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Coincidence and Common Fixed Point Theorems for Hybrid Mappings

Abdelkrim Aliouche and Valeriu Popa

Abstract. We prove common fixed point theorems for two pairs of hybrid mappings satisfying implicit relations in metric spaces using the concept of $T-$weak commutativity and we correct errors of [1], [4], [5] and [12]. Our Theorems generalize results of [1-5], [12] [16], [17-20] and [26].

1. Introduction and Preliminaries

Let (X, d) be a metric space. For $x \in X$ and $A \subset X$, $D(x, A) = \inf \{d(x, y),\ y \in A\}$.

Let $CB(X)$ be the set of all nonempty closed and bounded subsets of X. Let H be the Hausdorff metric with respect to d defined by

$$H(A, B) = \max \left\{ \sup_{a \in A} D(a, B),\ \ \sup_{b \in B} D(A, b) \right\} \text{ for all } A, B \in CB(X).$$

It is well known that $(CB(X), H)$ is a metric space and if (X, d) is complete, then $(CB(X), H)$ is also complete

Lemma 1.1 ([14]). *If $A, B \in CB(X)$ and $k > 1$, then for each $a \in A$, there exists $b \in B$ such that $d(a, b) \leq kH(A, B)$.*

Let $f : X \to X$ be a single-valued mapping and $T : X \to CB(X)$ be a multi-valued mapping.

Definition 1.2. 1) A point $x \in X$ is said to be a coincidence point of f and T if $fx \in Tx$. We denote by $C(f, T)$ the set of all coincidence points of f and T.
2) A point $x \in X$ is a fixed point of T if $x \in Tx$.

Definition 1.3. 1) f and T are said to be commuting [4] in X if for all $x \in X$, $fTx \subset Tfx$.

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2) f and T are said to be weakly commuting on X [21, 22] if for all $x \in X$, $fTx \in CB(X)$ and

$$H(fTx, Tfx) \leq D(fx, Tx)$$

3) f and T are said to be compatible [7, 11] if for all $x \in X$, $fTx \in CB(X)$ and

$$\lim_{n\to\infty} H(fTx_n, Tfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$ for some $t \in X$ and $A \in CB(X)$.

Commuting implies weakly commuting implies compatible, but the converse is not true in general, see [9].

Definition 1.4. 1) f and T are said to be weakly compatible [8,9] if they commute at their coincidence points; i.e., $fx \in Tx$ implies that $fTx = Tfx$.

2) f and T are said to be R-weakly commuting at $x \in X$ [15, 23], if $fTx \in CB(X)$ and there exists an $R > 0$ such that

$$H(fTx, Tfx) \leq RD(fx, Tx) \tag{1.1}$$

f and T are said to be pointwise R-weakly commuting on X if for all $x \in X$, $fTx \in CB(X)$ and (1.1) holds for some $R > 0$.

3) f and T are said to be (IT)-commuting at $x \in X$ [25] if $fTx \subset Tfx$.

It is proved in [25] that a pointwise R-weakly commuting hybrid pair is not weakly compatible in general and IT-commutativity of f and T at a coincidence point is more general than their weak compatibility at the same point. However, pointwise R-weak commutativity at a coincidence point is equivalent to (IT) commutativity at this point.

Definition 1.5 ([10])**.** f is T-weakly commuting at $x \in X$ if $ffx \in Tfx$.

Remark 1.6. 1) For a hybrid pair (f, T), (IT) commuting at coincidence points implies that f is T-weakly commuting at these points, but T-weakly commuting hybrid pair is neither IT-commuting nor compatible nor weakly commuting nor weakly compatible in general, see [10].

2) If T is a single-valued mapping, then T-weak commutativity at coincidence points is equivalent to weak compatibility of f and T.

3) If f and T are single-valued maps then weak compatibility of f and T is equivalent to R-weak commutativity of f and T at their coincidence points.

Lemma 1.7. a) *If f is T-weakly commuting at $x \in X$, then $fx \in C(f, T)$.*

b) *If f is T-weakly commuting at $x \in X$ and $fx = ffx$, then fx is a common fixed point of f and T.*

The following Theorem was proved by [12].

Theorem 1.8. *Let* (X,d) *be a complete metric space,* $T : X \to X$ *and* $F,G : X \to CB(X)$ *satisfying*

$$F(X) \cup G(X) \subset T(X), \tag{1.2}$$

$$\begin{aligned} &T \text{ is } F\text{-weakly commuting and} \\ &T \text{ is } G\text{-weakly commuting at their coincidence points.} \end{aligned} \tag{1.3}$$

$$H(Fx,Gy) \leq a\frac{D^2(Fx,Ty)+D^2(Gy,Tx)}{D(Fx,Ty)+D(Gy,Tx)} + bd(Tx,Ty), \tag{1.4}$$

for all $x, y \in X$, $x \neq y$, $Fx \neq Fy$ *and* $Gx \neq Gy$, *where* $a, b > 0$ *and* $a + 2b < 1$, *whenever* $D(Fx,Ty) + D(Gy,Tx) \neq 0$ *and* $H(Fx,Gy) = 0$ *whenever* $D(Fx,Ty) + D(Gy,Tx) = 0$. *Then, there exists* $z \in X$ *such that* $z = Tz \in Fz \cap Gz$.

In [17] and [18], the study of fixed points for mappings satisfying implicit relations was introduced and the study of a pair of hybrid mappings satisfying implicit relations was initiated in [19].

It is our purpose in this paper to prove coincidence and common fixed point theorems for two pairs of hybrid mappings satisfying implicit relations which generalize results of [1-5], [12], [16], [17-20] and [26].

2. Implicit relations

Let Φ_6 the family of all real continuous mappings $\phi(t_1,t_2,t_3,t_4,t_5,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(ϕ_1) : ϕ is increasing in variable t_1 and decreasing in variables t_3, t_4, t_5 and t_6.

(ϕ_2) : there exists $0 \leq h < 1$ and $k > 1$ such that
  (ϕ_a) : $u \leq kt$ and $\phi(t,v,v,u,u+v,0) \leq 0$ or
  (ϕ_b) : $u \leq kt$ and $\phi(t,v,u,v,0,u+v) \leq 0$
  implies $u \leq hv$.

Example 2.1. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b(t_3+t_4) - c(t_5+t_6)$, $a, c > 0$, $b \geq 0$ and $a + 2b + 2c < 1$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $1 < k < \dfrac{1}{a+2b+2c}$, $u \leq kt$ and

$$\phi(t,v,v,u,u+v,0) = t - av - b(v+u) - c(u+v) \leq 0.$$

Then, $u \leq kt \leq u \leq kav + kb(v+u) + kc(u+v)]$ and so $u \leq hv$, where $h = \dfrac{k(a+b+c)}{1-(kb+kc)} < 1$. Similarly, $u \leq kt$ and $\phi(t,v,u,v,0,u+v) \leq 0$ implies $u \leq hv$.

Example 2.2. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a\max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}$, $0 < a < 1$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $1 < k < \frac{1}{a}$, $u \le kt$ and

$$\phi(t, v, v, u, u+v, 0) = t - a\max\left\{v, u, \frac{u+v}{2}\right\} \le 0.$$

Then,

$$u \le kt \le ka\max\left\{v, u, \frac{u+v}{2}\right\} \le ka\max\{v, u\}.$$

If $u > 0$ and $u \ge v$, it follows that $u \le kau < u$ which is a contradiction and so $u \le hv$, where $h = ka < 1$. If $u = 0$, therefore $u \le hv$. Similarly, $u \le kt$ and $\phi(t, v, u, v, 0, u+v) \le 0$ implies $u \le hv$.

Example 2.3. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a\max\{t_2^2, t_3t_4, t_5t_6, t_3t_5, t_4t_6\}^{\frac{1}{2}}$, $0 < a < \frac{1}{\sqrt{2}}$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $1 < k < \frac{1}{a\sqrt{2}}$, $u \le kt$ and

$$\phi(t, v, v, u, u+v, 0) = t - a\max\{v^2, uv, v(u+v)\}^{\frac{1}{2}} \le 0.$$

Then,

$$u \le kt \le ka\max\left\{v^2, uv, v(u+v)\right\}^{\frac{1}{2}}.$$

If $u > 0$ and $u \ge v$, it follows that $u \le ka\sqrt{2}u < u$ which is a contradiction and so $u \le hv$, where $h = ka\sqrt{2} < 1$. If $u = 0$, therefore $u \le hv$. Similarly, $u \le kt$ and $\phi(t, v, u, v, 0, u+v) \le 0$ implies $u \le hv$.

Example 2.4. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 + \frac{t_1}{1+t_5t_6} - at_2^2 - bt_3^2 - ct_4^2$, $a > 0$, $b, c \ge 0$ and $a + b + c < 1$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $1 < k < \frac{1}{\sqrt{a+b+c}}$, $u \le kt$ and

$$\phi(t, v, v, u, u+v, 0) = t^2 + t - av^2 - bv^2 - cu^2 \le 0.$$

Then, $t^2 \le av^2 + bv^2 + cu^2$ and $u^2 \le k^2t^2 \le k^2\left(av^2 + bv^2 + cu^2\right)$.

It follows that $u \le h_1 v$, where $h_1 = k\sqrt{\dfrac{a+b}{1-k^2c}} < 1$. Similarly, $u \le kt$ and $\phi(t,v,u,v,0,u+v) \le 0$ implies $u \le h_2 v$, where $h_2 = k\sqrt{\dfrac{a+c}{1-k^2b}} < 1$. If $h = \max\{h_1, h_2\}$, then $u \le hv$.

Example 2.5. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) =$
$= t^p - \max\{at_2t_3^{p-1}, at_2^{p-1}t_4, at_3^{p-1}t_4, ct_5^{p-1}t_6\}$, $p \ge 2$, $0 < a < 1$ and $c \ge 0$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $1 < k < \dfrac{1}{\sqrt[p]{a}}$, $u \le kt$ and

$$\phi(t,v,v,u,u+v,0) = t^p - \max\{av^p, av^{p-1}u\} \le 0.$$

Then, $u^p \le k^p t^p \le k^p \max\{av^p, av^{p-1}u\}$. If $u > 0$ and $u \ge v$, it follows that $u^p \le ak^p u^p < u^p$ which is a contradiction and so $u \le hv$, where $h = k\sqrt[p]{a} < 1$. If $u = 0$, therefore $u \le hv$. Similarly, $u \le kt$ and $\phi(t,v,u,v,0,u+v) \le 0$ implies $u \le hv$.

Example 2.6. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - b[a\max\{t_2,t_3,t_4,\dfrac{t_5+t_6}{2}\} -$
$-(1-a)\max\{t_2^2, t_3t_4, t_5t_6, \frac{1}{2}t_3t_6, \frac{1}{2}t_4t_5\}^{\frac{1}{2}}]$, $0 < b < 1$ and $0 \le a < 1$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $1 < k < \dfrac{1}{b}$, $u \le kt$ and

$$\phi(t,v,v,u,u+v,0) =$$
$$= t - b\left[a\max\left\{v,u,\frac{u+v}{2}\right\} - (1-a)\max\left\{v^2, uv, \frac{1}{2}u(u+v)\right\}^{\frac{1}{2}}\right] \le 0.$$

Then,

$$u \le kt \le kb\left[a\max\left\{v,u,\frac{u+v}{2}\right\} + (1-a)\max\left\{v^2, uv, \frac{1}{2}u(u+v)\right\}^{\frac{1}{2}}\right].$$

If $u > 0$ and $u \ge v$, it follows that $u \le kbu < u$ which is a contradiction and so $u \le hv$, where $h = kb < 1$. If $u = 0$, therefore $u \le hv$. Similarly, $u \le kt$ and $\phi(t,v,u,v,0,u+v) \le 0$ implies $u \le hv$.

Example 2.7. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b\dfrac{t_5^2+t_6^2}{t_5+t_6} - c(t_3+t_4)$, $t_5+t_6 \ne 0$, $a,b > 0$, $c \ge 0$ and $a+2b+2c < 1$.

Example 2.8. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b\dfrac{t_3^2+t_4^2}{t_3+t_4} - c(t_5+t_6)$, $t_3+t_4 \ne 0$, $a,b,c > 0$ and $a+2b+2c < 1$.

They follow as in Example 2.1 since $\frac{t_5^2+t_6^2}{t_5+t_6} \le t_5+t_6$ and $\frac{t_3^2+t_4^2}{t_3+t_4} \le t_3+t_4$ if $t_5+t_6 \neq 0$ and $t_3+t_4 \neq 0$.

3. Main Results

Theorem 3.1. *Let (X,d) be a metric space, $S,T : X \to X$ and $F,G : X \to CB(X)$ satisfying*

$$F(X) \subset T(X) \quad \text{and} \quad G(X) \subset S(X) \tag{3.1}$$

$$\phi\Big(H(Fx,Gy), d(Sx,Ty), D(Sx,Fx), D(Ty,Gy), D(Sx,Gy), D(Fx,Ty)\Big) \le 0 \tag{3.2}$$

for all $x,y \in X$, where $\phi \in \Phi_6$, whenever $D(Sx,Gy)+D(Fx,Ty) \neq 0$ and $H(Fx,Gy)=0$ whenever $D(Sx,Gy)+D(Fx,Ty)=0$. Suppose that one of $S(X)$ or $T(X)$ is complete. Then

a) *there exists $p,q \in X$ such that $Sp \in Fp$ and $Tq \in Gq$.*

Further, if S is F-weakly commuting and T is G-weakly commuting at their coincidence points, therefore

b) *There exists $z \in X$ such that $Sz \in Fz$ and $Tz \in Gz$.*
c) *In the case (b), if $Sz = Tz$, then $Sz = Tz \in Fz \cap Gz$.*
d) *In the case (c), if $Sz = Tz = z$, then z is a common fixed point of S,T,F and G.*

Proof. First, assume that there exists $p,q \in X$ such that $D(Sp,Gq) + D(Fp,Tq) = 0$. So, $D(Sp,Gq) = 0$ and $D(Fp,Tq) = 0$ which implies that $Sp \in Gq$ and $Tq \in Fp$. Since $H(Fp,Gq) = 0$, it follows that $D(Sp,Fp) \le H(Fp,Gq) = 0$ and hence $Sp \in Fp$. In a similar manner, we get $Tq \in Gq$.

Now, assume that $D(Sx,Gy) + D(Fx,Ty) \neq 0$ for all $x,y \in X$. Let $x_0 \in X$ be an arbitrary point. By (3.1) and Lemma 1.1, we define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} \in Gx_{2n-1}, \qquad y_{2n+1} = Tx_{2n+1} \in Fx_{2n}$$

and

$$d(y_{2n}, y_{2n+1}) \le kH(Fx_{2n}, Gx_{2n-1}),$$
$$d(y_{2n+1}, y_{2n+2}) \le kH(Fx_{2n}, Gx_{2n+1}), \quad \text{for} \quad n = 1,2,\ldots$$

Using (3.2) and (ϕ_1, we have

$$0 \geq \phi\Big(H(Fx_{2n}, Gx_{2n-1}), d(Sx_{2n}, Tx_{2n-1}), D(Sx_{2n}, Fx_{2n}),$$
$$D(Tx_{2n-1}, Gx_{2n-1}), D(Sx_{2n}, Gx_{2n-1}), D(Fx_{2n}, Tx_{2n-1})\Big)$$
$$\geq \phi\Big(H(Fx_{2n}, Gx_{2n-1}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}),$$
$$d(y_{2n-1}, y_{2n}), 0, d(y_{2n-1}, y_{2n+1})\Big).$$
$$\geq \phi\Big(H(Fx_{2n}, Gx_{2n-1}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}),$$
$$d(y_{2n-1}, y_{2n}), 0, d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\Big).$$

By (ϕ_b), we obtain

$$d(y_{2n}, y_{2n+1}) \leq hd(y_{2n-1}, y_{2n}).$$

In the same manner, applying (3.3) we get

$$0 \geq \phi\Big(H(Fx_{2n}, Gx_{2n+1}), d(Sx_{2n}, Tx_{2n+1}), D(Sx_{2n}, Fx_{2n}),$$
$$D(Tx_{2n+1}, Gx_{2n+1}), D(Sx_{2n}, Gx_{2n+1}), D(Fx_{2n}, Tx_{2n+1})\Big)$$
$$\geq \phi\Big(H(Fx_{2n}, Gx_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}),$$
$$d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}), 0\Big).$$

By (ϕ_a, we obtain

$$d(y_{2n+1}, y_{2n+2}) \leq hd(y_{2n}, y_{2n+1})$$

and hence

$$d(y_n, y_{n+1}) \leq hd(y_{n-1}, y_n).$$

Therefore, $\{y_n\}$ is a Cauchy sequence in X. As $S(X)$ is complete, it converges to $z \in S(X)$ and so there exists $p \in X$ such that $z = Sp$. Using (3.2) and (ϕ_1) we have

$$0 \geq \phi\Big(H(Fp, Gx_{2n-1}), d(Sp, Tx_{2n-1}), D(Sp, Fp),$$
$$D(Tx_{2n-1}, Gx_{2n-1}), D(Sp, Gx_{2n-1}), D(Fp, Tx_{2n-1})\Big)$$
$$\geq \phi\Big(D(Fp, y_{2n}), d(Sp, y_{2n-1}), D(Sp, Fp),$$
$$d(y_{2n-1}, y_{2n}), d(Sp, y_{2n}), D(y_{2n-1}, Fp)\Big).$$

Letting n tend to infinity, we get

$$\phi(D(Fp, Sp), 0, D(Fp, Sp), 0, 0, D(Fp, Sp)) \leq 0.$$

By (ϕ_b) we obtain $Sp \in Fp$. Similarly, as $F(X) \subset T(X)$, there exists $q \in X$ such that $z = Sp = Tq$. Applying (3.2) and (ϕ_1) we have

$$\begin{aligned} 0 &\geq \phi\Big(H(Fx_{2n}, Gq), d(Sx_{2n}, Tq), D(Sx_{2n}, Fx_{2n}), \\ &\qquad D(Tq, Gq), D(Sx_{2n}, Gq), D(Fx_{2n}, Tq)\Big) \\ &\geq \phi\Big(D(y_{2n+1}, Gq), d(y_{2n}, Tq), d(y_{2n}, y_{2n+1}), \\ &\qquad D(Tq, Gq), D(y_{2n}, Gq), d(y_{2n+1}, Tq)\Big) \end{aligned}$$

Letting n tend to infinity, we get

$$\phi(D(Tq, Gq), 0, 0, D(Tq, Gq), D(Tq, Fq), 0) \leq 0.$$

By (ϕ_a) we obtain $Tq \in Gq$. Since S is F-weakly commuting at $p \in C(S, T)$ and T is G-weakly commuting at $q \in C(G, T)$ it follows by Lemma 1.7 (a) that $z = Sp \in C(F, T)$ and $z = Tq \in C(G, T)$. Hence, $Sz \in Fz$ and $Tz \in Gz$. If $Sz = Tz$, then $Sz = Tz \in Fz \cap Gz$ and if $Sz = Tz = z$, then z is a common fixed point of S, T, F and G. □

Corollary 3.2. *Let (X, d) be a metric space, $S, T : X \to X$ and $F, G : X \to CB(X)$ satisfying* (3.1) *and*

$$H(Fx, Gy) \leq ad(Sx, Ty) + b(D(Sx, Fx) + D(Ty, Gy)) + c(D(Sx, Gy), D(Fx, Ty))$$

for all $x, y \in X$, where $a, c > 0$, $b \geq 0$ and $a + 2b + 2c < 1$. Suppose that one of $S(X)$ or $T(X)$ is complete. Then, the conclusions (a), (b), (c) and (d) of Theorem 3.1 hold.

Proof. It follows from Theorem 3.1 and Example 2.1. □

Remark 3.3. In Theorems of [1] and [12], to prove that $z = Tz$, the authors used: "$Tx_{2n} \in Gx_{2n-1}$ and $Tz \in Fz$ implies that $d(Tx_{2n}, Tz) \leq H(Gx_{2n-1}, Fz)$" which is false because "$a \in A$ and $b \in B$ implies $d(a, b) \leq H(A, B)$" is not true in general as it shown by the following example.

Example 3.4. Let $d(x, y) = |x - y|$, $A = [0, \frac{1}{2}]$ and $B = [\frac{1}{4}, 1]$. We have $0 \in A$ and $1 \in B$, but $d(0, 1) = 1 > H(A, B) = \frac{1}{2}$. Therefore, Theorem 1.8 of [12] is false as it is proved by the following Example.

Example 3.5. Let $(X, d) = ([1, \infty), |.|)$, $Sx = Tx = x^2 + 1$ and $Fx = Gx = [2, x + 3]$ for all $x \in X$. It is easy to verify that for all $x, y \in X$

$$\begin{aligned} d(Sx, Sy) &= \left|x^2 - y^2\right| \\ &\geq 2\,|x - y| \\ &= H(Fx, Fy) \end{aligned}$$

and hence

$$H(Fx, Fy) \leq \frac{1}{2}d(Sx, Sy)$$
$$\leq \frac{1}{2}d(Sx, Sy) + \frac{1}{8}\frac{D^2(Fx, Sy) + D^2(Sx, Fy)}{D(Fx, Sy) + D(Sx, Fy)}.$$

It is easy to see that the other conditions of Theorem 1.8 of [12] are satisfied, but S and F have no common fixed point.

The following Corollary is the correct form of Theorem 1.8 of [12].

Corollary 3.6. *Let* (X, d) *be a complete metric space,* $T : X \to X$ *and* $F, G : X \to CB(X)$ *satisfying* (1.2) *and*

$$H(Fx, Gy) \leq ad(Tx, Ty) + c\frac{D^2(Fx, Ty) + D^2(Tx, Gy)}{D(Fx, Ty) + D(Tx, Gy)}$$

for all $x, y \in X$*, where* $a, c > 0$ *and* $a + 2c < 1$*, whenever* $D(Tx, Gy) + D(Fx, Ty) \neq 0$ *and* $H(Fx, Gy) = 0$ *whenever* $D(Tx, Gy) + D(Fx, Ty) = 0$*. Then, (a) holds. Further, if* T *is* F*-weakly commuting and* T *is* G*-weakly commuting at their coincidence points, therefore the conclusions (b), (c) and (d) of Theorem 3.1 hold.*

Proof. It follows from The fact that

$$\frac{D^2(Fx, Ty) + D^2(Tx, Gy)}{D(Fx, Ty) + D(Tx, Gy)} \leq D(Fx, Ty) + D(Tx, Gy)$$

if $D(Tx, Gy) + D(Fx, Ty) \neq 0$ and Corollary 3.2. □

Remark 3.7. In [17] Remark 3 and [12] Remark 5, we have: "the conditions in the hypothesis of Theorem 3.1 of [1] and Theorem 1 of [12], $x \neq y, Fx \neq Fy$ and $Gx \neq Gy$ are necessary since the Theorem fails for F and G taken as constant mappings". This is demonstrated by the following example.

Example 3.8. Let $X = \{0, 1\}, Tx = 1 - x$ and $Fx = Gx = \{0, 1\}$ for all $x \in X$. It is easy to verify that the mappings satisfy all the hypothesis except $x \neq y, Fx \neq Fy$.

Remark 3.9.
1) In Example 3.8, we have $T(0) \in F(0)$ and $T(1) \in F(1)$; i.e., T and F have coincidence points. Since $T^2(0) \neq T(0)$ and $T^2(1) \neq T(1)$, T and F have no common fixed point.
2) In Theorems of [1], [4] and [12], $x \neq y, Fx \neq Fy$ and $Gx \neq Gy$ are not necessary as it is shown by the following Example.
3) In Theorem 1 of [26], S and g are compatible should be the pairs (S, f) and (T, G) are compatible and in Corollary 2, g should be replaced by f and the pair (S, f) is compatible should be added.
4) In the paper of Imdad and J. Ali [5], the condition (ϕ_b) should be added in (G_2) in order to prove that $d(y_{2n}, y_{2n+1}) \leq hd(y_{2n-1}, y_{2n})$

and the condition (G_3) should be deleted because it can be deduced by ϕ_b for $v = 0$.

5) In [16], the authors made the following remark . It is not yet known whether their Theorem remains true if one of the mappings f and T is not continuous and Theorem 2 of [25] yields that the answer is affirmative.

Example 3.10. Let $X = \{0, 1, \frac{1}{2}\}$, $Tx = 1-x$ and $Fx = Gx = \{0, \frac{1}{2}, 1\}$ for all $x \in X$. It is easy to verify that the mappings satisfy the conditions of Theorems of [1], [4] and [12] except $x \neq y$, $Fx \neq Fy$, but $T(\frac{1}{2}) = \frac{1}{2} \in F(\frac{1}{2})$ and so $\frac{1}{2}$ is a common fixed point of T and F.

As $x \neq y, Fx \neq Fy$ and $Gx \neq Gy$ are not necessary, it follows that Theorem of [1] and Theorems 3.2 and 3.3 of [4] part (a) are false, it suffices to take Example 3.8 for [1] and $X = \{0, 1\}, Tx = 1 - x, Sx = Ix = Jx = x$ and $Fx = Gx = \{0, 1\}$ for all $x \in X$ for [4].

We can also prove the following Theorem which generalizes Theorems 3.2 and 3.3 of [4].

Theorem 3.11. *Let (X, d) be a metric space, $S, T : X \to X$ and $F, G : X \to CB(X)$ satisfying*

$$F(X) \subset Tg(X) \text{ and } G(X) \subset Sf(X)$$

$$\phi\Big(H(Fx, Gy), d(Sfx, Tgy), D(Sfx, Fx), D(Tgy, Gy), D(Sfx, Gy), D(Fx, Tgy)\Big) \leq 0$$

for all $x, y \in X$, where $\phi \in \Phi_6$, whenever $D(Sfx, Gy) + D(Fx, Tgy) \neq 0$ and $H(Fx, Gy) = 0$ whenever $D(Sfx, Gy) + D(Fx, Tgy) = 0$. Suppose that one of $S(X)$ or $T(X)$ is complete. Then

a) *There exists $p, q \in X$ such that $Sfp \in Fp$ and $Tgq \in Gq$.*

Further, if Sf is F-weakly commuting and Tg is G-weakly commuting at their coincidence points, therefore

b) *There exists $z \in X$ such that $Sfz \in Fz$ and $Tgz \in Gz$.*

c) *In the case (b), if $Sfz = Tgz$, then $Sfz = Tgz \in Fz \cap Gz$.*

d) *In the case (c), if $Sfz = Tgz = z$, (S, f), (Sf, S), (T, g), (Tg, T) commute, $S^2z = Sz$, $f^2z = fz$, $T^2z = Tz$ and $g^2z = gz$, then z is a common fixed point of f, S, T, g, Sf, Tg, F and G.*

The following Theorem generalizes Theorems of Popa [17-20] and Imdad et al [3].

Theorem 3.12. *Let (X, d) be a metric space, $S, T : X \to X$ and $F, G : X \to CB(X)$ satisfying* (3.1) *and*

$$\phi(H(Fx, Gy), d(Sx, Ty), D(Sx, Fx), D(Ty, Gy), D(Sx, Gy), D(Fx, Ty)) \leq 0$$

for all $x, y \in X$, where $\phi \in \Phi_6$. Suppose that one of $S(X)$ or $T(X)$ is complete. Then, (a) holds. Further, if S is F-weakly commuting and T is G-weakly commuting at their coincidence points, therefore the conclusions (b), (c) and (d) of Theorem 3.1 hold.

In the same manner, we can prove the following Theorem which extend and improve Theorem 3.1 of Imdad and Ali [5].

Theorem 3.13. *Let $\{F_n\}_{n\geq 1}$ be a sequence of multi-valued mappings from a metric space (X, d) into $CB(X)$ and $S, T : X \to X$ satisfying*

$$F_i(X) \subset T(X) \text{ and } F_j(X) \subset S(X)$$

$$\phi\Big(H(F_ix, F_jy), d(Sx, Ty), D(Sx, F_ix), D(Ty, F_jy), D(Sx, F_jy), D(F_ix, Ty)\Big) \leq 0$$

for all $x, y \in X$, where $\phi \in \Phi_6$ and $i = 2n-1$, $j = 2n$, $n \geq 1$. Suppose that one of $S(X)$ or $T(X)$ is complete. Then

a) *There exists $p, q \in X$ such that $Sp \in F_ip$ and $Tq \in F_jq$.*

Further, if S is F_i-weakly commuting and T is F_j-weakly commuting at their coincidence points, therefore

b) *There exists $z \in X$ such that $Sz \in F_iz$ and $Tz \in F_jz$.*
c) *In the case (b), if $Sz = Tz$, then $Sz = Tz \in F_iz \cap F_jz$.*
d) *In the case (c), if $Sz = Tz = z$, then z is a common fixed point of S, T, F_i and G_j.*

Theorem 3.14. *Let $\{F_n\}_{n\geq 1}$ be a sequence of mappings from a metric space (X, d) into $CB(X)$ and $S, T : X \to X$ satisfying*

$$F_1(X) \subset T(X) \quad \text{and} \quad F_n(X) \subset S(X), \quad n > 1$$

$$\phi\Big(H(F_1x, F_ny), d(Sx, Ty), D(Sx, F_1x), D(Ty, F_ny), D(Sx, F_ny), D(F_1x, Ty)\Big) \leq 0$$

for all $x, y \in X$, where $\phi \in \Phi_6$, whenever $D(Sx, F_iy) + D(F_1x, Ty) \neq 0$ and $H(Fx, Gy) = 0$ whenever $D(Sx, F_iy) + D(F_1x, Ty) = 0$. Suppose that one of $S(X)$ or $T(X)$ is complete. Then

a) *There exists $p, q \in X$ such that $Sp \in F_1p$ and $Tq \in F_nq$, $n > 1$.*

Further, if S is F_1-weakly commuting and T is F_n-weakly commuting at their coincidence points for $n > 1$, therefore

b) *There exists $z \in X$ such that $Sz \in F_1z$ and $Tz \in F_iz$.*
c) *In the case (b), if $Sz = Tz$, then $Sz = Tz \in F_1z \cap F_iz$.*
d) *In the case (c), if $Sz = Tz = z$, then z is a common fixed point of T_n, F and G.*

The following Theorem generalizes Theorems of Popa [17-20], Imdad et al [3] and Djoudi and Aliouche [2].

Theorem 3.15. *Let $\{F_n\}_{n\geq 1}$ be a sequence of multi-mappings from a metric space (X,d) into $CB(X)$ and $S,T : X \to X$ satisfying*

$$F_1(X) \subset T(X) \quad \text{and} \quad F_n(X) \subset S(X), \quad n > 1$$

$$\phi\Big(H(F_1x, F_ny), d(Sx,Ty), D(Sx,F_1x), D(Ty,F_ny), D(Sx,F_ny), D(F_1x,Ty)\Big) \leq 0$$

for all $x, y \in X$, where $\phi \in \Phi_6$. Suppose that one of $S(X)$ or $T(X)$ is complete. Then, (a) holds. Further, if S is F_1-weakly commuting and T is F_n-weakly commuting at their coincidence points for $n > 1$, therefore, the conclusions (b), (c) and (d) of Theorem 3.14 hold.

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Abdelkrim Aliouche
Department of Mathematics
University of Larbi Ben M'Hidi
Oum-El-Bouaghi, 04000
Algeria
E-mail address: alioumath@yahoo.fr

Valeriu Popa
Department of Mathematics
University of Bacău
Str. Spiru Haret nr. 8
600114 Bacău
Romania
E-mail address: vpopa@ub.ro