Mathematica Moravica
Vol. **10** (2006), 77–90

Distributive Lattices and Hurewicz families

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Abstract. Hurewicz and Rothberger respectively introduced prototypes of the selection properties $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ and $\mathsf{S}_1(\mathcal{A},\mathcal{B})$. In the series of papers titled "Combinatorics of open covers" (see the bibliography) we learned that for various topologically significant families $\mathcal{A}$ and $\mathcal{B}$ these selection properties are intimately related to game theory and Ramsey theory. The similarity in techniques used there to explore these relationships suggests that there should be a general, unified way to obtain these results. In this paper we pursue one possibility by considering the selection principle $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ for distributive lattices. The selection principle $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ for distributive lattices will be treated in [3]. We use two examples throughout to illustrate the generality of the methods developed here.

1. Basic notions

Fix a distributive lattice $\mathbb{L}$. Except when explicitly stated otherwise, our lattices are *not* assumed to be complemented, nor to have a largest or a least element, nor to have any completeness properties beyond those implied by the basic definition of a distributive lattice.

Let families $\mathcal{A}$ and $\mathcal{B}$ of subsets of the lattice $\mathbb{L}$ be given. The selection principle $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ states that for each sequence $(O_n : n \in \mathbb{N})$ from $\mathcal{A}$ there is a sequence $(T_n : n \in \mathbb{N})$ such that T_n is for each n a finite subset of O_n, and $\cup_{n\in\mathbb{N}} T_n$ is an element of $\mathcal{B}$. It is evident from the definition that $\mathsf{S}_{fin}(\cdot,\cdot)$ is *antimonotonic* in the first variable, and *monotonic* in the second variable: More precisely, let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be families of subsets of $\mathbb{L}$ such that $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{B} \subseteq \mathcal{D}$. Then we have the implications

$$\mathsf{S}_{fin}(\mathcal{C},\mathcal{B}) \Rightarrow \mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$$

and

$$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B}) \Rightarrow \mathsf{S}_{fin}(\mathcal{A},\mathcal{D}).$$

We denote these interrelationships by a diagram as below, where the property at the initial point of an arrow implies the property at the arrow's terminal point:

2000 *Mathematics Subject Classification.* Primary: 03E05, 05D10, 06B99, 91A44.

Key words and phrases. game, selection principle, lattice, $\mathcal{A}$-tree, Hurewicz $\mathcal{A}$-tree, Hurewicz pairs, Ramsey family, partition relation, selectable pair.

$$\begin{tikzcd}
\mathsf{S}_{fin}(C,D) \arrow[r] & \mathsf{S}_{fin}(A,D) \\
\mathsf{S}_{fin}(C,B) \arrow[r] \arrow[u] & \mathsf{S}_{fin}(A,B) \arrow[u]
\end{tikzcd}$$

There is a natural game, $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$, associated with this selection principle: Players ONE and TWO play an inning per positive integer. In the n-th inning ONE first chooses a set $O_n \in \mathcal{A}$; then TWO responds by choosing a finite subset T_n of O_n. TWO wins a play $O_1, T_1, \ldots, O_n, T_n, \ldots$ if $\cup_{n\in\mathbb{N}} T_n \in \mathcal{B}$; otherwise, ONE wins.

If ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$, then the selection principle $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ holds. The converse implication is not always true.When it is in fact true, the game is a powerful tool to analyse the combinatorial properties of the families $\mathcal{A}$ and $\mathcal{B}$. In this paper we identify circumstances under which this converse is true. The concepts of an $\mathcal{A}$-tree and of *Hurewicz pairs* are central to this. In Sections 2 we develop the notion of a Hurewicz $\mathcal{A}$-tree, and prove the Fundamental Theorem of Hurewicz $\mathcal{A}$-trees (Theorem 2) from which much of the theory of $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ can be derived. To derive game-theoretic results from Theorem 2 the pair $(\mathcal{A},\mathcal{B})$ should have properties permitting various constructions in $\mathbb{L}$. The notion of a Hurewicz pair, introduced in Section 3, is intended to capture these requirements. We derive the Fundamental Theorem for Hurewicz Pairs (Theorem 7): If $(\mathcal{A},\mathcal{B})$ is a Hurewicz pair, then $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ holds if, and only if, ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$.

Next we take up the connections of $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ with Ramsey theory. For k a positive integer the symbol $\mathcal{A} \to \lceil\mathcal{B}\rceil^2_k$ denotes that for each $A \in \mathcal{A}$ and for each function $f : [A]^2 \to \{1,\ldots,k\}$ there is a set $B \subset A$, an $i \in \{1,\ldots,k\}$, and a partition $B = \cup_{n\in\mathbb{N}} B_n$ of B into pairwise disjoint finite sets, such that for all $\{a,b\} \in [B]^2$ with for each n $|\{a,b\} \cap B_n| \leq 1$, we have $f(\{a,b\}) = i$. This is an example of a partition relation. We shall call it the Baumgartner-Taylor partition relation since a version of it was introduced by these mathematicians in [4]. They introduced it to give a Ramsey-theoretic characterization of P-point ultrafilters. In Sections 4 and 5 we identify circumstances under which truth of the partition relation is equivalent to truth of the selection principle $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$. The concepts of a Ramsey family and of a selectable pair are central to this task. In Section 4 we prove the Fundamental Theorem of Ramsey Families (Theorem 12), stating that for certain Ramsey families $\mathcal{A}$, if ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$, then $\mathcal{A} \to \lceil\mathcal{B}\rceil^2_k$ holds.

In Section 5 we prove the Fundamental Theorem of Selectable Pairs (Theorem 17), stating that under appropriate hypotheses, if $\mathcal{A} \to \lceil\mathcal{B}\rceil^2_k$ holds, then $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ holds.

For each of the three situations ($\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$, the game $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ or partition relation $\mathcal{A} \rightarrow \lceil\mathcal{B}\rceil^2_k$), the only members of $\mathcal{B}$ that matter are those which are subsets of $\cup\mathcal{A}$. Thus, we will make the further assumption throughout this paper about $\mathcal{A}$ and $\mathcal{B}$:

Hypothesis 1. $\cup\mathcal{B} \subset \cup\mathcal{A}$.

Various well-known concepts are important in the study of selection principles, and are now introduced. For elements a and b of lattice $\mathbb{L}$ define $a \leq b$ if $b = a \vee b$. We write $a < b$ to denote that $a \leq b$ and $a \neq b$. Then $<$ defines a partial ordering on the set $\mathbb{L}$. For subsets A and B of $\mathbb{L}$ we define:

Definition 1. (1) A *refines* B, written $A \prec B$, if there is for each $a \in A$ a $b \in B$ such that $a \leq b$. If $A \prec B$ and $B \subset A$, B is *cofinal in* A.
(2) $\mathcal{A}$ is *refinement closed* if whenever $B \subset \cup\mathcal{A}$ is refined by an element of $\mathcal{A}$, then $B \in \mathcal{A}$. $\mathcal{A}$ is *cofinally closed* if every cofinal subset of an element of $\mathcal{A}$ is also an element of $\mathcal{A}$.

In the definition of "A refines B" we do not require that A be a subset of B.

2. Hurewicz $\mathcal{A}$-trees

Definition 2. (1) A family $(T_\tau : \tau \in {}^{<\omega}\mathbb{N})$ of elements of $\mathbb{L}$ is said to be a $\mathcal{A}$-*tree* if:
For each $\tau \in {}^{<\omega}\mathbb{N}$ $\{T_{\tau\frown n} : n \in \mathbb{N}\} \in \mathcal{A}$.
(2) For f in ${}^{\omega}\mathbb{N}$ the subset $\{T_{f\lceil_n} : n \in \mathbb{N}\}$ of an $\mathcal{A}$-tree is said to be a branch. It is said to be a $\mathcal{B}$–branch if $\{T_{f\lceil_n} : n \in \mathbb{N}\}$ is a member of $\mathcal{B}$.
(3) An $\mathcal{A}$–tree $(T_\tau : \tau \in {}^{<\omega}\mathbb{N})$ is said to be a *Hurewicz $\mathcal{A}$-tree* if:
HT1 For each τ if $m < n$ then $T_{\tau\frown m} \leq T_{\tau\frown n}$;
HT2 For each τ and for each n, $T_\tau \leq T_{\tau\frown n}$.

Several basic constructions play an important role in determining when an $\mathcal{A}$-tree has a $\mathcal{B}$-branch.

Definition 3. Let $(T_\tau : \tau \in {}^{<\omega}\mathbb{N})$ be an $\mathcal{A}$–tree.
(1) For each n and for each k in $\mathbb{N}$ put

$$C^n_k = \begin{cases} T_k, & \text{if} \quad n = 1 \\ (\wedge\{C^{n-1}_k \wedge T_{\tau\frown k} : \tau \in {}^{n-1}\mathbb{N}\}), & \text{otherwise.} \end{cases}$$

(2) For each n in $\mathbb{N}$ put $\mathcal{U}_n = \{C^n_k : k \in \mathbb{N}\}$. The sequence $(\mathcal{U}_n : n \in \mathbb{N})$ is said to be the *linearization* of the $\mathcal{A}$–tree.

The definition of C^n_k for $n > 1$ may appear to require that the lattice in question has a certain degree of completeness under infinitary operations. This is only appearances, since only finitely many terms in the definition have an effect on computing C^n_k:

Lemma 1. *Let $(T_\tau : \tau \in {}^{<\omega}\mathbb{N})$ be a Hurewicz $\mathcal{A}$–tree.*

(1) *For each* n *and for each* $(i_1, \dots, i_n)$ *such that* $k \leq \max\{i_1, \dots, i_n\}$ *we have*

$$C_k^n \leq T_{(i_1,\dots,i_n)}$$

(2) *For each* n*: If* $k < \ell$*, then* $C_k^n \leq C_\ell^n$.

Proof. The proof of 1 is by induction on n.

$\underline{n=1}$: If $k \leq i_1$ then by HT1, $T_k \leq T_{i_1}$, *i.e.*, $C_k^1 \leq T_{i_1}$.

$\underline{n>1}$: Put $n' = n-1$. Consider C_k^n and $T_{i_1,\dots,i_n}$ with $k \leq \max\{i_1, \dots, i_n\}$.

Case 1: $i_n < k$.

Then $k \leq \max\{i_1, \dots, i_{n'}\}$ and so by HT2

$$C_k^{n'} \leq T_{i_1,\dots,i_{n'}} \leq T_{i_1,\dots,i_{n'},i_n}.$$

But since $C_k^n \leq C_k^{n'}$ we are done.

Case 2: $k \leq i_n$.

Then by definition and by HT1, $C_k^n \leq T_{i_1,\dots,i_{n'},k} \leq T_{i_1,\dots,i_{n'},i_n}$ and we are done.

Also the proof of 2 is by induction on n.

Case 1: $n = 1$. Then $C_k^n = T_k$ and $C_\ell^n = T_\ell$. Since these are members of a Hurewics $\mathcal{A}$–tree we have from HT1 that $C_k^n \leq C_\ell^n$.

Case 2: $n > 1$ and the result is true below n. Thus, $C_k^{n-1} \leq C_\ell^{n-1}$. Also, by HT1,

$$\wedge\{T_{\sigma\frown k} : \sigma \in {}^n\mathbb{N}\} \leq \wedge\{T_{\sigma\frown\ell} : \sigma \in {}^n\mathbb{N}\}.$$

The result follows. ☐

Much of the theory of the selection principle $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ can be derived from the following fact:

Theorem 2 (Fundamental Theorem for Hurewicz $\mathcal{A}$-trees)**.** *Assume that* $(\mathcal{A}, \mathcal{B})$ *has the following properties:*

(1) *Each term of the linearization of each Hurewicz* $\mathcal{A}$*–tree is a member of* $\mathcal{A}$*,*

(2) $\mathcal{B}$ *is refinement closed (and thus cofinally closed)*

If $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ *holds, then each Hurewicz* $\mathcal{A}$*-tree has a* $\mathcal{B}$*-branch.*

Proof. Let $(T_\tau : \tau \in {}^{<\omega}\mathbb{N})$ be a Hurewicz $\mathcal{A}$-tree. Let $(U_n : n \in \mathbb{N})$ be its linearization. Each U_n is a member of $\mathcal{A}$. Apply $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ to the sequence $(U_n : n \in \mathbb{N})$ and choose for each n a finite set $F_n \subset U_n$ such that $\cup_{n\in\mathbb{N}} F_n$ is a member of $\mathcal{B}$. For each n choose a finite subset G_n of U_n such that $F_n \subset G_n$, and with k_n maximal with $C_{k_n}^n \in G_n$ we have for $j < \ell$ that $k_j < k_\ell$. Then by 3 also $\cup_{n\in\mathbb{N}} G_n$ is a member of $\mathcal{B}$.

By Lemma 1 part 2, each element of G_n is $\leq C_{k_n}^n$. Thus, $\{C_{k_n}^n : n \in \mathbb{N}\}$ is cofinal in $\cup_{n\in\mathbb{N}} G_n$, and so by 2 is also a member of $\mathcal{B}$. Since $k_1 < k_2 < \dots < k_n < \dots$ we have from Lemma 1 part 1 that for each n also

$$C_{k_n}^n \leq T_{k_1,\dots,k_n}.$$

Again applying 3 we see that $\{T_{k_1,\dots,k_n} : n \in \mathbb{N}\}$ is a member of $\mathcal{B}$. But this set is a $\mathcal{B}$-branch of the Hurewicz $\mathcal{A}$–tree. ☐

3. HUREWICZ PAIRS AND THE GAME $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$: FROM $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ TO $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$.

Game-theoretic applications of the fundamental theorem of Hurewicz $\mathcal{A}$-trees depend on being able to construct from strategies of player ONE such trees whose linearizations have appropriate properties. The notion of a Hurewicz pair, which we now define, captures some of the requirements for such constructions. $\mathcal{A}$ is said to be *Lindelöf* if each element of $\mathcal{A}$ has a countable subset which is in $\mathcal{A}$. It is evident that $\mathsf{S}_{fin}(\mathcal{A},\mathcal{A})$ implies that $\mathcal{A}$ is Lindelöf; the converse need not be true.

Definition 4. $(\mathcal{A},\mathcal{B})$ is a *Hurewicz pair* if it has the following properties:

H1 $\mathcal{A}$ is Lindelöf;
H2 For each A in $\mathcal{A}$, $\{\wedge\mathcal{F} : \emptyset \neq \mathcal{F} \subset A \text{ finite}\}$ is an element of $\mathcal{A}$;
H3 $\mathcal{A}$ is refinement closed;
H4 For each nonempty finite set $F \subset \cup\mathcal{A}$, $\vee F$ is in $\cup\mathcal{A}$.
H5 Each term of the linearization of a Hurewicz $\mathcal{A}$–tree is a member of $\mathcal{A}$.
H6 For each $B \in \mathcal{B}$ and each $C \subset \cup\mathcal{A}$, if $B \subset C$, then $C \in \mathcal{B}$
H7 For each $B \subset \cup\mathcal{A}$, if $\{\vee\mathcal{F} : \emptyset \neq \mathcal{F} \subset B \text{ finite }\} \in \mathcal{B}$, then $B \in \mathcal{B}$.
H8 $\mathcal{B}$ is refinement closed

$\mathcal{A}$ is a Hurewicz family if $(\mathcal{A},\mathcal{A})$ is a Hurewicz pair.

Definition 5. For a given family $\mathcal{A}$ let $\mathcal{A}_\Omega$ denote the set of $A \in \mathcal{A}$ which have the following property:

For each k and each partition $A = A_1 \cup \cdots \cup A_k$ of A there is a $j \leq k$ with $A_j \in \mathcal{A}$.

Lemma 3. *Let $(\mathcal{A},\mathcal{B})$ be a Hurewicz pair.*

1. *For each $A \in \mathcal{A}$, for each nonempty finite subset F of A, $\wedge F \in \cup\mathcal{A}$.*
2. *For $A \in \mathcal{A}$ and $b \in \cup\mathcal{A}$, $\{a \vee b : a \in A\} \in \mathcal{A}$.*
3. *For each $A \in \mathcal{A}$ the set $A^* := \{\vee F : \emptyset \neq F \subset A \text{ finite}\}$ is in $\mathcal{A}_\Omega$.*
4. *Each $A \in \mathcal{A}$ contains a countable subset B such that whenever $a, b \in B$, then either $a < b$ or $b < a$, and every infinite subset of B is in $\mathcal{A}$.*
5. *If $\mathcal{F}$ is a family of elements of $\mathcal{A}$, then $\cup\mathcal{F}$ is an element of $\mathcal{A}$.*

Proof. 1 follows immediately from [H2].
2 follows from [H3] and [H4].
3 follows from [H3] and [H4] that A^* is in $\mathcal{A}$. To see that A^* is in $\mathcal{A}_\Omega$, choose a partition $A^* = B_1 \cup \cdots \cup B_k$. Then A refines some B_j (For suppose the contrary and choose for each j an $a_j \in A$ but there is no $b \in B_j$ with $a_j \leq b$. Then put $c = \vee_{j\leq k} a_j$. This is an element of A^*, so a member of some B_j. But then $a_j \leq c$, contradicting the choice of a_j). By [H3], B_j is in $\mathcal{A}$.
4 By [H1], let $\{a_n : n = 1, 2, 3, \dots\}$ be a subset of A which is in $\mathcal{A}$. For each n put $b_n = \vee_{j\leq n} a_j$. By [H4] each b_n is an element of $\cup\mathcal{A}$. Put $B = \{b_n : n = 1, 2, 3, \dots\}$. Then as A refines B, [H3] implies that $B \in \mathcal{A}$. B is as required.
5 Each element of $\mathcal{F}$ refines $\cup\mathcal{F}$. Apply [H3]. □

We now consider for the families $\mathcal{A}$ and $\mathcal{B}$ of subsets of $\mathbb{L}$ the infinite game $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ that was defined in the introduction. First we analyze properties of winning strategies for player ONE in this game. We shall call finite sequences $(F_1,\dots,F_n)$ *admissible* if

(1) $j \leq n$ there is a $A_j \in \mathcal{A}$ such that $F_j \subseteq A_j$, and
(2) each F_j is finite.

Lemma 4. *Let $(\mathcal{A},\mathcal{B})$ be a Hurewicz pair. If ONE has a winning strategy in the game $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ then ONE has a winning strategy F such that for each admissible sequence $(F_1,\dots,F_n)$, for each $b \in F(F_1,\dots,F_n)$, we have $\vee_{j\leq n}(\vee F_j) \leq b$.*

Proof. Let σ be a winning strategy for ONE in $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$. Let $\prec$ be a well-order of the set of finite subsets of S. Define a strategy F for ONE as follows:

- $F(\emptyset) = \sigma(\emptyset)$
- For $F_1 \subset F(\emptyset)$ finite, first compute $\sigma(F_1)$ and $b = \vee F_1$. By [H4] there is an $A \in \mathcal{A}$ with $b \in A$. Put
$$F(F_1) = \{b \vee c : c \in \sigma(F_1)\}$$
By 2 of Lemma 3, $F(F_1) \in \mathcal{A}$.
- For $F_2 \subset F(F_1)$ finite, let $W_2 \subset \sigma(F_1)$ be the $\prec$-first finite set with $F_2 = \{W \vee (\vee F_1) : W \in W_2\}$. Compute $\sigma(F_1, W_2)$ and define $F(F_1,F_2) = \{(\vee F_2) \vee c : c \in \sigma(F_1,W_2)\}$. By 1 of Lemma 3 and by H4 $F(F_1,F_2) \in \mathcal{A}$.
- For $F_{n+1} \subset F(F_1,\dots,F_n)$ finite, for $2 \leq j \leq n$ let $W_{j+1} \subset \sigma(F_1,W_2,\dots,W_j)$ be the $\prec$-first finite set with $F_{j+1} = \{W \vee (\vee_{i\leq j} \vee F_i) : W \in W_{j+1}\}$. Compute $\sigma(F_1,W_2,\dots,W_{n+1})$ and define $F(F_1,F_2,\dots,F_{n+1}) = \{(\vee F_{n+1}) \vee c : c \in \sigma(F_1,W_2,\dots,W_{n+1})\}$. By 1 of Lemma 3 and by [H4] $F(F_1,F_2,\dots,F_{n+1}) \in \mathcal{A}$.

To see that F is a winning strategy for ONE, consider an F-play
$$F(\emptyset), T_1, F(T_1), T_2, F(T_1,T_2), \dots$$
From the definition of F recursively choose finite sets $W_1,\dots,W_n,\dots$ such that

(1) $W_1 = T_1$,
(2) W_{n+1} is the$\prec$-least finite set with $W_{n+1} \subseteq \sigma(W_1,\dots,W_n)$ and $T_{n+1} = \{(\vee T_n) \vee b : b \in W_{n+1}\}$.

Then
$$\sigma(\emptyset), W_1, \sigma(W_1), W_2, \sigma(W_1,W_2), \dots$$
is a σ-play of $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$, so won by ONE, so $\cup_{n\in\mathbb{N}} W_n \notin \mathcal{B}$. Then by the contrapositive of [H7] also $\{\vee F : \emptyset \neq F \subseteq \cup_{n\in\mathbb{N}} W_n \text{ finite}\}$ is not in $\mathcal{B}$. Then by [H8] also $\cup_{n\in\mathbb{N}} T_n$ is not in $\mathcal{B}$. Thus ONE wins the F-play we are considering. $\square$

Lemma 5. *Let $(\mathcal{A},\mathcal{B})$ be a Hurewicz pair. If σ is a winning strategy for ONE in $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$, then each strategy τ of ONE which satisfies*
$$\tau(T_1,\dots,T_n) \subseteq \sigma(T_1,\dots,T_n),$$
each admissible sequence $(T_1,\dots,T_n)$ is a winning strategy.

Proof. Consider a τ–play

$$\tau(\emptyset), T_1, \tau(T_1), T_2, \tau(T_1, T_2), \ldots$$

We have $\tau(\emptyset) \subseteq \sigma(\emptyset)$, and so $T_1 \subseteq \sigma(\emptyset)$. For each n, $T_{n+1} \subseteq \tau(T_1, \ldots, T_n) \subseteq \sigma(T_1, \ldots, T_n)$. Thus

$$\sigma(\emptyset), T_1, \sigma(T_1), T_2, \sigma(T_1, T_2), \ldots$$

is a σ-play of $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$, and so won by ONE. This means $\cup_{n \in \mathbb{N}} T_n \notin \mathcal{B}$. Thus the original τ–play is won by ONE. $\square$

Lemma 6. *Let $(\mathcal{A}, \mathcal{B})$ be a Hurewicz pair. If ONE has a winning strategy in the game $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$, then ONE has a winning strategy F which calls in each inning on ONE to play a member of $\mathcal{A}$ which under the partial order $<$ is an ω-sequence.*

Proof. Let σ be a winning strategy for ONE. By Lemma 5 and by [H1] we may assume that in each inning σ calls on ONE to play a countable element of $\mathcal{A}$.

Define a new strategy G for ONE as follows:

(1) With $\sigma(\emptyset) = \{b_n : n \in \mathbb{N}\}$ define $G(\emptyset) = \{\vee_{j \leq n} b_j : n \in \mathbb{N}\}$. By [H3] and the proof of 4 of Lemma 3, $G(\emptyset) \in \mathcal{A}$.

(2) To define $G(T_1, \ldots, T_n)$, choose for each $j \leq n$ a minimal finite set F_j with $F_1 \subset \sigma(\emptyset)$ and the elements of T_1 sups of elements of F_1, and for $j > 1$, $F_j \subset \sigma(F_1, \ldots, F_{j-1})$ and the elements of T_j sups of elements of F_j. Suppose that $\sigma(F_1, \ldots, F_n) = \{b_m : m \in \mathbb{N}\} \in \mathcal{A}$ and define

$$G(T_1, \ldots, T_n) = \{\vee_{j \leq m} b_j : m \in \mathbb{N}\}.$$

As before, $G(T_1, \ldots, T_n)$ is an element of $\mathcal{A}$.

If σ is a winning strategy for ONE, then by [H7] and [H8] also G is a winning strategy for ONE. $\square$

We now show that for Hurewicz pairs $(\mathcal{A}, \mathcal{B})$ the selection principle $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ is characterized by the game $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$.

Theorem 7 (Fundamental Theorem for Hurewicz Pairs)**.** *If $(\mathcal{A}, \mathcal{B})$ is a Hurewicz pair, then the following are equivalent:*

(1) $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$,

(2) *ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$.*

Proof. $2 \Rightarrow 1$: Let $(A_n : n \in \mathbb{N})$ be a sequence of elements of $\mathcal{A}$. Define for ONE the strategy σ so that in the n-th inning σ calls on ONE to play A_n. By 2 this is not a winning strategy for ONE. Consider a σ-play lost by ONE, say

$$A_1, T_1, \ldots, A_n, T_n, \ldots$$

where for each n the set T_n is a finite subset of A_n. Since ONE loses this play we have $\cup_{n \in \mathbb{N}} T_n \in \mathcal{B}$.

$1 \Rightarrow 2$: Let σ be a strategy for ONE. By Lemmas 5 and 6 we may assume that σ has the properties that

- In each inning, σ calls on ONE to play an ω-chain which is a member of $\mathcal{A}$ and
- For each finite sequence $(T_1, \dots, T_n)$ of finite sets such that $T_1 \subseteq \sigma(\emptyset)$ and for $1 < j \leq n$ $T_j \subset \sigma(T_1, \dots, T_{j-1})$, we have $\vee T_n \leq C$ for each $C \in \sigma(T_1, \dots, T_n)$.

Define the following array:

(1) $(T_n : n \in \mathbb{N})$ enumerates $\sigma(\emptyset)$ in such a way that $m < n \Rightarrow T_m < T_n$

(2) With $T_{i_1,\dots,i_n}$ defined for each $(i_1, \dots, i_n) \in {}^n\mathbb{N}$, let $(T_{i_1,\dots,i_n,k} : k \in \mathbb{N})$ enumerate $\sigma(T_{i_1}, T_{i_1,i_2}, \dots, T_{i_1,\dots,i_n})$ in such a way that for $j < k$ we have $T_{i_1,\dots,i_n,j} < T_{i_1,\dots,i_n,k}$.

Then $(T_\tau : \tau \in {}^{<\omega}\mathbb{N})$ is, by the rules of the game $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$, a Hurewicz $\mathcal{A}$–tree. Since $(\mathcal{A}, \mathcal{B})$ is a Hurewicz pair, and since $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ holds, Theorem 2 implies that this tree has a $\mathcal{B}$–branch. Let $f \in {}^\omega\mathbb{N}$ be given such that $(T_{f\lceil_n} : n \in \mathbb{N})$ is a $\mathcal{B}$-branch. Then

$$\sigma(\emptyset), T_{f(1)}, \sigma(T_{f(1)}), T_{f(1),f(2)}, \sigma(T_{f(1)}, T_{f(1),f(2)}), \dots$$

is a σ-play of $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$, and is lost by ONE. $\square$

For our applications below we will also need the following result:

Theorem 8. *Let $(\mathcal{A}, \mathcal{B})$ be a Hurewicz family. Then the following are equivalent:*

(1) $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ *holds.*

(2) $\mathsf{S}_{fin}(\mathcal{A}_\Omega, \mathcal{B})$ *holds.*

Proof. $1 \Rightarrow 2$ holds because $\mathcal{A}_\Omega \subseteq \mathcal{A}$. To see that $2 \Rightarrow 1$ holds, let a sequence $(A_n : n = 1, 2, 3, \dots)$ of elements of $\mathcal{A}$ be given. For each n define: $A_n^* = \{\vee F : F \subset A_n\}$ By 3 of Lemma 3, each A_n^* is in $\mathcal{A}_\Omega$. Apply $\mathsf{S}_{fin}(\mathcal{A}_\Omega, \mathcal{B})$ to $(A_n^* : n = 1, 2, 3, \dots)$: For each n fix a finite set $F_n \subset A_n^*$ such that $\mathbf{B} = \cup_{n<\infty} F_n$ is in $\mathcal{B}$. For each n, and for each $x \in F_n$, choose a finite subset G_x of A_n with $x = \vee G_x$. Put $V_n = \cup\{G_x : x \in F_n\}$, and finally put $V = \cup_{n<\infty} V_n$. Observe that for each n, V_n is a finite subset of A_n. Since V is a subset of $\cup\mathcal{A}$, and $\mathbf{V} = \{\vee F : F \subset V \text{ finite}\}$ is refined by the member $\mathbf{B}$ of $\mathcal{B}$, [H8] implies that $\mathbf{V}$ is an element of $\mathcal{B}$. But then [H7] implies that V is an element of $\mathcal{B}$. $\square$

Applications. Let (X, τ) be a topological space and let Y be a subset of X.

Let $\mathcal{O}_X$ denote the set of open covers of X, and let $\mathcal{O}_{XY}$ denote the set of covers of Y by sets open in X. If X is a Lindelöf space then $(\mathcal{O}_X, \mathcal{O}_{XY})$ is a Hurewicz pair. In particular, $\mathcal{O}_X$ is a Hurewicz family. Thus $\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_{XY})$ holds if, and only if, ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathcal{O}_X, \mathcal{O}_{XY})$. This result for the case $X = Y$ is Theorem 10 of [8]. The general case is Theorem 3.4.4 of [1].

Let $\mathcal{O}_{\Omega(X)}$ denote $(\mathcal{O}_X)_\Omega$ as per Definition 5 and let the symbol Ω_X denotes the collection of ω-covers of X. An open cover $\mathcal{U}$ of X is an ω-cover if $X \notin \mathcal{U}$, and if there is for each finite subset F of X a $U \in \mathcal{U}$ such that $F \subset U$. One can show that Ω_X consists of those members of $\mathcal{O}_{\Omega(X)}$ which do not have X as a member. Theorem 8 implies that $\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_{XY})$ holds if, and only if,

$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_{XY})$ holds. Observe that since Ω_X is a subset of $\mathcal{O}_X$, it follows that if ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{O}_X, \mathcal{O}_{XY})$, then also ONE has no winning strategy in the game $\mathsf{G}_{fin}(\Omega_X, \mathcal{O}_{XY})$, because in the latter game ONE is even more restricted in possible moves. This in turn implies that the selection hypothesis $\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_{XY})$ holds. Thus we obtain:

Corollary 9. *If (X, τ) is a Lindelöf space and Y is a subspace of X, the following are equivalent:*

(1) $\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_{XY})$ *holds;*
(2) *ONE has no winning strategy in the game* $\mathsf{G}_{fin}(\mathcal{O}_X, \mathcal{O}_{XY})$*;*
(3) *ONE has no winning strategy in the game* $\mathsf{G}_{fin}(\Omega_X, \mathcal{O}_{XY})$*;*
(4) $\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_{XY})$ *holds.*

The hard implication here was that $1 \Rightarrow 2$.

Let $\mathcal{D}_X$ be the collection of families $\mathcal{U}$ of open sets with $\cup\mathcal{U}$ dense in X. Let $\mathcal{D}_{\Omega(X)}$ denote $(\mathcal{D}_X)_\Omega$ as per Definition 5. Extending the notation of [14], let $\Omega_{\mathcal{D}_X}$ denote the collection of elements $\mathcal{U}$ of $\mathcal{D}_X$ such that no element of $\mathcal{U}$ is a dense subset of X and for each finite set $\mathcal{F}$ of nonempty open subsets of the space there is an $A \in \mathcal{U}$ such that for each $F \in \mathcal{F}$, $F \cap A \neq \emptyset$. One can show that $\Omega_{\mathcal{D}_X}$ consists of those members of $\mathcal{D}_{\Omega(X)}$ which do not have dense subsets of X as members. For the subspace Y of X let $\mathcal{D}_{XY}$ denote the set of the families $\mathcal{U}$ of open subsets of X for which $(\cup\mathcal{U}) \cap Y$ is a dense subset of Y.

Then $(\mathcal{D}_X, \mathcal{D}_{XY})$ is a Hurewicz pair. In particular, $\mathcal{D}_X$ is a Hurewicz family. Thus, $\mathsf{S}_{fin}(\mathcal{D}_X, \mathcal{D}_{XY})$ holds if, and only if, ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathcal{D}_X, \mathcal{D}_{XY})$. The version where $Y = X$ of this result is Theorem 2 of [14].

Since $\Omega_{\mathcal{D}_X}$ is a subset of $\mathcal{D}_X$, it follows that if ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{D}_X, \mathcal{D}_{XY})$, then ONE has no winning strategy in $\mathsf{G}_{fin}(\Omega_{\mathcal{D}_X}, \mathcal{D}_{XY})$. This in turn implies in an elementary way that $\mathsf{S}_{fin}(\Omega_{\mathcal{D}_X}, \mathcal{D}_{XY})$ holds. Theorem 8 implies that $\mathsf{S}_{fin}(\mathcal{D}_{\Omega(X)}, \mathcal{D}_{XY})$ holds if, and only if, $\mathsf{S}_{fin}(\mathcal{D}_X, \mathcal{D}_{XY})$ holds. And one can also show that $\mathsf{S}_{fin}(\Omega_{\mathcal{D}_X}, \mathcal{D}_{XY})$ is equivalent to $\mathsf{S}_{fin}(\mathcal{D}_{\Omega(X)}, \mathcal{D}_{XY})$. Thus we obtain the following generalization of Theorem 4 of [14]:

Corollary 10. *If (X, τ) is a Lindelöf space and Y is a subspace of X, the following are equivalent:*

(1) $\mathsf{S}_{fin}(\mathcal{D}_X, \mathcal{D}_{XY})$ *holds;*
(2) *ONE has no winning strategy in the game* $\mathsf{G}_{fin}(\mathcal{D}_X, \mathcal{D}_{XY})$*;*
(3) *ONE has no winning strategy in the game* $\mathsf{G}_{fin}(\Omega_{\mathcal{D}_X}, \mathcal{D}_{XY})$*;*
(4) $\mathsf{S}_{fin}(\Omega_{\mathcal{D}_X}, \mathcal{D}_{XY})$ *holds.*

Again, the hard implication is $1 \Rightarrow 2$.

As a further application we obtain the following result on Pixley-Roy duality: For space X and subspace Y of X, let $\mathsf{PR}(X)$ and $\mathsf{PR}(Y)$ respectively denote the Pixley-Roy hyperspaces of X and Y. See for example [16] for an elementary introduction. Above we introduced already Ω_X. The symbol Ω_{XY} denotes those families $\mathcal{U}$ of open subsets of X with the property that there is for each finite subset F of Y an element $U \in \mathcal{U}$ with $F \subset U$. In Theorem 3.3.6 of [1] it was shown that the following two statements are equivalent:

(1) For each n, $\mathsf{S}_{fin}(\mathcal{O}_{X^n}, \mathcal{O}_{X^nY^n})$ holds;
(2) $\mathsf{S}_{fin}(\Omega_X, \Omega_{XY})$ holds.

Using the techniques in the proof of Theorem 6 of [16] one obtains the following generalization of corresponding results of [5] and [14]:

Corollary 11. *Let X be a separable metric space and let Y be a subspace of X. The following are equivalent:*

(1) $\mathsf{S}_{fin}(\mathcal{D}_{\mathsf{PR}(X)}, \mathcal{D}_{\mathsf{PR}(X)\mathsf{PR}(Y)})$ *holds;*
(2) *For each n,* $\mathsf{S}_{fin}(\mathcal{O}_{X^n}, \mathcal{O}_{X^nY^n})$ *holds;*
(3) $\mathsf{S}_{fin}(\Omega_X, \Omega_{XY})$ *holds.*

4. RAMSEY FAMILIES: FROM $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$ TO $\mathcal{A} \to \lceil \mathcal{B} \rceil^2_k$.

In this section we show how the game $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$ can be used to verify that the partition relation $\mathcal{A} \to \lceil \mathcal{B} \rceil^2_k$ is true in when $\mathcal{A}$ has some special properties.

Definition 6. A family $\mathcal{A}$ of subsets of a lattice $\mathbb{L}$ is said to be a *Ramsey family* if for each $A \in \mathcal{A}$ and each k, for each partition $A = \cup_{j \le k} A_j$ of A into k pieces, there is a $j \le k$ such that $A_j \in \mathcal{A}$.

Thus, in our earlier notation of Definition 5, $\mathcal{A}_\Omega$ is a Ramsey family.

Theorem 12 (Fundamental Theorem of Ramsey Families)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of the lattice $\mathbb{L}$ such that $\mathcal{A}$ is Lindelöf. If $\mathcal{A}$ is a Ramsey family and if ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$, then for each $k \in \mathbb{N}$ the partition relation $\mathcal{A} \to \lceil \mathcal{B} \rceil^2_k$ holds.*

Proof. Let $A \in \mathcal{A}$ as well as a positive integer k be given. We may assume that A is countable. Enumerate A bijectively as $(a_n : n < \infty)$. Fix a function $f : [A]^2 \to \{1, \dots, k\}$. Recursively define a sequence $(A_n : n < \infty)$ of subsets of A, and a sequence $(i_n : n < \infty)$ of elements of $\{1, \dots, k\}$ so that:

(1) For each n, $A_n \in \mathcal{A}$ and $A_{n+1} \subset A_n$;
(2) For each n, $A_{n+1} = \{a_j \in A_n : j > n+1 \quad \text{and} \quad f(\{a_{n+1}, a_j\}) = i_{n+1}\}$.

To see that this can be done, first observe that putting $B_j = \{a_i \in A : i > 1 \text{ and } f(\{a_1, a_i\}) = j\}$ we get a partition of $A \setminus \{a_1\}$ into finitely many pieces. Since $\mathcal{A}$ is a Ramsey family there is a j for which B_j is in $\mathcal{A}$. Fix such a j and set $i_1 = j$, $A_1 = B_j$. Next observe that by similar argument the remaining terms of an infinite sequence as above can be selected consecutively.

Next, for each $j \in \{1, \dots, k\}$ define $E_j = \{a_n : i_n = j\}$. For each n, $A_n \cap E_1$, $\dots$, $A_n \cap E_k$ partitions A_n into finitely many pieces, and so there is a j_n with $A_n \cap E_{j_n}$ in $\mathcal{A}$. Since for each n we have $A_n \supset A_{n+1}$, we may assume that the same j works for all A_n's. Fix such a j.

Now define the following strategy, σ, for ONE in the game $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$: In the first inning, $\sigma(\emptyset)=A_1\cap E_j$. If TWO responds by choosing a finite set $T_1\subset\sigma(\emptyset)$, let n_1 be $1+\max\{n: a_n\in T_1\}$. Then ONE plays $\sigma(T_1)=A_{n_1}\cap E_j$. If TWO responds with finite set $T_2\subset\sigma(T_1)$, and put $n_2=1+\max\{n: a_n\in T_2\}$. Then ONE plays $\sigma(T_1,T_2)=A_{n_2}\cap E_j$, and so on. (Observe that $n_1<n_2$.)

Since ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$, there is a play during which ONE used σ but lost. Let

$$\sigma(\emptyset),T_1,\sigma(T_1),T_2,\sigma(T_1,T_2),T_3,\dots$$

be such a play lost by ONE. Then as TWO won we have $B=\cup_{n\in\mathbb{N}}T_n\in\mathcal{B}$. By the definition of the strategy σ we also have for $r\neq s$ that $n_r\neq n_s$, and $f(\{a,b\})=j$ whenever a and b are from distinct T_j's. $\square$

Applications. There are many examples of Ramsey families in the literature. We give some applications of Theorem 12 to these.

With (X,τ) a Lindelöf topological space and let Y be a subset of X, $(\mathcal{O}_X,\mathcal{O}_Y)$ is a Hurewicz pair and $\mathcal{O}_X$ is a Hurewicz family. Moreover, Ω_X is a Ramsey family. According to Gerlits and Nagy (see [7]) a space is said to be an ϵ-space if Ω_X is a Lindelöf family.

Corollary 13. *If (X,τ) is an ϵ-space and Y is a subspace of X, then $\mathsf{S}_{fin}(\mathcal{O}_X,\mathcal{O}_Y)$ implies that $\Omega_X\rightarrow\lceil\mathcal{O}_Y\rceil^2_k$ holds for each k.*

Proof. We saw in Corollary 9 that $\mathsf{S}_{fin}(\mathcal{O}_X,\mathcal{O}_Y)$ is equivalent to ONE not having a winning strategy in $\mathsf{G}_{fin}(\Omega_X,\mathcal{O}_Y)$. Apply Theorem 12. $\square$

The case when $X=Y$ of this corollary was obtained in Theorem 6 of [17].

It was noted in [14] (p. 21) that if each finite power of a space has countable cellularity then for that space the family $\mathcal{D}_\Omega$ is a Lindelöf family. In particular, separable metric spaces have this property. The case $X=Y$ of the following corollary was obtained in Theorem 10 of [14]:

Corollary 14. *Let (X,τ) is a Lindelöf space such that $\mathcal{D}_{\Omega(X)}$ is a Lindelöf family. Let Y be a subspace of X. If $\mathsf{S}_{fin}(\mathcal{D}_X,\mathcal{D}_Y)$ holds then $\mathcal{D}_{\Omega(X)}\rightarrow\lceil\mathcal{D}_Y\rceil^2_k$ holds for each k.*

Proof. By Corollary 10 $\mathsf{S}_{fin}(\mathcal{D}_X,\mathcal{D}_Y)$ implies that ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{D}_{\Omega(X)},\mathcal{D}_Y)$. Apply Theorem 12. $\square$

And this in turn immediately gives the following generalization of (11) of Corollary 11 of [14]:

Corollary 15. *Let X be a separable metric space and let Y be a subspace of X. If $\mathsf{S}_{fin}(\mathcal{D}_{\mathsf{PR}(X)},\mathcal{D}_{\mathsf{PR}(Y)})$ holds then for each k, $\mathcal{D}_{\Omega(\mathsf{PR}(X))}\rightarrow\lceil\mathcal{D}_{\mathsf{PR}(Y)})\rceil^2_k$ holds.*

5. Selectable pairs: From $\mathcal{A} \to \lceil \mathcal{B} \rceil^2_k$ to $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$.

In this section we show how the partition relation $\mathcal{A} \to \lceil \mathcal{B} \rceil^2_k$ implies $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ appropriate $\mathcal{A}$ and $\mathcal{B}$. We require the following notion: An element A of $\mathcal{A}$ is said to be *large* if it is nonempty and for each finite subset F of $\mathbb{L}$, $A \setminus F$ is in $\mathcal{A}$. The symbol $\mathcal{A}_\Lambda$ denotes the set $\{A \in \mathcal{A} : A \text{ is large}\}$.

Definition 7. $(\mathcal{A}, \mathcal{B})$ is a *selectable pair* if

- S1 $\mathcal{A}$ is Lindelöf;
- S2 The union of countably many members of $\mathcal{A}$ is a member of $\mathcal{A}$;
- S3 If A is a countable element of $\mathcal{A}$, $f : A \to \mathbb{N}$ is a function and $(B_n : n \in \mathbb{N})$ is a sequence of elements of $\mathcal{A}$, then $\{a \wedge b : a \in A \text{ and } b \in B_{f(a)}\}$ is an element of $\mathcal{A}$;
- S4 For each $a \in \cup\mathcal{A}$, $\mathcal{P}(\{b \in \mathbb{L} : b \leq a\}) \cap \mathcal{B} = \emptyset$;
- S5 $\mathcal{B} = \mathcal{B}_\Lambda$;
- S6 If C is a countable subset of $\cup\mathcal{A}$ such that there is a $B \in \mathcal{B}$ with $\{b \in B : (\exists x \in C)(b \leq x)\} \in \mathcal{B}$, then C is a member of $\mathcal{B}$.

Certain Hurewicz pairs are selectable pairs.

Lemma 16. *If $(\mathcal{A}, \mathcal{B})$ is a Hurewicz pair, then it also has properties S1, S2 and S6.*

S1. is [H1]. [H3] implies [S2] as follows: Let $(A_n : n \in \mathbb{N})$ be a sequence of elements of $\mathcal{A}$ and put $A = A1$ and $B = \cup_{n<\infty} A_n$. Apply [H3].

[S6] follows from [H8] as follows: Let C be a countable subset of $\mathbb{L}$ and let $B \in \mathcal{B}$ be such that $A := \{b \in B : (\exists x \in C)(b \leq x)\}$ is in $\mathcal{B}$. Then A refines C, and now apply [H8]. $\square$

Theorem 17 (Fundamental Theorem of Selectable Pairs)**.** *If $(\mathcal{A}, \mathcal{B})$ is a selectable pair and $\mathcal{A} \to \lceil \mathcal{B} \rceil^2_2$, then $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ holds.*

Proof. Let $(\mathcal{U}_n : n < \infty)$ be a sequence of elements of $\mathcal{A}$. By [S1] we may assume each $\mathcal{U}_n$ is countable, and enumerate it as $(u^n_k : k < \infty)$. Define $\mathcal{V}$ to be the collection of elements of $\mathbb{L}$ of the form $u^1_n \wedge u^n_k$. By [S3] $\mathcal{V}$ is an element of $\mathcal{A}$. Choose for each element of $\mathcal{V}$ a representation of the form $u^1_n \wedge u^n_k$.

Define a function $f : [\mathcal{V}]^2 \to \{1, 2\}$ by

$$f(\{u^1_{n_1} \wedge u^{n_1}_k, u^1_{n_2} \wedge u^{n_2}_j\}) = \begin{cases} 1 & \text{if} \quad n_1 = n_2, \\ 2 & \text{otherwise.} \end{cases}$$

Choose a nearly homogeneous of color j $\mathcal{W} \subseteq \mathcal{V}$ with $\mathcal{W} \in \mathcal{B}$. Let $(\mathcal{W}_k : k < \infty)$ be a sequence of finite sets, disjoint from each other and with union $\mathcal{W}$, such that for a and b from distinct $\mathcal{W}_k$'s, $f(\{a, b\}) = j$.

<u>Case 1:</u> $j = 1$. Then there is an n such that for all $a \in \mathcal{W}$ we have $a \leq u^1_n$. Since [S4] then implies that $\mathcal{W}$ is not an element of $\mathcal{B}$, Case 1 does not hold.

<u>Case 2:</u> $j = 2$. For each $k > 1$ define $\mathcal{G}_k$ to be the set of u^k_j which occur as second coordinate in the chosen representations of elements of $\mathcal{W}$. Then each $\mathcal{G}_k$ is a finite subset of $\mathcal{U}_k$. Put $\mathcal{G} = \bigcup_{k<\infty} \mathcal{G}_k$.

We have $\mathcal{W} = \{b \in \mathcal{W} : (\exists x \in \mathcal{G})(b \leq x)\}$ is in $\mathcal{B}$, and each element of $\mathcal{G}$ is an element of some member of $\mathcal{A}$. Thus by [S6], $\mathcal{G}$ is an element of $\mathcal{B}$. By letting $\mathcal{G}_k$ be empty for those k for which no $\mathcal{G}_k$ was defined, we find that the sequence $(\mathcal{G}_k : k < \infty)$ witnesses $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ for $(\mathcal{U}_k : k < \infty)$. □

6. Applications

Let X be a space and Y a subspace of X. Let Ω_{XY} be the collection of ω-covers $\mathcal{U}$ of X such that no element of $\mathcal{U}$ covers Y (so, Y is infinite). Let Λ_Y be the collection of large covers of Y by sets open in X. If X is an ϵ-space, then (Ω_{XY}, Λ_Y) is a Hurewicz pair. Indeed, (Ω_{XY}, Λ_Y) is a selectable pair. Thus we obtain:

Corollary 18. *If X is an ϵ-space then $\Omega_{XY} \to \lceil \Lambda_Y \rceil^2_2$ implies that $\mathsf{S}_{fin}(\Omega_{XY}, \Lambda_Y)$ holds.*

If for a space X the family $\mathcal{D}_{\Omega(X)}$ is a Lindelöf family then the pair $(\mathcal{D}_{\Omega(X)}, \mathcal{D}_X)$ is a selectable pair. Thus we obtain Theoem 10 of [14]:

Corollary 19. *If $\mathcal{D}_{\Omega(X)}$ is a Lindelöf family then the following are equivalent:*

(1) $\mathcal{D}_{\Omega(X)} \to \lceil \mathcal{D}_X \rceil^2_2$;
(2) $\mathsf{S}_{fin}(\mathcal{D}_X, \mathcal{D}_X)$ *holds.*

For a space non-compact X, $\mathcal{K}$ denotes the compact-covering open covers of X, also said to be the k-covers of X in [6]. An open cover $\mathcal{U}$ of X is a k-cover if there is for every compact subset K of X an element U of $\mathcal{U}$ such that $K \subseteq U$, and $X \notin mathcalU$. $(\mathcal{K}, \mathcal{K})$ is a selectable pair, and thus we have the implication(2) $\Rightarrow$ (1) in Theorem 7 of [6]:

Corollary 20. *Let X is a non-compact space for which $\mathcal{K}$ is a Lindelöf family. If $\mathcal{K} \to \lceil \mathcal{K} \rceil^2_k$ for some $k \geq 2$, then $\mathsf{S}_{fin}(\mathcal{K}, \mathcal{K})$ holds.*

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