Mathematica Moravica
Vol. **11** (2007), 9–16

Common Fixed Point Theorems of Meir and Keeler Type for Weakly Compatible Maps

H. Bouhadjera and Brian Fisher

Abstract. The subject of this paper is to prove a common fixed point theorem for four weakly compatible maps which extends and improves Theorem 1 of [2] and others by removing the assumption of continuity and relaxing the property of compatibility to weak compatibility. Also we give another extension of the same theorem.

1. Introduction and preliminaries

In 1986, Jungck [3] introduced generalized commuting maps, called compatible maps, which are more general than the concept of weakly commuting maps. Let $\mathcal{S}$ and $\mathcal{T}$ be two self maps of a metric space $(\mathcal{X}, d)$. He defines $\mathcal{S}$ and $\mathcal{T}$ to be compatible if

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n) = 0 \tag{1}$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t$ for some $t \in \mathcal{X}$. This concept has been useful as a tool for obtaining more comprehensive fixed point theorems. In general, commuting maps are weakly commuting and weakly commuting maps are compatible, but the converses are not necessarily true ([3]).

In 1993, G. Jungck, P.P. Murthy and Y.J. Cho [6] introduced the concept of compatible mappings of type (A) as follows: $\mathcal{S}$ and $\mathcal{T}$ above are compatible of type (A) if in lieu of (1), we have

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n) = 0.$$

The notions of compatible and compatible maps of type (A) are independent (see [6]).

In 1995, H. K. Pathak and M. S. Khan [8] gave a generalization of compatible maps of type (A) by introducing the concept of compatible maps of type (B).

2000 *Mathematics Subject Classification.* Primary: 7H10, 54H25.
Key words and phrases. Weakly compatible maps, compatible and compatible maps of type (A), (B), (P) and (C), common fixed point theorems.

$\mathcal{S}$ and $\mathcal{T}$ above are said to be compatible of type (B) if instead of (1) we have the two inequalities

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n)\right]$$

and

$$\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{T}t) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{T}^2x_n)\right].$$

This definition is equivalent to the concept of compatible mappings of type (A) under some conditions.

In [7] Pathak et al. introduced the compatibility of type (P) and compared with the compatibility and the compatibility of type (A). They define $\mathcal{S}$ and $\mathcal{T}$ above to be compatible of type (P) if in place of (1), we have

$$\lim_{n\to\infty} d(\mathcal{S}^2x_n, \mathcal{T}^2x_n) = 0.$$

In 1998, Pathak, Cho, Kang and Madharia [8] introduced a new extension of compatible maps of type (A) by giving the notion of compatible maps of type (C). They defined $\mathcal{S}$ and $\mathcal{T}$ above to be compatible of type (C) if we have in lieu of (1) the two inequalities:

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{T}^2x_n) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n)\right]$$

and

$$\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{T}t) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{S}^2x_n) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{T}^2x_n)\right].$$

As is obvious from the definitions, compatible maps of type (B) and (C) generalize those of compatible maps of type (A).

Recently, Jungck gave in his paper [4] a new generalization of the compatibility by introducing the concept of weakly compatible mappings. He defines $\mathcal{S}$ and $\mathcal{T}$ above to be weakly compatible if $\mathcal{S}t = \mathcal{T}t$, $t \in \mathcal{X}$ implies $\mathcal{S}\mathcal{T}t = \mathcal{T}\mathcal{S}t$.

It is clear that every compatible (compatible of type (A), (B), (P), (C)) pair of mappings is weakly compatible. The following example is an example of weakly compatible maps which are not compatible (compatible of type (A), (B), (P), (C)).

Example 1.1. Let $\mathcal{X} = [0, \infty)$ be with the absolute value metric. Define mappings $\mathcal{S}$, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{S}x = \begin{cases} x & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1, \\ 3 & \text{if } x \in (1,\infty), \end{cases} \qquad \mathcal{T}x = \begin{cases} 2-x & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1, \\ 6 & \text{if } x \in (1,\infty). \end{cases}$$

$\mathcal{S}$ and $\mathcal{T}$ are weakly compatible since they commute at their coincidence point $t = 1$. We assert that $\mathcal{S}$ and $\mathcal{T}$ are neither compatible nor compatible of type (A), (B), (P) and (C). For that purpose let us suppose that $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $x_n = 1 - \frac{1}{n}$ for $n \in \mathbb{N}^*$. We have

$$\begin{aligned} \mathcal{S}x_n &= x_n \to 1; & \mathcal{T}x_n &= 2 - x_n \to 1 \text{ as } n \to \infty, \\ \mathcal{S}\mathcal{T}x_n &= \mathcal{S}(2 - x_n) = 3; & \mathcal{T}\mathcal{S}x_n &= \mathcal{T}(x_n) = 2 - x_n, \\ \mathcal{S}\mathcal{S}x_n &= \mathcal{S}(x_n) = x_n; & \mathcal{T}\mathcal{T}x_n &= \mathcal{T}(2 - x_n) = 6. \end{aligned}$$

Thus we have

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n) = \lim_{n\to\infty} |3 - 2 + x_n| = 2 \neq 0$$

and so the pair $\{\mathcal{S}, \mathcal{T}\}$ is not compatible. Also, we have

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) = \lim_{n\to\infty} |3 - 6| = 3 \neq 0,$$

which implies that the pair $\{\mathcal{S}, \mathcal{T}\}$ is not compatible of type (A). Further, we have

$$\begin{aligned} 3 &= \lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) \\ &\nleq \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n)\right] \\ &= \frac{1}{2}\left[\lim_{n\to\infty} |3 - 1| + \lim_{n\to\infty} |1 - x_n|\right] = 1, \end{aligned}$$

and so $\mathcal{S}$ and $\mathcal{T}$ are not compatible of type (B).

Again, we have

$$\lim_{n\to\infty} d(\mathcal{S}^2x_n, \mathcal{T}^2x_n) = \lim_{n\to\infty} |x_n - 6| = 5 \neq 0,$$

which tells us that $\mathcal{S}$ and $\mathcal{T}$ are noncompatible of type (P).

Finally, we have

$$\begin{aligned} 3 &= \lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) \\ &\nleq \frac{1}{3}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{T}^2x_n)\right] \\ &= \frac{1}{3}\left[\lim_{n\to\infty} |3 - 1| + \lim_{n\to\infty} |1 - x_n| + \lim_{n\to\infty} |1 - 6|\right] = \frac{7}{3}, \end{aligned}$$

and so the pair $\{\mathcal{S}, \mathcal{T}\}$ is noncompatible of type (C).

For our main results we will need the following lemma given in [3].

Lemma 1.1. *Let* $(\mathcal{A}, \mathcal{S})$ *and* $(\mathcal{B}, \mathcal{T})$ *be self mappings of a complete metric space* $(\mathcal{X}, d)$ *such that* $\mathcal{A}(\mathcal{X}) \subset \mathcal{T}(\mathcal{X})$ *and* $\mathcal{B}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$. *Assume further that given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for all* x, y *in* $\mathcal{X}$

$$\epsilon \leq M(x,y) < \epsilon + \delta \Rightarrow d(\mathcal{A}x, \mathcal{B}y) < \epsilon$$

and

$$d(\mathcal{A}x, \mathcal{B}y) < M(x,y), \text{ whenever } M(x,y) > 0,$$

where

$$M(x,y) = \max \{d(\mathcal{S}x, \mathcal{T}y), d(\mathcal{A}x, \mathcal{S}x), d(\mathcal{B}y, \mathcal{T}y), [d(\mathcal{S}x, \mathcal{B}y) + d(\mathcal{A}x, \mathcal{T}y)]\}.$$

Then for each x_0 *in* $\mathcal{X}$, *the sequence* $\{y_n\}$ *in* $\mathcal{X}$ *defined by the rule*

$$y_{2n} = \mathcal{A}x_{2n} = \mathcal{T}x_{2n+1} \text{ and } y_{2n+1} = \mathcal{B}x_{2n+1} = \mathcal{S}x_{2n+2} \text{ for } n \in \mathbb{N}.$$

is a Cauchy sequence.

The following theorem was stated in [3].

Theorem 1.1. *Let* $(\mathcal{A}, \mathcal{S})$ *and* $(\mathcal{B}, \mathcal{T})$ *be compatible pairs of self mappings of a complete metric space* $(\mathcal{X}, d)$ *such that*

(*i*) $\mathcal{A}(\mathcal{X}) \subset \mathcal{T}(\mathcal{X})$ *and* $\mathcal{B}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$,

(*ii*) *given* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for all* x, y *in* $\mathcal{X}$,

$$\epsilon \leq M(x,y) < \epsilon + \delta \Rightarrow d(\mathcal{A}x, \mathcal{B}y) < \epsilon$$

and

(*iii*) $d(\mathcal{A}x, \mathcal{B}y) <$

$$< k\big[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{A}x, \mathcal{S}x) + d(\mathcal{B}y, \mathcal{T}y) + d(\mathcal{S}x, \mathcal{B}y) + d(\mathcal{A}x, \mathcal{T}y)\big]$$

for all x, y *in* $\mathcal{X}$ *where* $0 \leq k < 1/3$.

If one of the mappings $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ *and* $\mathcal{T}$ *is continuous, then* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ *and* $\mathcal{T}$ *have a unique common fixed point.*

Note that Theorem 1.1 is not correct as it stands since if $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ had a fixed point z, then (*iii*) could not hold when $x = y = z$. The inequality $<$ in (*iii*) should be replaced by $\leq$.

2. Main results

Theorem 2.1. *Let* $(\mathcal{A}, \mathcal{S})$ *and* $(\mathcal{B}, \mathcal{T})$ *be weakly compatible pairs of self mappings of a complete metric space* $(\mathcal{X}, d)$ *such that*

(*a*) $\mathcal{A}(\mathcal{X}) \subset \mathcal{T}(\mathcal{X})$ *and* $\mathcal{B}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$,

(*b*) *one of* $\mathcal{A}(\mathcal{X})$, $\mathcal{B}(\mathcal{X})$, $\mathcal{S}(\mathcal{X})$ *or* $\mathcal{T}(\mathcal{X})$ *is closed,*

(*c*) *given* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for all* x, y *in* $\mathcal{X}$,

$$\epsilon \leq M(x,y) < \epsilon + \delta \Rightarrow d(\mathcal{A}x, \mathcal{B}y) < \epsilon$$

where

$$M(x,y) = \max \{d(\mathcal{S}x, \mathcal{T}y), d(\mathcal{A}x, \mathcal{S}x), d(\mathcal{B}y, \mathcal{T}y), [d(\mathcal{S}x, \mathcal{B}y) + d(\mathcal{A}x, \mathcal{T}y)]/2\}$$

and

(*d*) $d(\mathcal{A}x, \mathcal{B}y) \leq$

$$\leq k \max\{d(\mathcal{S}x, \mathcal{T}y), d(\mathcal{A}x, \mathcal{S}x), d(\mathcal{B}y, \mathcal{T}y), d(\mathcal{S}x, \mathcal{B}y), d(\mathcal{A}x, \mathcal{T}y)\}$$

for all x, y *in* X *where* $0 \leq k < 1$.

Then, $\mathcal{A}$*,* $\mathcal{B}$*,* $\mathcal{S}$ *and* $\mathcal{T}$ *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in $\mathcal{X}$, then, since (a) holds, we can define inductively a sequence

$$\{\mathcal{A}x_0, \mathcal{B}x_1, \mathcal{A}x_2, \mathcal{B}x_3, \ldots, \mathcal{A}x_{2n}, \mathcal{B}x_{2n+1}, \ldots\} \tag{2}$$

such that

$$y_{2n} = \mathcal{A}x_{2n} = \mathcal{T}x_{2n+1} \text{ and } y_{2n+1} = \mathcal{B}x_{2n+1} = \mathcal{S}x_{2n+2} \text{ for } n \in \mathbb{N}.$$

By Lemma 1.1, it follows that $\{y_n\}$ is a Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is complete, $\{y_n\}$ converges to some element $z \in \mathcal{X}$, as do the subsequences $\{\mathcal{A}x_{2n}\} = \{\mathcal{T}x_{2n+1}\}$, $\{\mathcal{S}x_{2n}\} = \{\mathcal{B}x_{2n-1}\}$ and $\{\mathcal{S}x_{2n+2}\} = \{\mathcal{B}x_{2n+1}\}$.

Suppose that $\mathcal{A}(\mathcal{X})$ is closed. Then since $\mathcal{A}(\mathcal{X}) \subset \mathcal{T}(\mathcal{X})$, there exists a point $u \in \mathcal{X}$ such that $z = \mathcal{T}u$. Using the inequality (d), we have

$$d(\mathcal{A}x_{2n}, \mathcal{B}u) \leq k \max\{d(\mathcal{S}x_{2n}, \mathcal{T}u), d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}), d(\mathcal{B}u, \mathcal{T}u),$$
$$d(\mathcal{S}x_{2n}, \mathcal{B}u), d(\mathcal{A}x_{2n}, \mathcal{T}u)\}.$$

Letting $n \to \infty$, we get

$$d(z, \mathcal{B}u) \leq kd(z, \mathcal{B}u),$$

which is a contradiction. Thus, $z = \mathcal{T}u = \mathcal{B}u$ and by the weak compatibility of $(\mathcal{B}, \mathcal{T})$, it follows that $\mathcal{B}\mathcal{T}u = \mathcal{T}\mathcal{B}u$ and so $\mathcal{B}z = \mathcal{B}\mathcal{T}u = \mathcal{T}\mathcal{B}u = \mathcal{T}z$.

We claim that z is a common fixed point of $\mathcal{B}$ and $\mathcal{T}$. Assume not. Then by inequality (d), we obtain

$$d(\mathcal{A}x_{2n}, \mathcal{B}z) \leq k \max\{d(\mathcal{S}x_{2n}, \mathcal{T}z), d(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}), d(\mathcal{B}z, \mathcal{T}z),$$
$$d(\mathcal{S}x_{2n}, \mathcal{B}z), d(\mathcal{A}x_{2n}, \mathcal{T}z)\}.$$

Letting n tends to infinity, it gives

$$d(z, \mathcal{B}z) \leq kd(z, \mathcal{B}z),$$

which implies that $z = \mathcal{B}z = \mathcal{T}z$.

Now, since $\mathcal{B}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, there exists a point $v \in \mathcal{X}$ such that $z = \mathcal{S}v$. Then, from inequality (d), we have

$$d(\mathcal{A}v, \mathcal{B}z) \leq k \max\{d(\mathcal{S}v, \mathcal{T}z), d(\mathcal{A}v, \mathcal{S}v), d(\mathcal{B}z, \mathcal{T}z), d(\mathcal{S}v, \mathcal{B}z), d(\mathcal{A}v, \mathcal{T}z)\}.$$

It follows that

$$d(\mathcal{A}v, z) \leq kd(\mathcal{A}v, z),$$

a contradiction, which implies that $\mathcal{A}v = z$. Also, since $\mathcal{A}v = \mathcal{S}v = z$ and by the weak compatibility of $\mathcal{A}$ and $\mathcal{S}$, it follows that $\mathcal{S}\mathcal{A}v = \mathcal{A}\mathcal{S}v$ and so $\mathcal{S}z = \mathcal{S}\mathcal{A}v = \mathcal{A}\mathcal{S}v = \mathcal{A}z$.

Again the use of inequality (d) gives

$$d(\mathcal{A}z, \mathcal{B}z) \leq k\{d(\mathcal{S}z, \mathcal{T}z), d(\mathcal{A}z, \mathcal{S}z), d(\mathcal{B}z, \mathcal{T}z), d(\mathcal{S}z, \mathcal{B}z), d(\mathcal{A}z, \mathcal{T}z)\},$$

i.e.

$$d(\mathcal{A}z, z) \leq kd(\mathcal{A}z, z).$$

Consequently, we have $\mathcal{A}z = z = \mathcal{S}z$. Hence, z is a common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$.

Finally, we prove the uniqueness of z. Indeed, suppose that w is a second distinct common fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$. Then again using inequality (d), we get

$$d(\mathcal{A}z, \mathcal{B}w) \leq k\{d(\mathcal{S}z, \mathcal{T}w), d(\mathcal{A}z, \mathcal{S}z), d(\mathcal{B}w, \mathcal{T}w), d(\mathcal{S}z, \mathcal{B}w), d(\mathcal{A}z, \mathcal{T}w)\}$$

that is

$$d(z, w) \leq kd(z, w),$$

a contradiction, which implies that $w = z$.

Similarly, we can obtain this conclusion by supposing $\mathcal{B}(\mathcal{X})$ (resp. $\mathcal{S}(\mathcal{X})$, $\mathcal{T}(\mathcal{X})$) is closed. □

Now, we give our second main result which is another generalization of Theorem 1 of [2] and for this, we need the following:

Lemma 2.1 ([4] (resp. [6], [7], [9])). *Let $\mathcal{S}$ and $\mathcal{T}$ be compatible and compatible of type (A) (resp. (B), (P)) self mappings of a metric space $(\mathcal{X}, d)$. If $\mathcal{S}t = \mathcal{T}t$ for some $t \in \mathcal{X}$, then $\mathcal{S}\mathcal{T}t = \mathcal{T}\mathcal{S}t$.*

Proposition 2.1 ([1]). *Let $\mathcal{S}$ and $\mathcal{T}$ be compatible mappings of type (C) from a metric space $(\mathcal{X}, d)$ into itself. Suppose that $\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t$ for some $t \in \mathcal{X}$. Then we have the following:*

(1) $\lim_{n\to\infty} \mathcal{T}\mathcal{T}x_n = \mathcal{S}t$ *if $\mathcal{S}$ is continuous at t,*
(2) $\lim_{n\to\infty} \mathcal{S}\mathcal{S}x_n = \mathcal{T}t$ *if $\mathcal{T}$ is continuous at t,*
(3) $\mathcal{S}\mathcal{T}t = \mathcal{T}\mathcal{S}t$ *and $\mathcal{S}t = \mathcal{T}t$ if $\mathcal{S}$ and $\mathcal{T}$ are continuous at t.*

Theorem 2.2. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ be mappings from a complete metric space $(\mathcal{X}, d)$ into itself satisfying conditions (a), (b), (c) and (d) of Theorem 2.1. Further, if the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are compatible and compatible of type (A) (resp. (B), (P) and (C)), then the four mappings have a unique common fixed point $z \in \mathcal{X}$.*

Proof. Define the sequence $\{y_n\}$ as in the proof of Theorem 2.1. Then $\{y_n\}$ is a Cauchy sequence in $\mathcal{X}$ and converges with its subsequences to $z \in \mathcal{X}$. Suppose that $\mathcal{B}(\mathcal{X})$ is closed. Then since $\mathcal{B}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, there exists an element $u \in \mathcal{X}$ such that $z = \mathcal{S}u$. Using (d) we obtain

$$d(\mathcal{A}u, \mathcal{B}x_{2n+1}) \leq k \max\{d(\mathcal{S}u, \mathcal{T}x_{2n+1}), d(\mathcal{A}u, \mathcal{S}u), d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}),$$
$$d(\mathcal{S}u, \mathcal{B}x_{2n+1}), d(\mathcal{A}u, \mathcal{T}x_{2n+1})\}.$$

By letting $n \to \infty$ in the above inequality, it gives

$$d(\mathcal{A}u, z) \leq kd(\mathcal{A}u, z),$$

which is a contradiction. Thus, $\mathcal{A}u = z = \mathcal{S}u$. But $\mathcal{A}$ and $\mathcal{S}$ are compatible of type (A) (resp. (B), (P)), then by Lemma 2.1 it follows that $\mathcal{A}\mathcal{S}u = \mathcal{S}\mathcal{A}u$ and thus, $\mathcal{A}z = \mathcal{S}z$.

We claim that z is a fixed point of $\mathcal{A}$ and $\mathcal{S}$. Suppose not, then by assumption (d), we get

$$d(\mathcal{A}z, \mathcal{B}x_{2n+1}) \leq k \max\{d(\mathcal{S}z, \mathcal{T}x_{2n+1}), d(\mathcal{A}z, \mathcal{S}z), d(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}), d(\mathcal{S}z, \mathcal{B}x_{2n+1}), d(\mathcal{A}z, \mathcal{T}x_{2n+1})\}.$$

Therefore as $n \to \infty$

$$d(\mathcal{A}z, z) \leq kd(\mathcal{A}z, z),$$

a contradiction which implies that $z = \mathcal{A}z = \mathcal{S}z$.

Now, since $\mathcal{A}(\mathcal{X}) \subset \mathcal{T}(\mathcal{X})$, then there exists a point $v \in \mathcal{X}$ such that $z = \mathcal{A}z = \mathcal{T}v$. The use of inequality (d) gives

$$d(\mathcal{A}z, \mathcal{B}v) \leq k \max\{[d(\mathcal{S}z, \mathcal{T}v), d(\mathcal{A}z, \mathcal{S}z), d(\mathcal{B}v, \mathcal{T}v), d(\mathcal{S}z, \mathcal{B}v), d(\mathcal{A}z, \mathcal{T}v)\}.$$

It follows that

$$d(z, \mathcal{B}v) \leq kd(z, \mathcal{B}v),$$

which is a contradiction, so we have $\mathcal{B}v = z = \mathcal{T}v$. Since $\mathcal{B}$ and $\mathcal{T}$ are compatible, compatible of type (A) (resp. (B), (P)), by Lemma 2.1 it follows that $\mathcal{B}\mathcal{T}v = \mathcal{T}\mathcal{B}v$ that is $\mathcal{B}z = \mathcal{T}z$.

Using (d) again, we get

$$d(\mathcal{A}z, \mathcal{B}z) \leq k \max\{d(\mathcal{S}z, \mathcal{T}z), d(\mathcal{A}z, \mathcal{S}z), d(\mathcal{B}z, \mathcal{T}z), d(\mathcal{S}z, \mathcal{B}z), d(\mathcal{A}z, \mathcal{T}z)\}$$

that is

$$d(z, \mathcal{B}z) \leq kd(z, \mathcal{B}z).$$

Consequently, we have $z = \mathcal{B}z = \mathcal{T}z$. Hence, $\mathcal{A}z = \mathcal{S}z = z = \mathcal{B}z = \mathcal{T}z$, and so z is a common fixed point of the four mappings.

Finally, we prove that z is unique. Suppose that w is a second distinct common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$. Then, using inequality (d), we obtain

$$d(\mathcal{A}z, \mathcal{B}w) \leq k \max\{d(\mathcal{S}z, \mathcal{T}w), d(\mathcal{A}z, \mathcal{S}z), d(\mathcal{B}w, \mathcal{T}w), d(\mathcal{S}z, \mathcal{B}w), d(\mathcal{A}z, \mathcal{T}w)\}$$

i.e.,

$$d(z, w) \leq kd(z, w).$$

Therefore, $z = w$.

Similarly, one can obtain this conclusion by supposing $\mathcal{A}(\mathcal{X})$, $\mathcal{S}(\mathcal{X})$ or $\mathcal{T}(\mathcal{X})$ is closed.

For compatibility of type (C), we use the same proof and condition (3) of Proposition 2.1. □

References

[1] H. Bouhadjera, *General common fixed point theorems for compatible mappings of type (C)*, Sarajevo J. Math., **1**(14)(2005), 261–270.

[2] J. Jachymski, *Common fixed point theorems for some families of mappings*, Indian J. Pure Appl. Math, **25**(1994), 925–937.

[3] K. Jha, R.P. Pant and S.L. Singh, *Common fixed points for compatible mappings in metric spaces*, Radovi Mat., **12**(2003), 107–114.

[4] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci., **9**(1986), 771-779.

[5] G. Jungck, *Common fixed points for non-continuous nonself mappings on a nonnumeric spaces*, Far East J. Math. Sci., **4**(2)(1996), 199–212.

[6] G. Jungck, P.P. Murthy and Y.J. Cho, *Compatible mappings of type* (A) *and common fixed points*, Math. Japon., **36**(1993), 381–390.

[7] H.K. Pathak, Y.J. Cho, S.M. Kang and B.S. Lee, *Fixed point theorems for compatible mappings of type* (P) *and applications to dynamic programming*, Le Matematiche, (Fasc. I)(50)(1995), 15–30.

[8] H.K. Pathak, Y.J. Cho, S.M. Kang, B. Madharia, *Compatible mappings of type* (C) *and common fixed point theorems of Greguš type*, Demonstratio Mathematica, **XXXI**(3)(1998), 499–518.

[9] H.K. Pathak and M.S. Khan, *Compatible mappings of type* (B) *and common fixed point theorems of Greguš type*, Czechoslovak Math. J., **45**(120)(1995), 685–698.

H. Bouhadjera
Département de Mathématiques
Faculté des Sciences
Université Badji Mokhtar B.P. 12
23000 Annaba
Algerie

E-mail address: b_hakima2000@yahoo.fr

Brian Fisher
Department of Mathematics
Leicester University
Leicester, LE1 7RH
England

E-mail address: fbr@le.ac.uk