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On a Generalization of Normal, Almost Normal and Mildly Normal Spaces–I

Erdal Ekici and Takashi Noiri

Abstract. In this paper, we introduce the notions of δp-normal spaces, almost δp-normal spaces, mildly δp-normal spaces, $g\delta p$-closed sets and the forms of generalized δ-preclosed functions. We obtain characterizations and the relationships of such normal spaces, properties of the forms of generalized δ-preclosed functions and preservation theorems.

1. Introduction

Levine [8] initiated the investigation of so-called g-closed sets in topological spaces, since then many modifications of g-closed sets were defined and investigated by a large number of topologists [1, 5, 14]. In 1996, Maki et al. [9] introduced the concepts of gp-closed sets. On the other hand, the notions of p-normal spaces, almost p-normal spaces and mildly p-normal spaces were introduced by Paul and Bhattacharyya [15]; Navalagi [11]; Navalagi [11], respectively.

In this paper we introduce five sections. In the first section, second section and third section, we introduce the notions of δp-normal spaces, almost δp-normal spaces and mildly δp-normal spaces which are the generalized forms of p-normal spaces, almost p-normal spaces and mildly p-normal spaces, respectively. Also, we obtain characterizations and properties of such generalizations of normal spaces. In fourth section, we introduce and study the concepts of $g\delta p$-closed sets and some new forms of generalized δ-preclosed functions. In the last section, we obtain the relationships between δp-normal spaces and generalized δ-preclosed functions.

2. Preliminaries

In this paper, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated and $f : (X, \tau) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f of a space (X, τ) into a space (Y, σ).

Let A be a subset of a space X. The closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

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Definition 1. A subset A of a space X is said to be:
(1) regular open [19] if $A = int(cl(A))$,
(2) α-open [12] if $A \subset int(cl(int(A)))$,
(3) preopen [10] or nearly open [7] if $A \subset int(cl(A))$.

The complement of an α-open (resp. preopen, regular open) set is called α-closed [12] (resp. preclosed [10], regular closed [19]). The intersection of all α-closed (resp. preclosed) sets containing A is called the α-closure (resp. preclosure) of A and is denoted by $\alpha\text{-}cl(A)$ (resp. $pcl(A)$). The α-interior (resp. preinterior) of A, denoted by $\alpha\text{-}int(A)$ (resp. $pint(A)$) is defined to be the union of all α-open (resp. preopen) sets contained in A.

The δ-interior [20] of a subset A of X is defined by the union of all regular open sets of X contained in A and is denoted by $\delta\text{-}int(A)$. A subset A is called δ-open [20] if $A = \delta\text{-}int(A)$, i.e. a set is δ-open if it is the union of regular open sets. The complement of a δ-open set is called δ-closed. Alternatively, a set A of (X, τ) is called δ-closed [20] if $A = \delta\text{-}cl(A)$, where $\delta\text{-}cl(A) = \{x \in X : A \cap int(cl(U)) \neq \emptyset, U \in \tau \text{ and } x \in U\}$. A subset A of a space X is said to be δ-preopen [16] if $A \subset int(\delta\text{-}cl(A))$.

The complement of a δ-preopen set is said to be δ-preclosed. The intersection of all δ-preclosed sets of X containing A is called the δ-preclosure [16] of A and is denoted by $\delta\text{-}pcl(A)$. The union of all δ-preopen sets of X contained in A is called the δ-preinterior of A and is denoted by $\delta\text{-}pint(A)$ [16]. A subset U of X is called a δ-preneighborhood of a point $x \in X$ if there exists a δ-preopen set V such that $x \in V \subset U$. Note that $\delta\text{-}pcl(A) = A \cup cl(\delta\text{-}int(A))$ and $\delta\text{-}pint(A) = A \cap int(\delta\text{-}cl(A))$.

The family of all δ-preopen (δ-preclosed, α-open, regular open, regular closed, δ-open, δ-closed, preopen) sets of a space X is denoted by $\delta PO(X)$ (resp. $\delta PC(X)$, $\alpha O(X)$, $RO(X)$, $RC(X)$, $\delta O(X)$, $\delta C(X)$, $PO(X)$). The family of all δ-preopen sets containing a point x is denoted by $\delta PO(X, x)$. It is shown in [13] that $\alpha O(X)$ is a topology and it is stronger than given topology on X.

Definition 2. A space X is said to be prenormal [13] or p-normal [15] if for any pair of disjoint closed sets A and B, there exist disjoint preopen sets U and V such that $A \subset U$ and $B \subset V$.

Definition 3. A space X is said to be almost normal [17] (resp. almost p-normal [11]) if for each closed set A and each regular closed set B such that $A \cap B = \emptyset$, there exist disjoint open (resp. preopen) sets U and V such that $A \subset U$ and $B \subset V$.

Definition 4. A space X is said to be mildly normal [18] (resp. mildly p-normal [11]) if for every pair of disjoint regular closed sets A and B of X, there exist disjoint open (resp. preopen) sets U and V such that $A \subset U$ and $B \subset V$.

Definition 5. A function $f : X \to Y$ is called

(1) R-map [4] if $f^{-1}(V)$ is regular open in X for every regular open set V of Y,

(2) completely continuous [2] if $f^{-1}(V)$ is regular open in X for every open set V of Y.

3. δp-NORMAL SPACES

Definition 6. A space X is said to be δp-normal if for any pair of disjoint closed sets A and B, there exist disjoint δ-preopen sets U and V such that $A \subset U$ and $B \subset V$.

Remark 1. The following implication holds for a topological space (X, τ):

$$\text{normal} \;\Rightarrow\; \text{p-normal} \;\Rightarrow\; \delta p\text{-normal}$$

None of these implications is reversible as shown by the following example.

Example 1. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the space (X, τ) is δp-normal but not p-normal.

For the other implication the example can be seen in [13, 15].

Theorem 1. *For a space X the following are equivalent:*

(1) *X is δp-normal,*

(2) *for every pair of open sets U and V whose union is X, there exist δ-preclosed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$,*

(3) *for every closed set F and every open set D containing F, there exists a δ-preopen set U such that $C \subset U \subset \delta$-pcl$(U) \subset D$.*

Proof. (1) $\implies$ (2) : Let U and V be a pair of open sets in a δp-normal space X such that $X = U \cup V$. Then $X \setminus U$, $X \setminus V$ are disjoint closed sets. Since X is δp-normal, there exist disjoint δ-preopen sets U_1 and V_1 such that $X \setminus U \subset U_1$ and $X \setminus V \subset V_1$. Let $A = X \setminus U_1$, $B = X \setminus V_1$. Then A and B are δ-preclosed sets such that $A \subset U$, $B \subset V$ and $A \cup B = X$.

(2) $\implies$ (3) : Let C be a closed set and D be an open set containing C. Then $X \setminus C$ and D are open sets whose union is X. Then by (2), there exist δ-preclosed sets M_1 and M_2 such that $M_1 \subset X \setminus C$ and $M_2 \subset D$ and $M_1 \cup M_2 = X$. Then $C \subset X \setminus M_1$, $X \setminus D \subset X \setminus M_2$ and $(X \setminus M_1) \cap (X \setminus M_2) = \emptyset$. Let

$$U = X \setminus M_1 \text{ and } V = X \setminus M_2.$$

Then U and V are disjoint δ-preopen sets such that $C \subset U \subset X \setminus V \subset D$. As $X \setminus V$ is δ-preclosed set, we have δ-pcl$(U) \subset X \setminus V$ and $C \subset U \subset \delta - pcl(U) \subset D$.

(3) $\implies$ (1) : Let C_1 and C_2 be any two disjoint closed sets of X. Put $D = X \setminus C_2$, then $C_2 \cap D = \emptyset$. $C_1 \subset D$ where D is an open set. Then by (3), there exists a δ-preopen set U of X such that

$$C_1 \subset U \subset \delta - pcl(U) \subset D.$$

It follows that

$$C_2 \subset X \setminus \delta - pcl(U) = V,$$

say, then V is δ-preopen and $U \cap V = \emptyset$. Hence, C_1 and C_2 are separated by δ-preopen sets U and V. Therefore X is δp-normal. □

Definition 7. A function $f : X \to Y$ is called strongly δ-preopen if $f(U) \in \delta PO(Y)$ for each $U \in \delta PO(X)$.

Definition 8. A function $f : X \to Y$ is called strongly δ-preclosed if $f(U) \in \delta PC(Y)$ for each $U \in \delta PC(X)$.

Theorem 2. *A function $f : X \to Y$ is strongly δ-preclosed if and only if for each subset B in Y and for each δ-preopen set U in X containing $f^{-1}(B)$, there exists a δ-preopen set V containing B such that $f^{-1}(V) \subset U$.*

Proof. $(\Rightarrow)$: Suppose that f is strongly δ-preclosed. Let B be a subset of Y and $U \in \delta PO(X)$ containing $f^{-1}(B)$. Put $V = Y \backslash f(X \backslash U)$, then V is a δ-preopen set of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

$(\Leftarrow)$: Let K be any δ-preclosed set of X. Then $f^{-1}(Y \backslash f(K)) \subset X \backslash K$ and $X \backslash K \in \delta PO(X)$. There exists a δ-preopen set V of Y such that $Y \backslash f(K) \subset V$ and $f^{-1}(V) \subset X \backslash K$. Therefore, we have $f(K) \supset Y \backslash V$ and $K \subset f^{-1}(Y \backslash V)$. Hence, we obtain $f(K) = Y \backslash V$ and $f(K)$ is δ-preclosed in Y. This shows that f is strongly δ-preclosed. □

Theorem 3. *If $f : X \to Y$ is a strongly δ-preclosed continuous function from a δp-normal space X onto a space Y, then Y is δp-normal.*

Proof. Let K_1 and K_2 be disjoint closed sets in Y. Then $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are closed sets. Since X is δp-normal, then there exist disjoint δ-preopen sets U and V such that $f^{-1}(K_1) \subset U$ and $f^{-1}(K_2) \subset V$. By the previous theorem, there exist δ-preopen sets A and B such that $K_1 \subset A$, $K_2 \subset B$, $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Also, A and B are disjoint. Thus, Y is δp-normal. □

Definition 9. A function $f : X \to Y$ is said to be almost δ-preirresolute if for each x in X and each δ-preneighborhood V of $f(x)$, $\delta\text{-}pcl(f^{-1}(V))$ is a δ-preneighborhood of x.

Lemma 1. *Let $f : X \to Y$ be a function. Then f is almost δ-preirresolute if and only if $f^{-1}(V) \subset \delta - pint(\delta - pcl(f^{-1}(V)))$ for every $V \in \delta PO(Y)$.*

Theorem 4. *A function $f : X \to Y$ is almost δ-preirresolute if and only if $f(\delta - pcl(U)) \subset \delta - pcl(f(U))$ for every $U \in \delta PO(X)$.*

Proof. $(\Rightarrow)$: Let $U \in \delta PO(X)$. Suppose $y \notin \delta - pcl(f(U))$. Then there exists $V \in \delta PO(Y, y)$ such that $V \cap f(U) = \emptyset$. Hence $f^{-1}(V) \cap U = \emptyset$. Since $U \in \delta PO(X)$, we have $\delta\text{-}pint(\delta - pcl(f^{-1}(V))) \cap \delta - pcl(U) = \emptyset$. Then by Lemma 1, $f^{-1}(V) \cap \delta - pcl(U) = \emptyset$ and hence $V \cap f(\delta - pcl(U)) = \emptyset$. This implies that $y \notin f(\delta - pcl(U))$.

$(\Leftarrow)$: If $V \in \delta PO(Y)$, then $M = X \backslash \delta - pcl(f^{-1}(V)) \in \delta PO(X)$. By hypothesis, $f(\delta - pcl(M)) \subset \delta - pcl(f(M))$ and hence $X \setminus \delta - pint(\delta - pcl(f^{-1}(V))) = \delta - pcl(M) \subset f^{-1}(\delta - pcl(f(M))) \subset f^{-1}(\delta - pcl(f(X \setminus f^{-1}(V)))) \subset f^{-1}(\delta - pcl(Y \setminus V)) = f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. Therefore , $f^{-1}(V) \subset \delta - pint(\delta - pcl(f^{-1}(V)))$. By Lemma 1, f is almost δ-preirresolute. □

Theorem 5. *If $f : X \to Y$ is a strongly δ-preopen continuous almost δ-preirresolute function from a δp-normal space X onto a space Y, then Y is δp-normal.*

Proof. Let A be a closed subset of Y and B be an open set containing A. Then by continuity of f, $f^{-1}(A)$ is closed and $f^{-1}(B)$ is an open set of X such that $f^{-1}(A) \subset f^{-1}(B)$. As X is δp-normal, there exists a δ-preopen set U in X such that $f^{-1}(A) \subset U \subset \delta - pcl(U) \subset f^{-1}(B)$ by Theorem 1. Then, $f(f^{-1}(A)) \subset f(U) \subset f(\delta - pcl(U)) \subset f(f^{-1}(B))$. Since f is strongly δ-preopen almost δ-preirresolute surjection, we obtain $A \subset f(U) \subset \delta - pcl(f(U)) \subset B$. Then again by Theorem 1 the space Y is δp-normal. □

4. Almost δp-normal spaces

Definition 10. A space X is said to be almost δp-normal if for each closed set A and each regular closed set B such that $A \cap B = \emptyset$, there exist disjoint δ-preopen sets U and V such that $A \subset U$ and $B \subset V$.

Remark 2. The following diagram holds for a topological space (X, τ):

$$\begin{array}{ccccc} \text{normal} & \Rightarrow & \text{p-normal} & \Rightarrow & \delta p\text{-normal} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{almost normal} & \Rightarrow & \text{almost p-normal} & \Rightarrow & \text{almost } \delta p\text{-normal} \end{array}$$

None of these implications is reversible as shown by the following example.

Example 2. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the space (X, τ) is almost δp-normal but not almost p-normal.

For the other implication the example can be seen in the related papers.
Question: Clearly every δp-normal space is almost δp-normal. Does there exist an almost δp-normal space which is not δp-normal?

Theorem 6. *For a space X the following statements are equivalent:*
(1) *X is almost δp-normal,*
(2) *For every pair of sets U and V, one of which is open and the other is regular open whose union is X, there exist δ-preclosed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$,*
(3) *For every closed set A and every regular open set B containing A, there exists a δ-preopen set V such that $A \subset V \subset \delta - pcl(V) \subset B$.*

Proof. (1) $\Longrightarrow$ (2) : Let U be an open set and V be a regular open set in an almost δp-normal space X such that $U \cup V = X$. Then $(X \setminus U)$ is a closed set and $(X \setminus V)$ is a regular closed set with $(X \setminus U) \cap (X \setminus V) = \emptyset$. By almost δp-normality of X, there exist disjoint δ-preopen sets U_1 and V_1 such that $X \setminus U \subset U_1$ and $X \setminus V \subset V_1$. Let $A = X \setminus U_1$ and $B = X \setminus V_1$. Then A and B are δ-preclosed sets such that $A \subset U$, $B \subset V$ and $A \cup B = X$.

(2) $\implies$ (3) : Let A be a closed set and B be a regular open set containing A. Then $X \setminus A$ is open and B is regular open sets whose union is X. Then by (2), there exist δ-preclosed sets M_1 and M_2 such that $M_1 \subset X \setminus A$ and $M_2 \subset B$ and $M_1 \cup M_2 = X$. Then $A \subset X \setminus M_1$, $X \setminus B \subset X \setminus M_2$ and $(X \setminus M_1) \cap (X \setminus M_2) = \emptyset$. Let $U = X \setminus M_1$ and $V = X \setminus M_2$.Then U and V are disjoint δ-preopen sets such that $A \subset U \subset X \setminus V \subset B$. As $X \setminus V$ is δ-preclosed set, we have δ-$pcl(U) \subset X \setminus V$ and $A \subset U \subset \delta - pcl(U) \subset B$.

(3) $\implies$ (1) : Let A_1 and A_2 be any two disjoint closed and regular closed sets, respectively. Put $D = X \setminus A_2$, then $A_2 \cap D = \emptyset$. $A_1 \subset D$ where D is a regular open set. Then by (3), there exists a δ-preopen set U of X such that $A_1 \subset U \subset \delta - pcl(U) \subset D$. It follows that $A_2 \subset X \setminus \delta - pcl(U) = V$,say, then V is δ-preopen and $U \cap V = \emptyset$. Hence, A_1 and A_2 are separated by δ-preopen sets U and V. Therefore X is almost δp-normal. □

Theorem 7. *If $f : X \to Y$ is a continuous strongly δ-preopen R-map and almost δ-preirresolute surjection from an almost δp-normal space X onto a space Y, then Y is almost δp-normal.*

Proof. Similar to Theorem 5. □

Corollary 1. *If $f : X \to Y$ is a completely continuous strongly δ-preopen and almost δ-preirresolute surjection from an almost δp-normal space X onto a space Y, then Y is almost δp-normal.*

5. Mildly δp-normal spaces

Definition 11. A space X is said to be mildly δp-normal if for every pair of disjoint regular closed sets A and B of X, there exist disjoint δ-preopen sets U and V such that $A \subset U$ and $B \subset V$.

Remark 3. The following diagram holds for a topological space (X, τ):

$$\begin{array}{ccccc} \text{normal} & \Rightarrow & \text{p-normal} & \Rightarrow & \delta p\text{-normal} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{almost normal} & \Rightarrow & \text{almost p-normal} & \Rightarrow & \text{almost } \delta p\text{-normal} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{mildly normal} & \Rightarrow & \text{mildly p-normal} & \Rightarrow & \text{mildly } \delta p\text{-normal} \end{array}$$

Question: Clearly every almost δp-normal space is mildly δp-normal. Does there exist a mildly δp-normal (resp. mildly δp-normal) space which is not mildly p-normal (resp. almost δp-normal)?

Theorem 8. *For a space X the following are equivalent:*

(1) *X is mildly δp-normal,*

(2) *For every pair of regular open sets U and V whose union is X, there exist δ-preclosed sets G and H such that $G \subset U$, $H \subset V$ and $G \cup H = X$,*
(3) *For any regular closed set A and every regular open set B containing A, there exists a δ-preopen set U such that $A \subset U \subset \delta - pcl(U) \subset B$,*
(4) *For every pair of disjoint regular closed sets A and B, there exist δ-preopen sets U and V such that $A \subset U$, $B \subset V$ and $\delta - pcl(U) \cap \delta - pcl(V) = \emptyset$.*

Proof. Similar to Theorem 1. □

Theorem 9. *If $f : X \to Y$ is an strongly δ-preopen R-map and almost δ-preirresolute function from a mildly δp-normal space X onto a space Y, then Y is mildly δp-normal.*

Proof. Let A be a regular closed set and B be a regular open set containing A. Then by R-map of f, $f^{-1}(A)$ is a regular closed set contained in the regular open set $f^{-1}(B)$. Since X is mildly δp-normal, there exists a δ-preopen set V such that

$$f^{-1}(A) \subset V \subset \delta - pcl(V) \subset f^{-1}(B)$$

by Theorem 8. As f is strongly δ-preopen and an almost δ-preirresolute surjection, it follows that $f(V) \in \delta PO(Y)$ and $A \subset f(V) \subset \delta - pcl(f(V)) \subset B$. Hence Y is mildly δp-normal. □

Theorem 10. *If $f : X \to Y$ is R-map, strongly δ-preclosed function from a mildly δp-normal space X onto a space Y, then Y is mildly δp-normal.*

Proof. Similar to Theorem 3. □

6. $g\delta p$-closed sets and generalized functions

Definition 12. A subset A of a space (X, τ) is said to be g-closed [8] (resp. gp-closed [9]) if $cl(A) \subset U$ (resp. $p\text{-}cl(A) \subset U$) whenever $A \subset U$ and $U \in \tau$. The complement of g-closed (resp. gp-closed) set is said to be g-open (resp. gp-open).

Definition 13. A subset A of a space (X, τ) is said to be $g\delta p$-closed if $\delta\text{-}pcl(A) \subset U$ whenever $A \subset U$ and $U \in \tau$. The complement of $g\delta p$-closed set is said to be $g\delta p$-open.

Remark 4. The following diagram holds for any subset of a topological space (X, τ):

δ-preclosed	$\Rightarrow$	$g\delta p$-closed
$\Uparrow$		$\Uparrow$
preclosed	$\Rightarrow$	gp-closed
$\Uparrow$		$\Uparrow$
closed		g-closed

None of these implications is reversible as shown by the following examples.

Example 3. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the set $\{a, d\}$ is $g\delta p$-closed but it is not gp-closed. Let $\tau = \{X, \varnothing, \{b\}, \{d\}, \{b, d\}\}$. Then the set $\{b, c\}$ is $g\delta p$-closed but it is not δ-preclosed.

For the other implications the examples can be seen in [3, 9, 10, 16].

Definition 14. A function $f : X \to Y$ is said to be

(1) δ-preclosed if $f(K)$ is δ-preclosed in Y for each closed set K of X,
(2) $g\delta p$-closed if $f(K)$ is $g\delta p$-closed in Y for each closed set K of X.

Definition 15. A function $f : X \to Y$ is said to be

(1) quasi δ-preclosed if $f(K)$ is closed in Y for each $K \in \delta PC(X)$,
(2) δp-$g\delta p$-closed if $f(K)$ is $g\delta p$-closed in Y for each $K \in \delta PC(X)$,
(3) almost $g\delta p$-closed if $f(K)$ is $g\delta p$-closed in Y for each $K \in RC(X)$.

Remark 5. The following diagram holds for a function $f : (X, \tau) \to (Y, \sigma)$:

$$
\begin{array}{ccc}
 & & \text{almost } g\delta p\text{-closed} \\
 & & \Uparrow \\
\delta p\text{-}g\delta p\text{-closed} & \Rightarrow & \text{g}\delta p\text{-closed} \\
\Uparrow & & \Uparrow \\
\text{strongly } \delta\text{-preclosed} & \Rightarrow & \delta\text{-preclosed} \\
\Uparrow & & \Uparrow \\
\text{quasi } \delta\text{-preclosed} & \Rightarrow & \text{closed}
\end{array}
$$

None of these implications is reversible as shown by the following examples.

Example 4. Let $X = Y = \{a, b, c, d\}$ and $\tau = \sigma = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined as follows: $f(a) = a$, $f(b) = c$, $f(c) = c$ and $f(d) = d$. Then f is almost $g\delta p$-closed but it is not $g\delta p$-closed.

If we define the function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = a$, $f(b) = a$, $f(c) = c$ and $f(d) = d$, then f is δ-preclosed but it is not closed. If we define the function f as follows: $f(a) = c$, $f(b) = a$, $f(c) = c$ and $f(d) = d$, then f is δ-preclosed but it is not strongly δ-preclosed. If we define the function f as an identity function, then f is closed and strongly δ-preclosed but it is not quasi δ-preclosed.

Example 5. Let $X = Y = \{a, b, c, d\}$ and $\tau = \sigma = \{X, \varnothing, \{b\}, \{d\}, \{b, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If we define the function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = c$, $f(b) = d$, $f(c) = c$ and $f(d) = d$, then f is δp-$g\delta p$-closed but it is not δp-δpg-closed. If we define the function f as follows: $f(a) = a$, $f(b) = a$, $f(c) = d$ and $f(d) = d$, then f is $g\delta p$-closed but it is neither δpg-closed nor δp-$g\delta p$-closed.

Definition 16. A function $f : X \to Y$ is said to be δp-$g\delta p$-continuous if $f^{-1}(K)$ is $g\delta p$-closed in X for every $K \in \delta PC(Y)$.

Theorem 11. *A function $f : X \to Y$ is δp-$g\delta p$-continuous if and only if $f^{-1}(V)$ is $g\delta p$-open in X for every $V \in \delta PO(Y)$.*

Theorem 12. *If $f : X \to Y$ is closed δp-$g\delta p$-continuous, then $f^{-1}(K)$ is $g\delta p$-closed in X for each $g\delta p$-closed set K of Y.*

Proof. Let K be a $g\delta p$-closed set of Y and U an open set of X containing $f^{-1}(K)$. Put $V = Y - f(X - U)$, then V is open in Y, $K \subset V$, and $f^{-1}(V) \subset U$. Therefore, we have δ-$pcl(K) \subset V$ and hence

$$f^{-1}(K) \subset f^{-1}(\delta - pcl(K)) \subset f^{-1}(V) \subset U.$$

Since f is δp-$g\delta p$-continuous, $f^{-1}(\delta\text{-}pcl(K))$ is $g\delta p$-closed in X and hence δ-$pcl(f^{-1}(K)) \subset \delta\text{-}pcl(f^{-1}(\delta\text{-}pcl(K))) \subset U$. This shows that $f^{-1}(K)$ is $g\delta p$-closed in X. □

Theorem 13. *A function $f : X \to Y$ is δp-$g\delta p$-closed if and only if for each subset B of Y and each $U \in \delta PO(X)$ containing $f^{-1}(B)$, there exists a $g\delta p$-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.*

Proof. $(\Rightarrow)$: Suppose that f is δp-$g\delta p$-closed. Let B be a subset of Y and $U \in \delta PO(X)$ containing $f^{-1}(B)$. Put $V = Y \backslash f(X \backslash U)$, then V is a $g\delta p$-open set of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

$(\Leftarrow)$: Let K be any δ-preclosed set of X. Then $f^{-1}(Y \backslash f(K)) \subset X \backslash K$ and $X \backslash K \in \delta PO(X)$. There exists a $g\delta p$-open set V of Y such that $Y \backslash f(K) \subset V$ and $f^{-1}(V) \subset X \backslash K$. Therefore, we have $f(K) \supset Y \backslash V$ and $K \subset f^{-1}(Y \backslash V)$. Hence, we obtain $f(K) = Y \backslash V$ and $f(K)$ is $g\delta p$-closed in Y. This shows that f is δp-$g\delta p$-closed. □

Theorem 14. *If $f : X \to Y$ is continuous δp-$g\delta p$-closed, then $f(H)$ is $g\delta p$-closed in Y for each $g\delta p$-closed set H of X.*

Proof. Let H be any $g\delta p$-closed set of X and V an open set of Y containing $f(H)$. Since $f^{-1}(V)$ is an open set of X containing H, δ-$pcl(H) \subset f^{-1}(V)$ and hence $f(\delta\text{-}pcl(H)) \subset V$. Since f is δp-$g\delta p$-closed and δ-$pcl(H) \in \delta PC(X)$, we have $\delta\text{-}pcl(f(H)) \subset \delta\text{-}pcl(f(\delta\text{-}pcl(H))) \subset V$. Therefore, $f(H)$ is $g\delta p$-closed in Y. □

Definition 17. A function $f : X \to Y$ is said to be δ-preirresolute [6] if $f^{-1}(V) \in \delta PO(X)$ for every $V \in \delta PO(Y)$.

Remark 6. A δ-preirresolute function is δp-$g\delta p$-continuous but not conversely.

Example 6. Let $X = Y = \{a, b, c, d\}$ and $\tau = \sigma = \{X, \varnothing, \{b\}, \{d\}, \{b, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined as follows: $f(a) = b$, $f(b) = d$, $f(c) = c$ and $f(d) = d$. Then f is δp-$g\delta p$-continuous but it is not δ-preirresolute.

Corollary 2. *If $f : X \to Y$ is closed δ-preirresolute, then $f^{-1}(K)$ is $g\delta p$-closed in X for each $g\delta p$-closed set K of Y.*

Theorem 15. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then*

(1) *the composition $gof : X \to Z$ is δp-$g\delta p$-closed if f is δp-$g\delta p$-closed and g is continuous δp-$g\delta p$-closed,*
(2) *the composition $gof : X \to Z$ is δp-$g\delta p$-closed if f is strongly δ-preclosed and g is δp-$g\delta p$-closed,*
(3) *the composition $gof : X \to Z$ is δp-$g\delta p$-closed if f is quasi δ-preclosed and g is $g\delta p$-closed.*

Theorem 16. *Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition $gof : X \to Z$ be δp-$g\delta p$-closed. Then, if f is an δ-preirresolute surjection, then g is δp-$g\delta p$-closed.*

Proof. Let $K \in \delta PC(Y)$. Since f is δ-preirresolute and surjective, $f^{-1}(K) \in \delta PC(X)$ and $(gof)(f^{-1}(K)) = g(K)$. Therefore, $g(K)$ is $g\delta p$-closed in Z and hence g is δp-$g\delta p$-closed. □

Theorem 17. *Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition $gof : X \to Z$ be δp-$g\delta p$-closed. Then, if g is a closed δp-$g\delta p$-continuous injection, then f is δp-$g\delta p$-closed.*

Proof. Let $H \in \delta PC(X)$. Then $(gof)(H)$ is $g\delta p$-closed in Z and $g^{-1}((gof)(H)) = f(H)$. By Theorem 12, $f(H)$ is $g\delta p$-closed in Y and hence f is δp-$g\delta p$-closed. □

7. FURTHER CHARACTERIZATIONS OF δp-NORMAL SPACES

Lemma 2. *A subset A of a space X is $g\delta p$-open if and only if $F \subset \delta$-$pint(A)$ whenever F is closed and $F \subset A$.*

Theorem 18. *For a topological space X, the following are equivalent:*

(a) *X is δp-normal,*
(b) *for any pair of disjoint closed sets A and B of X, there exist disjoint $g\delta p$-open sets U and V of X such that $A \subset U$ and $B \subset V$,*
(c) *for each closed set A and each open set B containing A, there exists a $g\delta p$-open set U such that $A \subset U \subset \delta$-$pcl(U) \subset B$,*
(d) *for each closed set A and each g-open set B containing A, there exists a δ-preopen set U such that $A \subset U \subset \delta$-$pcl(U) \subset int(B)$,*
(e) *for each closed set A and each g-open set B containing A, there exists a $g\delta p$-open set G such that $A \subset G \subset \delta$-$pcl(G) \subset int(B)$,*
(f) *for each g-closed set A and each open set B containing A, there exists a δ-preopen set U such that $cl(A) \subset U \subset \delta$-$pcl(U) \subset B$,*
(g) *for each g-closed set A and each open set B containing A, there exists a $g\delta p$-open set G such that $cl(A) \subset G \subset \delta$-$pcl(G) \subset B$.*

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c)$: Since every δ-preopen set is $g\delta p$-open, it is obvious.

$(d) \Rightarrow (e)$ and $(f) \Rightarrow (g) \Rightarrow (c)$: Since every closed (resp. open) set is g-closed (resp. g-open), it is obvious.

$(c) \Rightarrow (d)$: Let A be any closed subset of X and B be a g-open set containing A. We have $A \subset int(B)$. Then there exists a $g\delta p$-open set G such that $A \subset G \subset \delta$-$pcl(G) \subset int(B)$. Since G is $g\delta p$-open, by Lemma 2 $A \subset \delta$-$pint(G)$. Put $U = \delta$-$pint(G)$, then U is δ-preopen and $A \subset U \subset \delta$-$pcl(U) \subset int(B)$.

$(e) \Rightarrow (f)$: Let A be any g-closed subset of X and B be an open set containing A. We have $cl(A) \subset B$. Then there exists a $g\delta p$-open set G such that $cl(A) \subset G \subset \delta$-$pcl(G) \subset B$. Since G is $g\delta p$-open and $cl(A) \subset G$, by Lemma 2 we have $cl(A) \subset \delta$-$pint(G)$, put $U = \delta$-$pint(G)$, then U is δ-preopen and $cl(A) \subset U \subset \delta$-$pcl(U) \subset B$. □

Theorem 19. *If $f : X \to Y$ is a continuous quasi δ-preclosed surjection and X is δp-normal, then Y is normal.*

Proof. Let M_1 and M_2 be any disjoint closed sets of Y. Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X. Since X is δp-normal, there exist disjoint U_1, $U_2 \in \delta PO(X)$ such that $f^{-1}(M_i) \subset U_i$ for $i = 1, 2$. Put $V_i = Y - f(X - U_i)$, then V_i is open in Y, $M_i \subset V_i$ and $f^{-1}(V_i) \subset U_i$ for $i = 1, 2$. Since $U_1 \cap U_2 = \varnothing$ and f is surjective, we have $V_1 \cap V_2 = \varnothing$. This shows that Y is normal. □

Theorem 20. *Let $f : X \to Y$ be a closed δp-$g\delta p$-continuous injection. If Y is δp-normal, then X is δp-normal.*

Proof. Let N_1 and N_2 be disjoint closed sets of X. Since f is a closed injection, $f(N_1)$ and $f(N_2)$ are disjoint closed sets of Y. By the δp-normality of Y, there exist disjoint V_1, $V_2 \in \delta PO(Y)$ such that $f(N_i) \subset V_i$ for $i = 1, 2$. Since f is δp-$g\delta p$-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint $g\delta p$-open sets of X and $N_i \subset f^{-1}(V_i)$ for $i = 1, 2$. Now, put $U_i = \delta\text{-}pint(f^{-1}(V_i))$ for $i = 1, 2$. Then, $U_i \in \delta PO(X)$, $N_i \subset U_i$ and $U_1 \cap U_2 = \varnothing$. This shows that X is δp-normal. □

Corollary 3. *If $f : X \to Y$ is a closed δ-preirresolute injection and Y is δp-normal, then X is δp-normal.*

Lemma 3. *A function $f : X \to Y$ is almost $g\delta p$-closed if and only if for each subset B of Y and each $U \in RO(X)$ containing $f^{-1}(B)$, there exists a $g\delta p$-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.*

Lemma 4. *If $f : X \to Y$ is almost $g\delta p$-closed, then for each closed set M of Y and each $U \in RO(X)$ containing $f^{-1}(M)$, there exists $V \in \delta PO(Y)$ such that $M \subset V$ and $f^{-1}(V) \subset U$.*

Theorem 21. *Let $f : X \to Y$ be a continuous almost $g\delta p$-closed surjection. If X is normal, then Y is δp-normal.*

Proof. Let M_1 and M_2 be any disjoint, closed sets of Y. Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X. By the normality of X, there exist disjoint open sets U_1 and U_2 such that $f^{-1}(M_i) \subset U_i$, where $i = 1, 2$. Now, put $G_i = int(cl(U_i))$ for $i = 1, 2$, then $G_i \in RO(X)$, $f^{-1}(M_i) \subset U_i \subset G_i$ and $G_1 \cap G_2 = \varnothing$. By Lemma 4, there exists $V_i \in \delta PO(Y)$ such that $M_i \subset V_i$ and $f^{-1}(V_i) \subset G_i$, where $i = 1, 2$. Since $G_1 \cap G_2 = \varnothing$ and f is surjective, we have $V_1 \cap V_2 = \varnothing$. This shows that Y is δp-normal. □

Corollary 4. *If $f : X \to Y$ is a continuous δ-preclosed surjection and X is normal, then Y is δp-normal.*

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Department of Mathematics
Canakkale Onsekiz Mart University
Terzioglu Campus
17020 Canakkale
Turkey
E-mail address: eekici@comu.edu.tr

Department of Mathematics
Yatsushiro College of Techology
Yatsushiro
Kumamoto, 866-8501
Japan
E-mail address: noiri@as.yatsushiro-nct.ac.jp