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On Fefferman's Non-existence Problems

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Abstract. Fefferman has propounded four open Cauchy problems A, B, C and D in Navier-Stokes equations. The problems A and B refer to sufficient condition for the existence of solutions, and the problems C and D to the ones for the non-existence of solutions. Here, we shall answer to the problems C and D.

1. Introduction

The Euler and Navier-Stokes equations describe the motion of a fluid in $\mathbb{R}^n$ (n=2 or n=3) ([2], [5]). The steady-state of 3-dimensional case of Navier-Stokes system of equations has been considered as a boundary value problem (BVP) in [1]. This BVP for stationary case is associated to flows with free surfaces, flows around bodies, channels and wakes behind bodies. In all these problems Navier-Stokes equations are investigated over (finite or infinite) domain with boundary conditions determined by physical considerations. But for non-stationary problems one needs besides the boundary conditions suitable initial conditions. In this cases the Navier-Stokes equations are to be solved for an unknown velocity vector $u(x,t) = (u_i(x,t))_{1\leq i\leq n} \in \mathbb{R}^n$ and pressure $p(x,t) \in \mathbb{R}$, defined for space-variable $x \in \mathbb{R}^n$ and time $t \geq 0$. We restrict attention here to incompressible fluids filling all of $\mathbb{R}^n$. The Navier-Stokes equations are then given by

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \quad (x \in \mathbb{R}^n, t \geq 0) \tag{1}$$

$$\text{div } u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \quad (x \in \mathbb{R}^n, t \geq 0) \tag{2}$$

with initial conditions

$$u(x,0) = u^0(x) \quad (x \in \mathbb{R}^n). \tag{3}$$

Here, $u^0(x)$ is a given C^∞ divergence-free vector field on $\mathbb{R}^n$, $f_i(x,t)$ are the components of a given externally applied force (e.g. gravity), ν is a positive coefficient (the viscosity) and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the space variables.

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Equation (1) is just Newton's gravity low, that is $f = ma$, for a fluid element subject to the external force $f = (f_i(x,t))_{1\le i\le n}$ and to the forces arising from pressure and friction. Equation (2) just says that fluid is incompressible. For physically requirements, we want to make sure $u(x,t)$ does now grow large as $|x| \longrightarrow \infty$. Hence, we will restrict to forces f and initial conditions u^0 that satisfy

$$|\partial_x^\alpha u^0(x)| \le C_{\alpha K}(1+|x|)^{-K} \quad \text{on } \mathbb{R}^n, \text{ for any } \alpha \text{ and } K \tag{4}$$

and

$$|\partial_x^\alpha \partial_t^m f(x,t)| \le C_{\alpha m K}(1+|x|+t)^{-K} \text{ on } \mathbb{R}^n \times [0,\infty), \text{ for any } \alpha, m, K. \tag{5}$$

For physical considerations we suppose a solution of (1), (2) and (3) which satisfies

$$p, u \in C^\infty(\mathbb{R}^n \times [0,\infty)) \tag{6}$$

and

$$\int_{\mathbb{R}^n} |u(x,t)|^2 \, \mathrm{d}\, x < C, \quad \text{for all } t \ge 0 \text{ (boundary energy)}. \tag{7}$$

Consequently, to rule out problems at infinity, we may look for spatially periodic solutions of (1), (2) and (3). Thus, we assume that $u^0(x)$, $f(x,t)$ satisfy the following periodic conditions

$$u^0(x+e_j) = u^0(x), \quad f(x+e_j,t) = f(x,t), \quad 1 \le j \le n \tag{8}$$

(e_j is the j-th unit vector in $\mathbb{R}^n$).

In place of (4) and (5), we assume that u^0 is a smooth function, and that

$$|\partial_x^\alpha \partial_t^m f(x,t)| \le C_{\alpha m K}(1+|t|)^{-K} \text{ on } \mathbb{R}^n \times [0,\infty), \text{ for any } \alpha, m, K. \tag{9}$$

We then accept a solution of (1), (2) and (3) as physically reasonable if it satisfies

$$u(x+e_j,t) = u(x,t), \text{ on } \mathbb{R}^n \times [0,\infty), \text{ and for } 1 \le j \le n \tag{10}$$

and

$$p, u \in C^\infty(\mathbb{R}^n \times [0,\infty)). \tag{11}$$

A fundamental problem in analysis is to decide whether such smooth, physically reasonable solutions exist for the Navier-Stokes equations ([3]-[5]). To give reasonable leeway to solvers while retaining the heart of the problem, Fefferman asked for a proof of one the following four statements ([2], [4]).

(A) Existence and Smoothness of Navier-Stokes Solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Let $u^0(x)$ be any smooth, divergence-free vector field satisfying (4). Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t)$, $u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (6) and (7).

(B) Existence and Smoothness of Navier-Stokes Solutions in $\mathbb{R}^3/\mathbf{Z}^3$. Take $\nu > 0$ and $n = 3$. Let $u^0(x)$ be any smooth, divergence-free vector field satisfying (8). Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t)$, $u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (10) and (11).

(C) Breakdown of Navier-Stokes Solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^0(x)$ on $\mathbb{R}^3$ and a smooth, $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ satisfying (4) and (5), for which there exist no solutions (p,u) of (1), (2), (3), (6) and (7) on $\mathbb{R}^3 \times [0,\infty)$.
(D) Breakdown of Navier-Stokes Solutions on $\mathbb{R}^3/\mathbf{Z}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^0(x)$ on $\mathbb{R}^3$ and a smooth, $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ satisfying (8) and (9), for which there exist no solutions (p,u) of (1), (2), (3), (10) and (11) on $\mathbb{R}^3 \times [0,\infty)$.

2. The Problem C

In this section we shall give some sufficient conditions for non-existence of solutions of the problem C.

Theorem 2.1. *If there exists an index i $(1 \le i \le 3)$ such that*

$$\int_{\mathbb{R}^3} \mathrm{d}\,x \left[\int_0^\infty f_i(x,t)\ \mathrm{d}\,t + u_i^0(x)\right] \neq 0,$$

then the problem C holds.

Proof. For $x \in \mathbb{R}^3$ and $t \ge 0$, we reform (1) in the following integral equation

$$\int_{\mathbb{R}^3} \mathrm{d}\,x \int_0^\infty \frac{\partial u_i(x,t)}{\partial t}\ \mathrm{d}\,t + \sum_{j=1}^{3} \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} u_j(x,t) \frac{\partial u_i(x,t)}{\partial x_j}\ \mathrm{d}\,x =$$
$$= \nu \int_0^\infty \mathrm{d}\,t \sum_{j=1}^{3} \int_{\mathbb{R}^3} \frac{\partial^2 u_i(x,t)}{\partial x_j^2}\ \mathrm{d}\,x - \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} \frac{\partial p(x,t)}{\partial x_i}\ \mathrm{d}\,x +$$
$$+ \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} f_i(x,t)\ \mathrm{d}\,x,$$

for $i = 1,2,3$. Now by using (3), (6) and (7) in this equation, we have

$$\int_{\mathbb{R}^3} \mathrm{d}\,x \left[\int_0^\infty f_i(x,t)\ \mathrm{d}\,t + u_i^0(x)\right] = 0.$$

□

Theorem 2.2. *If there exists an index i $(1 \le i \le 3)$ such that*

$$\int_{\mathbb{R}^3} \mathrm{d}\,x \int_0^\infty [u_i(x,t) + t f_i(x,t)]\ \mathrm{d}\,t \neq 0,$$

then the problem C holds.

Proof. First we multiple the equation (1) by t and then we integrate it in the following form

$$\int_{\mathbb{R}^3} \mathrm{d}\,x \int_0^\infty t\frac{\partial u_i(x,t)}{\partial t}\ \mathrm{d}\,t + \sum_{j=1}^{3} \int_0^\infty t\ \mathrm{d}\,t \int_{\mathbb{R}^3} u_j(x,t)\frac{\partial u_i(x,t)}{\partial x_j}\ \mathrm{d}\,x =$$
$$\nu\sum_{j=1}^{3}\int_0^\infty t\ \mathrm{d}\,t\int_{\mathbb{R}^3}\frac{\partial^2 u_i(x,t)}{\partial x_j^2}\ \mathrm{d}\,x - \int_0^\infty t\ \mathrm{d}\,t\int_{\mathbb{R}^3}\frac{\partial p(x,t)}{\partial x_i}\ \mathrm{d}\,x +$$
$$+\int_0^\infty t\ \mathrm{d}\,t\int_{\mathbb{R}^3} f_i(x,t)\ \mathrm{d}\,x,$$

for $i = 1, 2, 3$. Now by integrating part by part and simplifying of the result, we obtain

$$(*)\qquad \int_{\mathbb{R}^3}\mathrm{d}\,x\int_0^\infty [u_i(x,t) + tf_i(x,t)]\ \mathrm{d}\,t = 0.$$

□

Theorem 2.3. *If there exist indexes i, k $(1 \le i, k \le 3)$ such that*

$$-\int_{\mathbb{R}^3} x_k u_i^0(x)\ \mathrm{d}\,x - \int_0^\infty \mathrm{d}\,t\int_{\mathbb{R}^3} u_k(x,t)u_i(x,t)\ \mathrm{d}\,x \neq$$
$$\delta_{ik}\int_0^\infty \mathrm{d}\,t\int_{\mathbb{R}^3} p(x,t)\ \mathrm{d}\,x + \int_0^\infty \mathrm{d}\,t\int_{\mathbb{R}^3} x_k f_i(x,t)\ \mathrm{d}\,x,$$

then the problem C holds.

Proof. First we multiple the equation (1) by x_k $(k = 1, 2, 3)$ and then we integrate the result in the following form

$$\int_{\mathbb{R}^3} x_k\ \mathrm{d}\,x\int_0^\infty \frac{\partial u_i(x,t)}{\partial t}\ \mathrm{d}\,t + \sum_{j=1}^{3}\int_0^\infty \mathrm{d}\,t\int_{\mathbb{R}^3} x_k u_j(x,t)\frac{\partial u_i(x,t)}{\partial x_j}\ \mathrm{d}\,x =$$
$$= \nu\sum_{j=1}^{3}\int_0^\infty \mathrm{d}\,t\int_{\mathbb{R}^3} x_k\frac{\partial^2 u_i(x,t)}{\partial x_j^2}\ \mathrm{d}\,x - \int_0^\infty \mathrm{d}\,t\int_{\mathbb{R}^3} x_k\frac{\partial p(x,t)}{\partial x_i}\ \mathrm{d}\,x +$$
$$+\int_0^\infty \mathrm{d}\,t\int_{\mathbb{R}^3} x_k f_i(x,t)\ \mathrm{d}\,x,$$

for $i = 1, 2, 3$. Now by integrating part by part and simplifying of this equation, we have

$$(**)\qquad \begin{aligned}-\int_{\mathbb{R}^3} x_k u_i^0(x)\ \mathrm{d}\,x - \int_0^\infty \mathrm{d}\,t\int_{\mathbb{R}^3} u_k(x,t)u_i(x,t)\ \mathrm{d}\,x &=\\ = \delta_{ik}\int_0^\infty \mathrm{d}\,t\int_{\mathbb{R}^3} p(x,t)\ \mathrm{d}\,x + \int_0^\infty \mathrm{d}\,t\int_{\mathbb{R}^3} x_k f_i(x,t)\ \mathrm{d}\,x.&\end{aligned}$$

□

Theorem 2.4. *If there exists an index i $(1 \leq i \leq 3)$ such that*

$$\int_{\mathbb{R}^3} \mathrm{d}\,x \int_0^\infty t f_i(x,t)) \ \mathrm{d}\,t \neq 0,$$

then the problem C holds.

Proof. First multiple the equation (2) by x_k $(k = 1,2,3)$ and then integrate it in the following form

$$\sum_{i=1}^{3} \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} x_k \frac{\partial u_i(x,t)}{\partial x_i} \ \mathrm{d}\,x = 0.$$

Hence,

$$(***) \qquad \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} u_k(x,t) \ \mathrm{d}\,x = 0,$$

for $k = 1,2,3$. Again, we can obtain the equation $(*)$ in a similar manner. Now by replacement $(***)$ in $(*)$, we have

$$\int_{\mathbb{R}^3} \mathrm{d}\,x \int_0^\infty t f_i(x,t) \ \mathrm{d}\,t = 0. \qquad \square$$

Theorem 2.5. *If there exist indexes i, k with $1 \leq i < k \leq 3$, such that*

$$\int_{\mathbb{R}^3} [x_i u_k^0(x) - x_k u_i^0(x)] \ \mathrm{d}\,x \neq \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} [x_k f_i(x,t) - x_i f_k(x,t)] \ \mathrm{d}\,x,$$

then the problem C holds.

Proof. For distinct indexes i and k $(1 \leq i, k \leq 3)$ in $(**)$, we have

$$-\int_{\mathbb{R}^3} x_k u_i^0(x) \ \mathrm{d}\,x - \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} u_k(x,t) u_i(x,t) \ \mathrm{d}\,x = \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} x_k f_i(x,t) \ \mathrm{d}\,x.$$

By replacement the indexes i and k in this equation, we obtain

$$-\int_{\mathbb{R}^3} x_i u_k^0(x) \ \mathrm{d}\,x - \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} u_i(x,t) u_k(x,t) \ \mathrm{d}\,x = \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} x_i f_k(x,t) \ \mathrm{d}\,x.$$

By subtracting of these equations, we have

$$\int_{\mathbb{R}^3} [x_i u_k^0(x) - x_k u_i^0(x)] \ \mathrm{d}\,x = \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} [x_k f_i(x,t) - x_i f_k(x,t)] \ \mathrm{d}\,x. \qquad \square$$

Proposition 2.6. *If for $k = 1$ or $k = 2$,*

$$\int_{\mathbb{R}^3} [x_3 u_3^0(x) - x_k u_k^0(x)] \ \mathrm{d}\,x + \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} [u_3^2(x,t) - u_k^2(x,t)] \ \mathrm{d}\,x \neq$$
$$\int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} [x_k f_k(x,t) - x_3 f_3(x,t)] \ \mathrm{d}\,x,$$

then the problem C holds.

Proof. If the indexes i, k are equal in $(**)$, we have

$$-\int_{\mathbb{R}^3} x_k u_k^0(x)\ \mathrm{d}\,x - \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} u_k^2(x,t)\ \mathrm{d}\,x =$$
$$\int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} p(x,t)\ \mathrm{d}\,x + \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} x_k f_k(x,t)\ \mathrm{d}\,x.$$

If we consider this equation for $k = 3$ and subtract it form the results for $k = 1$ and $k = 2$, we obtain

$$\int_{\mathbb{R}^3} [x_3 u_3^0(x) - x_k u_k^0(x)]\ \mathrm{d}\,x + \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} [u_3^2(x,t) - u_k^2(x,t)]\ \mathrm{d}\,x =$$
$$\mathrm{d}\,s \int_0^\infty \mathrm{d}\,t \int_{\mathbb{R}^3} [x_k f_k(x,t) - x_3 f_3(x,t)]\ \mathrm{d}\,x,$$

for $k = 1, 2$. □

Theorem 2.7. *Let $\varphi(x,t)$ be a function with continuous first order partial derivatives such that*

$$\sum_{i=1}^{3} \int_{\mathbb{R}^3} u_i(x,t) \frac{\partial \varphi(x,t)}{\partial x_i}\ \mathrm{d}\,x \neq 0,$$

for all $t > 0$. Then, the problem C holds.

Proof. First multiple the equation (2) by $\varphi(x,t)$ and then integrate it. By simplification of the result, we obtain

$$\sum_{i=1}^{3} \int_{\mathbb{R}^3} u_i(x,t) \frac{\partial \varphi(x,t)}{\partial x_i}\ \mathrm{d}\,x = 0,$$

for all $t > 0$. □

3. The Problem D

In this section we shall give some sufficient conditions for non-existence of solutions of the problem D.

Theorem 3.1. *If there exists an index i $(1 \leq i \leq 3)$ such that*

$$\int_{[0,1]^3} \left[\int_0^\infty f_i(x,t)\ \mathrm{d}\,t + u_i^0(x) \right] \mathrm{d}\,x \neq 0,$$

then the problem D holds.

Proof. The proof is based on the proof of Theorem 2.1, by replacing integration in x on $\mathbb{R}^3$ with the one on $[0,1]^3$. □

Theorem 3.2. *If there exists an index i $(1 \leq i \leq 3)$ such that*

$$\int_{[0,1]^3} \mathrm{d}\,x \int_0^\infty [u_i(x,t) + t f_i(x,t)]\ \mathrm{d}\,t \neq 0,$$

then the problem D holds.

Proof. The proof is based on the proof of Theorem 2.2, by replacing integration in x on $\mathbb{R}^3$ with the one on $[0,1]^3$. $\square$

Theorem 3.3. *Let $\varphi(x,t)$ be a function with continuous first order partial derivatives such that*

$$\sum_{i=1}^{3} \int_{[0,1]^3} u_i(x,t) \frac{\partial \varphi(x,t)}{\partial x_i} \, \mathrm{d}\, x \neq 0,$$

for all $t > 0$. Then, the problem D holds.

Proof. The proof is based on the proof of Theorem 2.7, by replacing integration in x on $\mathbb{R}^3$ with the one on $[0,1]^3$. $\square$

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