear topological spaces E and F. Let $f : X \times Y \to \mathbb{R}$ satisfy: (a) $y \mapsto f(x, y)$ is lower semicontinuous and general quasi-concave for each fixed $x \in X$. (b) $x \mapsto f(x, y)$ is upper semicontinuous and general quasi-convex for each fixed $y \in Y$. Then the following equality holds in the form as*

$$\max_{x\in X}\min_{y\in Y} f(x, y) = \min_{y\in Y}\max_{x\in X} f(x, y).$$

The proof of this statement is a totally analogous with the proof of Theorem 22. In this proof we application of Theorem 48. Thus the proof we omit.

10. Geometry of general concavity

This paragraph is primarily devoted to illustring the use of the results and facts on general concavity in this paper. In this sense first we give a result for general concave sets.

Theorem 50. *Let* $K = \{x_1, \dots, x_m\}$ *be a finite set of points in the space* $\mathbb{R}^n$*. If* $n+2 \le m$*, then* K *can be division on two disjoint subsets* A *and* B *such that* $\text{g.conc}(A) \cap \text{g.conc}(B)$ *is a nonempty set.*

The proof of this statement we omit because which is it very analogous to the proof of Theorem 23. A proof of this statement may be found in T a s k o v i ć [44].

Theorem 51. (Statement of division). *Let* $K := \{K_1, \dots, K_m\}$ *be a finite family of general concave sets in* $\mathbb{R}^n$ *with* $m \ge n+1$*. If every subfamily with* $n+1$ *elements of the family* K *has a nonempty intersection, then* $K_1 \cap K_2 \cap \cdots\cdots \cap K_m$ *is a nonempty set.*

For the proof of this statement we applying Theorem 50. A proof of this statement may be found in: T a s k o v i ć [44].

An annotation. In connection with the preceding, let $\mathfrak{R}$ be a family of closed general concave sets in $\mathbb{R}^n$ with more or equally of $n+1$ elements. If at least one member of family $\mathfrak{R}$ is a compact set and if all $n+1$ members of family $\mathfrak{R}$ have a nonempty intersection, then the family $\mathfrak{R}$ has a nonempty intersection.

Proof. For given family $\mathfrak{R}$ set that is $\mathfrak{R} = \{K \cap R : K \in \mathfrak{R}\}$ where $R \in \mathfrak{R}$ a given compact set. The elements of family $\mathfrak{R}$ are compact sets (as general concave closed sets contained in the compact R). Application of Theorem 51 for the case of finite family of general concave sets, we obtain that the intersection of *centered family* $\mathfrak{R}$ is a nonempty set. The proof is complete. □

Otherwise, the preceding statement of finite division has a version on finite intersection in the following form as.

Indeed, let $\{K_1, \dots, K_m\}$ be a finite family of compact general concave sets K_i $(i = 1, \dots, m)$ in $\mathbb{R}^n$. Then the intersection of all this sets is nonempty if and only if the intersection of mostly $n+1$ $(\le m)$ sets K_i is nonempty.

An extension of Krasnoselskij's theorem. In this sense, the set $Z \subset \mathbb{R}^n$ with the respect on to point $p \in Z$ is called **general lower starred** iff for every point $x \in Z$ the following fact holds as

$$\mathfrak{D}(p, x) := \{G(x, p, \lambda) : \lambda \in [0, 1]\} \subset Z,$$

where $G(x, p, \lambda)$ is a general concave structure on Z. The set of all points of Z for which it is general lower starred is called *general lower kernel* of Z. Otherwise, if for two points a and $b \in Z$ is $G(a, b, \lambda) \subset Z$, then we recall that *from the point* b *to see the point* a *over the set* Z. We have the following result.

Theorem 52. *Let* $K \subset \mathbb{R}^n$ *be a compact set with least of all* $n+1$ *points. If for every subset of* $n+1$ *points of the set* K *there exists a point in* K *from which to see all* $n+1$ *points over* K*, then the set* K *is general lower starred.*

For the proof of this statement may be application Theorem 51. In this case the proof is analogous to the proof of Theorem 25. A proof of this statement may be found in: T a s k o v i ć [44].

Annotations. We notice that in the case $G(x, y, \lambda) = [x, y] := \{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}$ directly we obtain well-known result for starred sets in 1946 of K r a s n o s e l s k i j [30].

In connection with the preceding, two nonempty general concave sets C and D in an arbitrary linear space X are **general lower complementary** if they form a partition of X, that is, $C \cap D = \varnothing$ and $C \cup D = X$. In this sense we have the following result.

Theorem 53. *Let A and B be disjoint general concave subsets of an arbitrary linear space X. Then there exist general lower complementary general concave sets C and D in X such that $A \subset C$ and $B \subset D$.*

We notice that a proof of this statement may be found in: T a s k o v i ć [44]. The proof of this statement is a totally analogous with the proof of Theorem 26.

11. Two open problems

Introduction and history.[1] The most famous of many problems in nonlinear analysis is Schauder's problem (*Scottish book*, problem **54**) of the following form, that if C is a nonempty convex compact subset of a linear topological space does every continuous mapping $f : C \to C$ has a fixed point?

The answer we give in this part is yes. In this connection this part proves and extends the Markoff-Kakutani theorem to arbitrary linear topological space as an immediate consequence of the preceding solution of Schauder's problem.

During the last twenty years this old conjecture was intensively examined by many mathematicians. For sets in normed spaces this has been proved by Schauder and for sets in locally convex spaces by Tychonoff.

[1]**History of Schauder's problem.** The most famous of many open problems in nonlinear analysis is Schauder's problem (in *Scottish book*, problem **54**). For some answers on this problem see papers of: Tychonoff, Fréchet, Leray, Borsuk, Steinhaus, Mazurkiewicz, Kuratowski, Knaster, Krasnoselskij, Ky Fan, Klee, Caristi, Kirk, Browder, Dugundji, Granas, and many others.

This problem J. Schauder is in a practical way to set down in one's own papers in years 1927 and 1930 which he published in: *Math. Zaitschrift* and *Studia Mathematica*. But this problem is especially to gain importance because him personal to set down S t e f a n B a n a c h in 1930 on World congress of mathematicians at Moscow.

First positive answer for locally convex spaces in 1930 is by Tychonoff. He had shown that if C is a nonempty convex compact subset of a locally convex space, then every continuous map $f : C \to C$ has a fixed point. Interestingly, the paper of T y c h o n o f f is to present personal J. Schauder in Zbl. für Math. **12** (1936), with number **308**. There are several thousand papers in connection with this preceding facts.

In this part we give that if C is a nonempty convex compact subset of a linear topological space, then every continuous mapping $f : C \to C$ has a fixed point.

Brouwer's theorem of fixed point is one of the oldest and best known results in mathematics. Schauder's theorem of fixed point is a generalization of Brouwer's theorem to infinite dimensional normed linear spaces. Schauder's theorem states that every continuous mapping of a compact convex subset of a normed linear space into itself has a fixed point.

Schauder's problem (*Scottish book*, problem **54**) is the following form: *Does every continuous mapping $f : C \to C$ of a nonempty convex compact subset C in arbitrary linear topological space have a fixed point?*

For locally convex space the answer is yes from Theorem 18. Namely, in 1935, Tychonoff had shown that if C is a nonempty convex compact subset of a locally convex space, then every continuous map $f : C \to C$ has a fixed point.

Schauder's theorem was further extended by Hukuhara [1950], Mazur [1938], Gohde [1965], Fan [1961], Dugundji [1976], Granas [1957], Klee [1960], Kirk [1965], Idzik [1988], Riedrich [1976], Eisenack-Fenske [1978], Jahn [1984], Browder [1965], Darbo [1955], Delenau [1961], Sadovskij [1967], Krasnoselskij [1955], Reinermann [1971], Tasković [1991], and many others.

Literature on applications of Schauder's theorem to nonlinear problems is extensive. The first result was proved by Markoff [1936] with the aid of the Schauder-Tychonoff fixed point theorem.

Kakutani [1941] found a direct elementary proof of the Markoff theorem. Extensions of Markoff-Kakutani theorem is due to Day [1961], Hahn [1978] and Ryll-Nardzewski [1966]. All this references in this part are from the book by Tasković [44].

In Tasković [46] we give the complete solution of the preceding well known Schauder's problem fixed point. Also, this solution is answering a question of S. Ulam. In connection with this, in this part, we extend the Markoff-Kakutani theorem to arbitrary linear topological spaces as an immediate consequence of the preceding solution of Schauder's problem.

On the other hand, in this sense, we extend and connect former results of Brouwer, Schauder, Tychonoff, Markoff, Kakutani, Darbo, Sadovskij, Krasnoselskij, Browder, Ky Fan, Reinermann, Hahn, Ryll-Nardzewski, Granas, Dugundji, Hukuhara, Mazur, Riedrich, Jahn, Eisenack-Fenske, Day, and some others.

Answer to Schauder's problem is affirmative. From the preceding statements and some further facts we are now in the position to formulate the following fact which is, also, an extension of the former results of Brouwer, Schauder, Tychonoff, Mazur, Hukuhara, Ky Fan, Browder, Sadovskij, Darbo, Krasnoselskij, Reinermann, Dugundji, Granas, Klee, Idzik, Riedrich, Eisenack-Fenske, Jahn, and some others.

Theorem 54. (Answer is yes for Schauder's problem, T a s k o v i ć [46]). *Let C be a nonempty convex compact subset of a linear topological space X and suppose that $T : C \to C$ is a continuous mapping. Then T has a fixed point in C.*

In connection with the preceding facts and results on locally general convex spaces and on locally general concave spaces we have the following two problems:

Open problem 1. (For general convexity). Does every continuous mapping $f : C \to C$ of a nonempty general convex compact subset C in arbitrary linear topological space have a fixed point?

For locally general convex space the answer is yes from Theorem 18. In connection with this see: T a s k o v i ć [44].

Open problem 2. (For general concavity). *Does every continuous mapping $f : C \to C$ of a nonempty general concave compact subset C in arbitrary linear topological space have a fixed point*?!

For locally general concave space the answer is yes from Theorem 45. For further facts in connection with this result see: T a s k o v i ć [44]. A fact in connection with the preceding considered may be found in: T a s k o v i ć [45].

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