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One Observation on (n,m)–Semigroups

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Abstract. In this paper one Čupona–Trpenovski's theorem about $n-$semigroups with neutral element is generalized.

1. Preliminaries

Definition 1 ([2])**.** Let $n \geq m+1$ and let $(Q;A)$ be an $(n,m)-$groupoid $(A : Q^n \to Q^m)$. We say that $(Q;A)$ is an $(n,m)-$group iff the following stetements hold:

(i) For every $i,j \in \{1,\dots,n-m+1\},\ i<j$, the following law holds

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-m}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-m})$$

[$:<i,j>-$associative law][1]; and

(ii) For every $i \in \{1,\dots,n-m+1\}$ and for every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds

$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n.$$

Remark 1. For $m=1$ $(Q;A)$ is an $n-$group. Cf. [5].

Definition 2. Let $(Q;B)$ be a $(2m,m)-$groupoid and $m \geq 2$. Then:

(α) $\overset{1}{B} \overset{def}{=} B$; and

(β) For every $s \in N$ and for every $x_1^{(s+2)m} \in Q$

$$\overset{s+1}{B}(x_1^{(s+2)m}) \overset{def}{=} B(\overset{s}{B}(x_1^{(s+1)m}), x_{(s+1)m+1}^{(s+2)m}).$$

Proposition 1. *Let $(Q;B)$ be a $(2m,m)-$semigroup, $m \geq 2$ and $s \in N$. Then, for every $x_1^{(s+2)m} \in Q$ and for every $t \in \{1,\dots,sm+1\}$ the following equality holds*

$$\overset{s+1}{B}(x_1^{(s+2)m}) = \overset{s}{B}(x_1^{t-1}, B(x_t^{t+2m-1}), x_{t+2m}^{(s+2)m}).$$

Proof. See the proof in [6]. □

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[1] $(Q;A)$ is an $(n,m)-$semigroup.

By 1, 2 and by 1, we obtain:

Proposition 2 ([2])**.** *Let $(Q;B)$ be a $(2m,m)-$semigroup, $m \geq 2$ and $(i,j) \in N^2$. Then, for every $x_1^{(i+j+1)m} \in Q$ and for all $t \in \{1,\dots,im+1\}$ the following equality holds*

$$\overset{i+j}{B}(x_1^{(i+j+1)m}) = \overset{i}{B}(x_1^{t-1}, \overset{j}{B}(x_t^{t+(j+1)m-1}), x_{t+(j+1)m}^{(i+j+1)m}).$$

2. MAIN RESULTS

Theorem 1. *Let $(Q;A)$ be an $(n,m)-$semigroup, $n = k \cdot m$, $k \geq 3$ and e_1^m be an element from the set Q^m. Also, let for all $x_1^{2m} \in Q$ the following equalities hold:*

(1) $A(x_1^m, \overset{k-1}{\overline{e_1^m}|}) = x_1^m$,

(2) $A(e_1^m, x_1^m, \overset{k-2}{\overline{e_1^m}|}) = x_1^m$ *and*

(3) $A(x_1^{2m-1}, \overset{k-2}{\overline{e_1^m}|}, x_{2m}) = A(x_1^{2m}, \overset{k-2}{\overline{e_1^m}|})$.

Then there is a $(2m,m)-$semigroup $(Q;B)$ such that the following statements hold

a) For all $x_1^m \in Q^m$ the following equality holds

$$B(x_1^m, e_1^m) = x_1^m;$$

b) For all $x_1^m \in Q^m$ the following equality holds

$$B(e_1^m, x_1^m) = x_1^m;$$

c) For all $x_1^m \in Q^m$ the following equality holds

$$B(x_1^m, e_1^m) = B(x_1^{m-1}, e_1^m, x_m); \quad \text{and}$$

d) For every $x_1^{k\cdot m} \in Q$ the following equality holds

$$A(x_1^{k\cdot m}) = \overset{k-1}{B}(x_1^{k\cdot m}).$$

Remark 2. i) For $m = 1$ Theorem 1 is proved in [1]. Further on, for $m = 1$ (3) is surplus. Cf. Chapter II-1 in [5].

ii) If $(Q;B)$ is a $(2m,m)-$group, then $e_1 = \ldots = e_m$. Cf. [3]].

Sketch of the proof. Let

(o) $$B(x_1^{2m}) \overset{def}{=} A(x_1^{2m}, \overset{k-2}{\overline{e_1^m}|})$$

for all $x_1^{2m} \in Q$.

1° $i \in \{1, \dots, m\}$:

$$
\begin{aligned}
B(x_1^{i-1}, B(x_i^{i+2m-1}), x_{i+2m}^{3m}) &= \\
&\overset{(o)}{=} A(x_1^{i-1}, A(x_i^{i+2m-1}, \overset{k-2}{\overline{e_1^m}|}), x_{i+2m}^{3m}, \overset{k-2}{\overline{e_1^m}|}) = \\
&\overset{1.1}{=} A(x_1^{i}, A(x_{i+1}^{i+2m-1}, \overset{k-2}{\overline{e_1^m}|}, x_{i+2m}), x_{i+2m+1}^{3m}, \overset{k-2}{\overline{e_1^m}|}) = \\
&\overset{(3)}{=} A(x_1^{i}, A(x_{i+1}^{i+2m}, \overset{k-2}{\overline{e_1^m}|}), x_{i+2m+1}^{3m}, \overset{k-2}{\overline{e_1^m}|}) = \\
&\overset{(o)}{=} B(x_1^{i}, B(x_{i+1}^{i+2m}), x_{i+2m+1}^{3m}).
\end{aligned}
$$

2° $B(x_1^m, e_1^m) \overset{(o)}{=} A(x_1^m, e_1^m, \overset{k-2}{\overline{e_1^m}|}) == A(x_1^m, \overset{k-1}{\overline{e_1^m}|}) \overset{(2)}{=} x_1^m.$

3° $B(e_1^m, x_1^m) \overset{(o)}{=} A(e_1^m, x_1^m, \overset{k-2}{\overline{e_1^m}|}) \overset{(2)}{=} x_1^m.$

4°

$$
\begin{aligned}
A(x_1^{k\cdot m}) &\overset{(1)}{=} A(A(x_1^{k\cdot m}), \overset{k-1}{\overline{e_1^m}|}) = \\
&\overset{1.1}{=} A(x_1^m, A(x_{m+1}^{k\cdot m}, e_1^m), \overset{k-2}{\overline{e_1^m}|}) = \\
&\overset{(o)}{=} B(x_1^m, A(x_{m+1}^{k\cdot m}, e_1^m)) = \\
&\overset{(1)}{=} B(x_1^m, A(A(x_{m+1}^{k\cdot m}, e_1^m), \overset{k-1}{\overline{e_1^m}|})) = \\
&\overset{1.1}{=} B(x_1^m, A(x_{m+1}^{2m}, A(x_{2m+1}^{k\cdot m}, \overset{2}{\overline{e_1^m}|}), \overset{k-2}{\overline{e_1^m}|}) = \\
&\overset{(o)}{=} B(x_1^m, B(x_{m+1}^{2m}, A(x_{2m+1}^{k\cdot m}, \overset{2}{\overline{e_1^m}|})) = \\
&\overset{1.3}{=} \overset{2}{B}(x_1^{2m}, A(x_{2m+1}^{k\cdot m}, \overset{2}{\overline{e_1^m}|})) = \\
&\vdots \\
&\vdots \\
&= \overset{k-1}{B}(x_1^{(k-1)\cdot m}, A(x_{(k-1)\cdot m+1}^{k\cdot m}, \overset{k-1}{\overline{e_1^m}|})) = \\
&\overset{(1)}{=} \overset{k-1}{B}(x_1^{(k-1)\cdot m}, x_{(k-1)\cdot m+1}^{k\cdot m}) = \\
&= \overset{k-1}{B}(x_1^{k\cdot m}).
\end{aligned}
$$

5°

$$B(x_1^m, e_1^m) \overset{2^\circ, 3^\circ}{=} B(e_1^m, x_1^m) =$$

$$\overset{(o)}{=} A(e_1^m, x_1^m, \overbrace{e_1^m}^{k-2}) =$$

$$\overset{(3)}{=} A(e_1^m, x_1^{m-1}, \overbrace{e_1^m}^{k-2}, x_m) =$$

$$\overset{4^\circ}{=} \overset{k-1}{B}(e_1^m, x_1^{m-1}, \overbrace{e_1^m}^{k-2}, x_m)$$

$\overline{a})$ $k = 3:$

$$\overset{3-1}{B}(e_1^m, x_1^{m-1}, e_1^m, x_m) \overset{1^\circ}{=} B(e_1^m, B(x_1^{m-1}, e_1^m, x_m)) \overset{3^\circ}{=} B(x_1^{m-1}, e_1^m, x_m)$$

$\overline{b})$ $k > 3:$

$$\overset{k-1}{B}(e_1^m, x_1^{m-1}, \overbrace{e_1^m}^{k-2}, x_m) \overset{1.4}{=} B(e_1^m, B(x_1^{m-1}, \overset{k-3}{B}(\overbrace{e_1^m}^{k-2}), x_m)) =$$

$$\overset{3^\circ}{=} B(x_1^{m-1}, \overset{k-3}{B}(\overbrace{e_1^m}^{k-2}), x_m)) \overset{(\beta)2^\circ}{=} B(x_1^{m-1}, e_1^m, x_m).$$

□

Theorem 2. *Let $(Q;B)$ be a $(2m,m)-$semigroup, $m > 1$, e_1^m be an element from the set Q^m and let the following statements hold*

(a) *For all $x_1^m \in Q^m$ the following equality holds*
$$B(x_1^m, e_1^m) = x_1^m;$$

(b) *For all $x_1^m \in Q^m$ the following equality holds*
$$B(e_1^m, x_1^m) = x_1^m;$$

(c) *For all $x_1^m \in Q^m$ the following equality holds*
$$B(x_1^{m-1}, e_1^m, x_m) = x_1^m.$$

(d) *Also let*
$$A(x_1^{k\cdot m}) \overset{def}{=} \overset{k-1}{B}(x_1^{k\cdot m})$$
for every $x_1^{k\cdot m} \in Q$, where $k \geq 3$.

Then the following statements hold

1) *$(Q;A)$ is an $(km,m)-$semigroup; and*
2) *For all $x_1^{2m} \in Q$ equalities (1)–(3) from Theorem 1 hold in $(Q;A)$.*

Remark 3. Cf. Chapter II-1 in [5].

Sketch of the proof. °1 Proof of 1): By (d) and by 1.4.

$\circ 2$ $A(x_1^m, \overbrace{e_1^m}^{k-1}) \overset{(d)}{=} \overset{k-1}{B}(x_1^m, \overbrace{e_1^m}^{k-1}) \overset{1.4}{=\!=} B(x_1^m, \overset{k-2}{B}(\overbrace{e_1^m}^{k-1})) \overset{(a)}{=} x_1^m.$

$\circ 3$ $A(e_1^m, x_1^m, \overbrace{e_1^m}^{k-2}) \overset{(d)}{=} \overset{k-1}{B}(e_1^m, x_1^m, \overbrace{e_1^m}^{k-2}) \overset{(b),(a)}{=} x_1^m.$

$\circ 4$

$$\begin{aligned} A(x_1^{2m-1}, \overbrace{e_1^m}^{k-2}, x_{2m}) &\overset{(d)}{=} \overset{k-1}{B}(x_1^{2m-1}, \overbrace{e_1^m}^{k-2}, x_{2m}) = \\ &\overset{1.4}{=} B(x_1^m, B(x_{m+1}^{2m-1}, \overset{k-3}{B}(\overbrace{e_1^m}^{k-2}), x_{2m}))^{\dagger} = \\ &\overset{(c)}{=} B(x_1^m, B(x_{m+1}^{2m-1}, x_{2m}, \overset{k-3}{B}(\overbrace{e_1^m}^{k-2}))) = \\ &\overset{1.4}{=} \overset{k-1}{B}(x_1^{2m}, \overbrace{e_1^m}^{k-2}) = \\ &\overset{(d)}{=} A(x_1^{2m}, \overbrace{e_1^m}^{k-2}). \end{aligned}$$

□

Proposition 3. *Let $(Q; B)$ be a $(2m, m)-$semigroup, $m > 1$, e_1^m be an element from the set Q^m and let for all $x_1^m \in Q^m$ the following equalities hold*

$(\widehat{a})$ $B(e_1^m, x_1^m) = x_1^m$ *and*

$(\widehat{b})$ $B(x_1^m, e_1^m) = x_1^m$.

Then, for all $i \in \{0, 1, \ldots, m\}$ and for every $x_1^m \in Q^m$ the following equality holds

$(\widehat{c})$ $B(x_1^i, e_1^m, x_{i+1}^m) = x_1^{m\ddagger}$

Remark 4. In [3] Proposition 3 is proved for $(2m, m)-$groups. See, also [4].

Sketch of the proof.

$$\begin{aligned} B(x_1^i, e_1^m, x_{i+1}^m) &\overset{(\widehat{a})}{=} B(e_1^m, B(x_1^i, e_1^m, x_{i+1}^m)) = \\ &\overset{1.1(i)}{=} B(e_1^i, B(e_{i+1}^m, x_1^i, e_1^m), x_{i+1}^m) = \\ &\overset{(\widehat{b})}{=} B(e_1^i, e_{i+1}^m, x_1^i, x_{i+1}^m) = \\ &= B(e_1^m, x_1^m) = \\ &\overset{(\widehat{a})}{=} x_1^m. \end{aligned}$$

□

† $\overset{k-3}{B}(\overbrace{e_1^m}^{k-2}) = e_1^m$.

‡ See (c) from Theorem 2.

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