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A Survey of $NP-$Polyagroups
Survey Article

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Abstract. This text is as an attempt to systemize the results about $NP-$polyagroups.

1. Notion and example

1.1. Definition *[12]: Let $k > 1$, $s \geq 1$, $n = k \cdot s + 1$ and let $(Q; A)$ be an $n-$groupoid. Then: we say that $(Q; A)$ is an $NP-$***polyagroup of the type** *$(s, n-1)$ iff the following statements hold:*

1° For all $i, j \in \{1, \ldots, n\}$ $(i < j)$ if $i, j \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}$, then the $< i, j > -$associative law holds in $(Q; A)$; and

2° For all $i \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}$ and for every $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the equality

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n$$

holds.

Remark: For $s = 1$ $(Q; A)$ is a $(k + 1)-$group, where $k + 1 \geq 3; k > 1$.

1.2. Example: *Let $(Q; \cdot)$ be a group and let α be a mapping of the set Q into the set Q. Also, let*

$$A(x_1^5) \overset{def}{=} x_1 \cdot \alpha(x_2) \cdot x_3 \cdot \alpha(x_4) \cdot x_5$$

for all $x_1^5 \in Q$. Then $(Q; A)$ is an $NP-$polyagroup of the type (2,4).

Remark: Consult Prop. 2.3 and Prop. 2.1.

2. Auxiliary proposition

2.1. Proposition *[12]: Let $k > 1, s > 1, n = k \cdot s + 1$ and let $(Q; A)$ be an $n-$groupoid. Also, let the following statements hold:*

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(|) For all $i, j \in \{1, \ldots, n\}$ $(i < j)$ if $i, j \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}$, then the $< i, j > -$associative law holds in $(Q; A)$;

(||) For every $a_1^n \in Q$ there is exactly one $x \in Q$ such that the following equality holds

$$A(a_1^{n-1}, x) = a_n; \text{ and}$$

(|||) For every $a_1^n \in Q$ there is exactly one $y \in Q$ such that the following equality holds

$$A(y, a_1^{n-2}) = a_n.$$

Then $(Q; A)$ is an $NP-$polyagroup of the type $(s, n-1)$.

Remark: For $s = 1$ see Prop. 2.2-III in [10]. See, also Prop. 4.1-XVI in [2003].

Sketch of a part of the proof.

a) $A(a, a_1^{s-1}, \overline{\overset{(t)}{a}, \overset{(t)}{a}{}_1^{s-1}}\Big|_{t=1}^{i-1}, x, \overline{\overset{(t)}{a}{}_1^{s-1}, \overset{(t)}{a}}\Big|_{t=i}^{k-2}, b_1^{s-1}, b)$ [1] $=$
$A(a, a_1^{s-1}, \overline{\overset{(t)}{a}, \overset{(t)}{a}{}_1^{s-1}}\Big|_{t=1}^{i-1}, y, \overline{\overset{(t)}{a}{}_1^{s-1}, \overset{(t)}{a}}\Big|_{t=i}^{k-2}, b_1^{s-1}, b) \Rightarrow$

$$A(\overline{\overset{(j)}{c}, \overset{(j)}{c}{}_1^{s-1}}\Big|_{j=i+1}^{k}, A(a, a_1^{s-1}, \overline{\overset{(t)}{a}, \overset{(t)}{a}{}_1^{s-1}}\Big|_{t=1}^{i-1}, x, \overline{\overset{(t)}{a}{}_1^{s-1}, \overset{(t)}{a}}\Big|_{t=i}^{k-2}, b_1^{s-1}, b), \overline{\overset{(j)}{c}{}_1^{s-1}, \overset{(j)}{c}}\Big|_{j=i}^{i}) =$$

$$A(\overline{\overset{(j)}{c}, \overset{(j)}{c}{}_1^{s-1}}\Big|_{j=i+1}^{k}, A(a, a_1^{s-1}, \overline{\overset{(t)}{a}, \overset{(t)}{a}{}_1^{s-1}}\Big|_{t=1}^{i-1}, y, \overline{\overset{(t)}{a}{}_1^{s-1}, \overset{(t)}{a}}\Big|_{t=i}^{k-2}, b_1^{s-1}, b), \overline{\overset{(j)}{c}{}_1^{s-1}, \overset{(j)}{c}}\Big|_{j=i}^{i}) \overset{(|)}{\Rightarrow}$$

$$A(A(\overline{\overset{(j)}{c}, \overset{(j)}{c}{}_1^{s-1}}\Big|_{j=i+1}^{k}, a, a_1^{s-1}, \overline{\overset{(t)}{a}, \overset{(t)}{a}{}_1^{s-1}}\Big|_{t=1}^{i-1}, x), \overline{\overset{(t)}{a}{}_1^{s-1}, \overset{(t)}{a}}\Big|_{t=i}^{k-2}, b_1^{s-1}, b, \overline{\overset{(j)}{c}{}_1^{s-1}, \overset{(j)}{c}}\Big|_{j=i}^{i}) =$$

$$A(A(\overline{\overset{(j)}{c}, \overset{(j)}{c}{}_1^{s-1}}\Big|_{j=i+1}^{k}, a, a_1^{s-1}, \overline{\overset{(t)}{a}, \overset{(t)}{a}{}_1^{s-1}}\Big|_{t=1}^{i-1}, y), \overline{\overset{(t)}{a}{}_1^{s-1}, \overset{(t)}{a}}\Big|_{t=i}^{k-2}, b_1^{s-1}, b, \overline{\overset{(j)}{c}{}_1^{s-1}, \overset{(j)}{c}}\Big|_{j=i}^{i}) \overset{(|||)}{\Rightarrow}$$

$A(\overline{\overset{(j)}{c}, \overset{(j)}{c}{}_1^{s-1}}\Big|_{j=i+1}^{k}, a, a_1^{s-1}, \overline{\overset{(t)}{a}, \overset{(t)}{a}{}_1^{s-1}}\Big|_{t=1}^{i-1}, x) =$
$A(\overline{\overset{(j)}{c}, \overset{(j)}{c}{}_1^{s-1}}\Big|_{j=i+1}^{k}, a, a_1^{s-1}, \overline{\overset{(t)}{a}, \overset{(t)}{a}{}_1^{s-1}}\Big|_{t=1}^{i-1}, y) \overset{(||)}{\Rightarrow} x = y.$

b) $A(a, a_1^{s-1}, \overline{\overset{(t)}{a}, \overset{(t)}{a}{}_1^{s-1}}\Big|_{t=i}^{k-2}, x, \overline{\overset{(t)}{a}{}_1^{s-1}, \overset{(t)}{a}}\Big|_{t=1}^{i-1}, b_1^{s-1}, b) =$
$A(a, a_1^{s-1}, \overline{\overset{(t)}{a}, \overset{(t)}{a}{}_1^{s-1}}\Big|_{t=i}^{k-2}, y, \overline{\overset{(t)}{a}{}_1^{s-1}, \overset{(t)}{a}}\Big|_{t=1}^{i-1}, b_1^{s-1}, b) \Rightarrow$

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[1] $i \in \{1, \ldots, k-1\}$.

$$A(\overline{\overset{(j)}{c},\overset{(j)}{c}{}_1^{s-1}}\Big|_{j=i+1}^{k},A(a,a_1^{s-1},\overline{\overset{(t)}{a},\overset{(t)}{a}{}_1^{s-1}}\Big|_{t=i}^{k-2},x,\overline{\overset{(t)}{a}{}_1^{s-1},\overset{(t)}{a}}\Big|_{t=1}^{i-1},b_1^{s-1},b),$$
$$\overline{\overset{(j)}{c}{}_1^{s-1},\overset{(j)}{c}}\Big|_{j=i+1}^{k})=$$

$$A(\overline{\overset{(j)}{c},\overset{(j)}{c}{}_1^{s-1}}\Big|_{j=i+1}^{k},A(a,a_1^{s-1},\overline{\overset{(t)}{a},\overset{(t)}{a}{}_1^{s-1}}\Big|_{t=i}^{k-2},y,\overline{\overset{(t)}{a}{}_1^{s-1},\overset{(t)}{a}}\Big|_{t=1}^{i-1},b_1^{s-1},b),$$
$$\overline{\overset{(j)}{c}{}_1^{s-1},\overset{(j)}{c}}\Big|_{j=i+1}^{k})\overset{(|)}{\Rightarrow}$$

$$A(A(\overline{\overset{(j)}{c},\overset{(j)}{c}{}_1^{s-1}}\Big|_{j=1}^{i},a,a_1^{s-1},\overline{\overset{(t)}{a},\overset{(t)}{a}{}_1^{s-1}}\Big|_{t=i}^{k-2},x),\overline{\overset{(t)}{a}{}_1^{s-1},\overset{(t)}{a}}\Big|_{t=1}^{i-1},b_1^{s-1},b,$$
$$\overline{\overset{(j)}{c}{}_1^{s-1},\overset{(j)}{c}}\Big|_{j=i+1}^{k})=$$

$$A(A(\overline{\overset{(j)}{c},\overset{(j)}{c}{}_1^{s-1}}\Big|_{j=1}^{i},a,a_1^{s-1},\overline{\overset{(t)}{a},\overset{(t)}{a}{}_1^{s-1}}\Big|_{t=i}^{k-2},y),\overline{\overset{(t)}{a}{}_1^{s-1},\overset{(t)}{a}}\Big|_{t=1}^{i-1},b_1^{s-1},b,$$
$$\overline{\overset{(j)}{c}{}_1^{s-1},\overset{(j)}{c}}\Big|_{j=i+1}^{k})\overset{(|||)}{\Rightarrow}$$

$$A(\overline{\overset{(j)}{c},\overset{(j)}{c}{}_1^{s-1}}\Big|_{j=1}^{i},a,a_1^{s-1},\overline{\overset{(t)}{a},\overset{(t)}{a}{}_1^{s-1}}\Big|_{t=i}^{k-2},x)=$$
$$A(\overline{\overset{(j)}{c},\overset{(j)}{c}{}_1^{s-1}}\Big|_{j=1}^{i},a,a_1^{s-1},\overline{\overset{(t)}{a},\overset{(t)}{a}{}_1^{s-1}}\Big|_{t=i}^{k-2},y)\overset{(||)}{\Rightarrow}x=y.$$

c) $A(a,a_1^{s-1},\overline{\overset{(t)}{a},\overset{(t)}{a}{}_1^{s-1}}\Big|_{t=1}^{i-1},x,\overline{\overset{(t)}{a}{}_1^{s-1},\overset{(t)}{a}}\Big|_{t=i}^{k-2},b_1^{s-1},b)=d\overset{b)}{\Leftrightarrow}$

$$A(\overline{\overset{(j)}{c},\overset{(j)}{c}{}_1^{s-1}}\Big|_{j=i+1}^{k},A(a,a_1^{s-1},\overline{\overset{(t)}{a},\overset{(t)}{a}{}_1^{s-1}}\Big|_{t=1}^{i-1},x,\overline{\overset{(t)}{a}{}_1^{s-1},\overset{(t)}{a}}\Big|_{t=i}^{k-2},b_1^{s-1},b),$$
$$\overline{\overset{(j)}{c}{}_1^{s-1},\overset{(j)}{c}}\Big|_{j=1}^{i})=$$

$$A(\overline{\overset{(j)}{c},\overset{(j)}{c}{}_1^{s-1}}\Big|_{j=i+1}^{k},d,\overline{\overset{(j)}{c}{}_1^{s-1},\overset{(j)}{c}}\Big|_{j=1}^{i})\overset{(|)}{\Leftrightarrow}$$
$$A(A(\overline{\overset{(j)}{c},\overset{(j)}{c}{}_1^{s-1}}\Big|_{j=i+1}^{k},a,a_1^{s-1},\overline{\overset{(t)}{a},\overset{(t)}{a}{}_1^{s-1}}\Big|_{t=1}^{i-1},x),\overline{\overset{(t)}{a}{}_1^{s-1},\overset{(t)}{a}}\Big|_{t=i}^{k-2},b_1^{s-1},b,$$
$$\overline{\overset{(j)}{c}{}_1^{s-1},\overset{(j)}{c}}\Big|_{j=1}^{i})=$$

$$A(\overline{\overset{(j)}{c},\overset{(j)}{c}{}_1^{s-1}}\Big|_{j=i+1}^{k},d,\overline{\overset{(j)}{c}{}_1^{s-1},\overset{(j)}{c}}\Big|_{j=1}^{i}),$$

where $\overset{(j)}{c}\Big|_{j=1}^{k}$ and $\overline{\overset{(j)}{c}{}_1^{s-1}}\Big|_{j=1}^{k}$ $(s>1)$ are arbitrary sequence over Q. □

2.2_1. **Proposition** *[12]: Let $k>1, s\geq 1, n=k\cdot s+1$ and let $(Q;A)$ be an $n-$groupoid. Also, let*

(a) The $<1,s+1>-$associative law holds in the $(Q;A)$; and

(b) For every $x,y,a_1^{n-1}\in Q$ the following implication holds

$$A(x,a_1^{n-1})=A(y,a_1^{n-1})\Rightarrow x=y.$$

Then statement 1° from Def. 1.1. holds in $(Q;A)$.

Remark: For $s=1$ Prop. 2.2_1 is proved in [7]. Cf. Prop. 2.1-III in [2003].

The sketch of a part of the proof.

$A(A(x_1^n),x_{n+1}^{2n-1})\overset{(a)}{=}A(x_1^s,A(x_{s+1}^{s+n}),x_{s+n+1}^{2n-1})\Rightarrow$

$A(y_1^s, A(A(x_1^n), x_{n+1}^{2n-1}), y_{s+1}^{n-1}) =$
$A(y_1^s, A(x_1^s, A(x_{s+1}^{s+n}), x_{s+n+1}^{2n-1}, y_{s+1}^{n-1}) \overset{(a)}{\Rightarrow}$
$A(A(y_1^s, A(x_1^n), x_{n+1}^{2n-1-s}), x_{2n-s}^{2n-1}, y_{s+1}^{n-1}) =$
$A(A(y_1^s, x_1^s, A(x_{s+1}^{s+n}), x_{s+n+1}^{2n-1-s}), x_{2n-s}^{2n-1}, y_{s+1}^{n-1}) \overset{(b)}{\Rightarrow}$
$A(y_1^s, A(x_1^n), x_{n+1}^{2n-1-s}) = A(y_1^s, x_1^s, A(x_{s+1}^{s+n}), x_{s+n+1}^{2n-1-s}).$
(Cf. the proof of Prop. 4.2_1 in [11].) □

Similarly, it posible to prove also the following two propositions.

2.2_2. **Proposition** *[12]: Let $k > 1, s \geq 1, n = k \cdot s + 1$ and let $(Q; A)$ be an $n-$groupoid. Also, let*

($\overline{a}$) *The $< (k-1) \cdot s + 1, k \cdot s + 1 > -$associative law holds in the $(Q; A)$; and*

($\overline{b}$) *For every $x, y, a_1^{n-1} \in Q$ the following implications holds*

$A(a_1^{n-1}, x) = A(a_1^{n-1}, y) \Rightarrow x = y.$

Then statement 1° from Def. 1.1. holds in $(Q; A)$.

Remark: Cf. Prop. 2.1-III in [2003].

2.2_3. **Proposition** *[9]: Let $k > 1, s > 1, n = k \cdot s + 1, i \in \{t \cdot s + 1 | t \in \{0, 1, \ldots, k\}\}$ and let $(Q; A)$ be an $n-$groupoid. Also, let*

(i) *The $< i - s, i > -$associative law holds in the $(Q; A)$;*

(ii) *The $< i, i + s > -$associative law holds in the $(Q;)$; and*

(iii) *For every $x, y, a_1^{n-1} \in Q$ the following implications holds*

$A(a_1^{i-1}, x, a_i^{n-1}) = A(a_1^{i-1}, y, a_1^{n-1}) \Rightarrow x = y.$

Then statement 1° from Def. 1.1. holds in $(Q; A)$.

2.3. Proposition *[7]: Let $(Q; A)$ be an $n-$groupoid and let $n \geq 2$. Further on, let the following statements hold:*

(a) *The $< 1, n > -$associative law holds in the $(Q; A)$;*

(b) *For every sequence a_1^{n-2} over Q, for every $a \in Q$ and for every $b \in Q$, there is at least one $x \in Q$ such that the equality $A(a, a_1^{n-2}, x) = b$ holds; and*

(c) *For every sequence a_1^{n-2} over Q, for every $a \in Q$ and for every $b \in Q$, such that the equality $A(y, a_1^{n-2}, a) = b$ holds.*

Then there are mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-1} and Q^{n-2} into Q such that for every sequence a_1^{n-2} over Q and for every $a, x \in Q$ the following equalities hold

(2L) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$,
(2R) $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$,
(3L) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$,
(3R) $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$,
(4L) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = x$ *and*
(4R) $A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = x$.

Remark: $\mathbf{e}$ *is an* $\{1, n\}-$*neutral operation of* $n-$*groupoid* $(Q; A)$ *iff algebra* $(Q; A, \mathbf{e})$ *[of the type* $< n, n-2 >$*] satisfies the laws* (2L) *and* (2R) *[5]. Operation* $^{-1}$ *from 2.3 is a generalization of the inverse operation in a group [6]. Cf. Chapter II and Chapter III in [2003].*

By Prop. 2.3 and by Def. 1.1, we obtain:

2.4. Proposition: *Let* $k > 1, s \geq 1, n = k \cdot s + 1$ *and let* $(Q; A)$ *be an* $NP-$*polyagroup of the type* $(s, n-1)$. *Then there are mappings* $^{-1}$ *and* $\mathbf{e}$, *respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into* Q *such that the laws:*
(2L) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$,
(2R) $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$,
(3L) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$,
(3R) $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$,
(4L) $A((a_1^{n-2}, b)^{-1}, a_1^{n-2}, A(b, a_1^{n-2}, x)) = x$ *and*
(4R) $A(A(x, a_1^{n-2}, b), a_1^{n-2}, (a_1^{n-2}, b)^{-1}) = x$.
hold in the algebra $(Q; A,^{-1}, \mathbf{e})$.

3. SOME CHARACTERIZATIONS OF $NP-$POLYAGROUPS

3.1_1. **Theorem**: *Let* $k > 1, s \geq 1, n = k \cdot s + 1$ *and let* $(Q; A)$ *be an* $n-$*groupoid. Then,* $(Q; A)$ *is an* $NP-$*polyagroup of the type* $(s, n-1)$ *iff there is a mapping* $^{-1}$ *of the set* Q^{n-1} *into the set* Q *such that the laws:*
(1Ls) $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^s, A(x_{s+1}^{s+n}), x_{s+n+1}^{2n-1})$,
(4L) $A((a_1^{n-2}, b)^{-1}, a_1^{n-2}, A(b, a_1^{n-2}, x)) = x$ *and*
(4R) $A(A(x, a_1^{n-2}, b), a_1^{n-2}, (a_1^{n-2}, b)^{-1}) = x$
hold in the algebra $(Q; A,^{-1})$.
Remark: For $s = 1$ see $1 - IX$ in [2003].

Proof. 1) $\Rightarrow$: By Def. 1.1 and by Prop. 2.4.

2) $\Leftarrow$: Firstly we prove the following statements:

$^\circ 1$ For every $x, y, a_1^{n-1} \in Q$ the following implication holds
$A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y$.

$^\circ 2$ The statement 1° from Def. 1.1 holds.

$^\circ 3$ For every $x, y, a_1^{n-1} \in Q$ the following implication holds
$A(a_1^{n-1}, x) = A(a_1^{n-1}, y) \Rightarrow x = y$.

$^\circ 4$ For every $a_1^{n-2}, b, c, x \in Q$ the following equivalences hold
$A(x, a_1^{n-2}, b) = c \Leftrightarrow x = A(c, a_1^{n-2}, (a_1^{n-2}, b)^{-1})$ and
$A(b, a_1^{n-2}, y) = c \Leftrightarrow y = A((a_1^{n-2}, b)^{-1}, a_1^{n-2}, c)$.

Sketch of the proof of $^\circ 1$:

$A(x, a_1^{n-2}, b) = A(y, a_1^{n-2}, b) \Rightarrow$
$A(A(x, a_1^{n-2}, b), a_1^{n-2}, (a_1^{n-2}, b)^{-1}) =$
$A(A(y, a_1^{n-2}, b), a_1^{n-2}, (a_1^{n-2}, b)^{-1}) \overset{(4R)}{\Rightarrow}$
$x = y$.

The proof of $^\circ 2$: By $(1Ls), ^\circ 1$ and by Prop. 2.2_1.

Sketch of the proof of $^\circ 3$:

$A(b, a_1^{n-2}, x) = A(b, a_1^{n-2}, y) \Rightarrow$
$A((a_1^{n-2}, b)^{-1}, a_1^{n-2}, A(b, a_1^{n-2}, x)) =$
$A((a_1^{n-2}, b)^{-1}, a_1^{n-2}, A(b, a_1^{n-2}, y)) \overset{(4L)}{\Rightarrow}$
$x = y$.

Sketch of the proof of $^\circ 4$:

a) $A(x, a_1^{n-2}, b) = c \overset{\circ 1}{\Leftrightarrow}$
$A(A(x, a_1^{n-2}, b), a_1^{n-2}, (a_1^{n-2}, b)^{-1}) =$
$A(c, a_1^{n-2}, (a_1^{n-2}, b)^{-1}) \overset{(4R)}{\Leftrightarrow}$
$x = A(c, a_1^{n-2}, (a_1^{n-2}, b)^{-1})$.

b) $A(b, a_1^{n-2}, y) = c \overset{\circ 3}{\Leftrightarrow}$
$A((a_1^{n-2}, b)^{-1}, a_1^{n-2}, A(b, a_1^{n-2}, y)) =$
$A((a_1^{n-2}, b)^{-1}, a_1^{n-2}, c) \overset{(4L)}{\Leftrightarrow}$
$y = A((a_1^{n-2}, b)^{-1}, a_1^{n-2}, c)$.

Finally, by $^\circ 1 - ^\circ 4$ and by Prop. 2.1 we conclude that $(Q; A)$ is an $NP-$polyagroup of the type $(s, n-1)$. Whence, by " $\Rightarrow$ ", we obtain Th. 3.1_1. $\square$

Similarly, it is posible to prove also the following proposition:

3.1_2. **Theorem**: *Let $k > 1, s \geq 1, n = k \cdot s + 1$ and let $(Q; A)$ be an $n-$groupoid. Then $(Q; A)$ is an $NP-$polyagroup of the type $(s, n-1)$ iff there is a mapping $^{-1}$ of the set Q^{n-1} into the set Q such that the laws:*

$(1Rs)$ $A(x_1^{(k-1)\cdot s}, A(x_{(k-1)\cdot s+1}^{(k-1)\cdot s+n}), x_{(k-1)\cdot s+n+1}^{2n-1}) = A(x_1^{k\cdot s}, A(x_{k\cdot s+1}^{2n-1}))$,

$(4L)$ $A((a_1^{n-2}, b)^{-1}, a_1^{n-2}, A(b, a_1^{n-2}, x)) = x$ *and*

$(4R)$ $A(A(x, a_1^{n-2}, b), a_1^{n-2}, (a_1^{n-2}, b)^{-1}) = x$

hold in the algebra $(Q; A,^{-1})$.

Remark: For $s = 1$ see $1 - IX$ in [2003].

3.2_1. **Theorem** *[13]: Let $k > 1, s \geq 1, n = k\cdot s+1$ and let $(Q; A)$ be an $n-$groupoid. Then: $(Q; A)$ is an $NP-$polyagroup of the type $(s, n-1)$ iff the following statements hold*

(1) *The $< 1, s+1 > -$associative law holds in $(Q; A)$;*

(2) *The $< 1, n > -$associative law holds in $(Q; A)$;*

(3) *For every $a_1^n \in Q$ there is* **at least one** *$x \in Q$ such that the following equality $A(a_1^{n-1}, x) = a_n$ holds; and*

(4) *For every $a_1^n \in Q$ there is* **at least one** *$y \in Q$ such that the following equality $A(y, a_1^{n-1}) = a_n$ holds.*

Remark: For $s = 1$ Th. 3.2_1 is proved in [7]. See, also, Th. 5.2_1 in [11].

Proof. $a)$ $\Rightarrow$: By Def. 1.1.

$b)$ $\Leftarrow$: Firstly we prove the following statement:

1^* There is mapping $^{-1}$ of the set Q^{n-1} into the set Q such that the following laws hold in the algebra $(Q; A,^{-1})$ *[*of the type $< n, n-1 >$*]*

(a) $A((a_1^{n-2}, b)^{-1}, a_1^{n-2}, A(b, a_1^{n-2}, x)) = x$ and

(b) $A(A(x, a_1^{n-2}, b), a_1^{n-2}, (a_1^{n-2}, b)^{-1}) = x$.

The proof of 1^* : By $(2) - (4)$ and by Prop. 2.4.

Finally, by (1), by 1^* and by Th. 3.1_1, we conclude that is an $NP-$polyagroup of the type $(s, n-1)$. Whence, by " $\Rightarrow$ ", we have Th. 3.2_1. □

Similarly, it is posible to prove also the following proposition:

3.2_2. **Theorem** *[13]: Let $k > 1, s \geq 1, n = k\cdot s+1$ and let $(Q; A)$ be an $n-$groupoid. Then: $(Q; A)$ is an $NP-$polyagroup of the type $(s, n-1)$ iff the following statements hold*

$(\overline{1})$ *The $< (k-1)\cdot s+1,\ k\cdot s+1 > -$associative law holds in $(Q; A)$;*

$(\overline{2})$ *The $< 1, n > -$associative law holds in $(Q; A)$;*

($\overline{3}$) *For every* $a_1^n \in Q$ *there is* **at least one** $x \in Q$ *such that the following equality* $A(a_1^{n-1}, x) = a_n$ *holds; and*

($\overline{4}$) *For every* $a_1^n \in Q$ *there is* **at least one** $y \in Q$ *such that the following equality* $A(y, a_1^{n-1}) = a_n$ *holds.*

Remark: For $s = 1$ Th. 3.2_2 is proved in [7].

3.3_1. **Theorem** *[12]: Let* $k > 1, s \geq 1, n = k \cdot s + 1$ *and let* $(Q; A)$ *be an* $n-$*groupoid. Then,* $(Q; A)$ *is an* $NP-$*polyagroup of the type* $(s, n-1)$ *iff there are mappings* $^{-1}$ *and* $\mathbf{e}$, *respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra* $(Q; A, ^{-1}, \mathbf{e})$ *[of the type* $< n, n-1, n-2 >$*]:*

$(1Ls)$ $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^s, A(x_{s+1}^{s+n}), x_{s+n+1}^{2n-1})$,

$(2R)$ $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ *and*

$(3R)$ $A(a, a_1^{n-2}, (a_1^{n-2})^{-1}) = \mathbf{e}(a_1^{n-2})$.

Remark: For $s = 1$ Th. 3.3_1 is proved in [7]. Cf. 3-III in [2003].

Proof. $a)$ $\Rightarrow$: By Def. 1.1. and by Prop. 2.4.

$b)$ $\Leftarrow$: Firstly we prove the following statements:

$\overset{\circ}{1}$ For every $x, y, a_1^{n-1} \in Q$ the following implication holds

$A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y$.

$\overset{\circ}{2}$ The statement 1° from Def. 1.1 holds.

$\overset{\circ}{3}$ Law

$(2L)$ $\quad A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$

holds in the algebra $(Q; A, ^{-1}, \mathbf{e})$.

$\overset{\circ}{4}$ Law

$(3L)$ $\quad A((a_1^{n-2})^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$

holds in the algebra $(Q; A, ^{-1}, \mathbf{e})$.

$\overset{\circ}{5}$ Law

$(4L)$ $\quad A((a_1^{n-2}, b)^{-1}, a_1^{n-2}, A(b, a_1^{n-2}, x)) = x$ and

$(4R)$ $\quad A(A(x, a_1^{n-2}, b), a_1^{n-2}, (a_1^{n-2}, b)^{-1}) = x$.

hold in the algebra $(Q; A, ^{-1}, \mathbf{e})$.

Sketch of the proof of $\overset{\circ}{1}$:

$$
\begin{aligned}
a)\ n - 2 - s &\geq 0 : \\
n - 2 - s &= k \cdot s + 1 - 2 - s \\
&= s(k-1) - 1 \\
&\overset{k>1}{\geq} s - 1 \\
&\overset{s>1}{\geq} 0.
\end{aligned}
$$

$b)$ $A(x, a_1^{s-1}, a, a_s^{n-2}) = A(y, a_1^{s-1}, a, a_s^{n-2}) \overset{a)}{\Rightarrow}$
$A(A(x, a_1^{s-1}, a, a_s^{n-2}), a_1^{s-1}, \mathbf{e}(a_s^{n-2}, a_1^{s-1}), \overset{n-2-s}{a}, \mathbf{e}(a_1^{s-1}, \overset{n-2-s+1}{a})) =$
$A(A(y, a_1^{s-1}, a, a_s^{n-2}), a_1^{s-1}, \mathbf{e}(a_s^{n-2}, a_1^{s-1}), \overset{n-2-s}{a}, \mathbf{e}(a_1^{s-1}, \overset{n-2-s+1}{a})) \overset{(1Ls)}{\Longrightarrow}$
$A(x, a_1^{s-1}, A(a, a_s^{n-2}, a_1^{s-1}, \mathbf{e}(a_s^{n-2}, a_1^{s-1})), \overset{n-2-s}{a}, \mathbf{e}(a_1^{s-1}, \overset{n-2-s+1}{a})) =$
$A(y, a_1^{s-1}, A(a, a_s^{n-2}, a_1^{s-1}, \mathbf{e}(a_s^{n-2}, a_1^{s-1})), \overset{n-2-s}{a}, \mathbf{e}(a_1^{s-1}, \overset{n-2-s+1}{a})) \overset{(2R)}{\Longrightarrow}$
$A(x, a_1^{s-1}, a, \overset{n-2-s}{a}, \mathbf{e}(a_1^{s-1}, \overset{n-2-s+1}{a})) =$
$A(y, a_1^{s-1}, a, \overset{n-2-s}{a}, \mathbf{e}(a_1^{s-1}, \overset{n-2-s+1}{a})) \overset{(2R)}{\Longrightarrow} x = y.$

The proof of $\overset{\circ}{2}$: By $(1L_s), \overset{\circ}{1}$ and by Prop. 2.2_1.

Sketch of the proof of $\overset{\circ}{3}$:

$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, a) = b \Rightarrow$
$A(A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) =$
$A(b, a_1^{n-2}, (a_1^{n-2})^{-1}) \overset{2^\circ}{\Longrightarrow}$ [2]
$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}))$
$A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \overset{(3R)}{\Longrightarrow}$
$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, \mathbf{e}(a_1^{n-2})) =$
$A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \overset{(2R)}{\Longrightarrow}$
$\mathbf{e}(a_1^{n-2}) = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \overset{(3R)}{\Longrightarrow}$
$A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) =$
$A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \overset{\overset{\circ}{1}}{\Longrightarrow} a = b.$

Sketch of the proof of $\overset{\circ}{4}$:

$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = b \Rightarrow$
$A(A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) =$
$A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \overset{\overset{\circ}{2}}{\Longrightarrow}$
$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1})) =$
$A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \overset{(3R)}{\Longrightarrow}$
$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) =$
$A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \overset{(2R), \overset{\circ}{3}}{\Longrightarrow}$
$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) =$
$A(b, a_1^{n-2}, (a_1^{n-2})^{-1}) \overset{\overset{\circ}{1}}{\Longrightarrow} b = \mathbf{e}(a_1^{n-2}).$

[2] $< 1, n >$ −associative law.

Sketch of the proof of $\overset{\circ}{5}$:

a) $A((a_1^{n-2},b)^{-1},a_1^{n-2},A(b,a_1^{n-2},x))\overset{\overset{\circ}{2}}{=}$
$A(A((a_1^{n-2},b)^{-1},a_1^{n-2},b),a_1^{n-2},x)\overset{\overset{\circ}{4}}{=}$
$A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},x)\overset{\overset{\circ}{3}}{=}x.$

b) $A(A(x,a_1^{n-2},b),a_1^{n-2},(a_1^{n-2},b)^{-1})\overset{\overset{\circ}{2}}{=}$
$A(x,a_1^{n-2},A(b,a_1^{n-2},(a_1^{n-2},b)^{-1}))\overset{(3R)}{=}$
$A(x,a_1^{n-2},\mathbf{e}(a_1^{n-2}))\overset{(2R)}{=}x.$

Finally, by $(1Ls)$, by $\overset{\circ}{5}$ and Th. 3.1_1, we conclude that $(Q;A)$ is an $NP-$polyagroup of the type $(s,n-1)$. Whence, by " $\Rightarrow$ ", we obtain Th. 3.3_1. $\Box$

Similarly, it is posible to prove also the following proposition:

3.3_2. **Theorem** *[12]: Let $k>1, s\geq 1, n=k\cdot s+1$ and let $(Q;A)$ be an $n-$groupoid. Then, $(Q;A)$ is an $NP-$polyagroup of the type $(s,n-1)$ iff there are mappings ${}^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q;A,{}^{-1},\mathbf{e})$ [of the type $<n,n-1,n-2>$]:*

$(1Rs)$ $A(x_1^{(k-1)\cdot s},A(x_{(k-1)\cdot s+1}^{(k-1)\cdot s+n},x_{(k-1)\cdot s+n+1}^{2n-1})=A(x_1^{k\cdot s},A(x_{k\cdot s+1}^{2n-1})),$

$(2L)$ $A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},x)=x,$

$(3L)$ $A((a_1^{n-2},a)^{-1},a_1^{n-2},a)=\mathbf{e}(a_1^{n-2}),$

Remark: For $s=1$ see $3-III$ in [10].

3.4_1. **Theorem** *[15]: Let $k>1, s\geq 1, n=k\cdot s+1$ and let $(Q;A)$ be an $n-$groupoid. Then, $(Q;A)$ is an $NP-$polyagroup of the type $(s,n-1)$ iff there is mapping $\mathbf{e}$ of the set Q^{n-2} into the set Q such that the laws*

$(1Ls)$ $A(A(x_1^n),x_{n+1}^{2n-1})=A(x_1^s,A(x_{s+1}^{s+n}),x_{s+n+1}^{2n-1}),$

$(2L)$ $A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},x)=x$ *and*

$(2R)$ $A(x,a_1^{n-2},\mathbf{e}(a_1^{n-2}))=x$

hold in the algebra $(Q;A,\mathbf{e})$ of the type $<n,n-2>$.

Remark: For $s=1$ Th. 3.4_1 is proved in [7]. Cf. 2-IX in [10].

Proof. 1) $\Rightarrow$: By Def. 1.1 and by Prop. 2.4.*[*Every $NP-$polyagroup of the type $(s,n-1)$ has an $\{1,n\}-$neutral operation.*]*

2) $\Leftarrow$: Firstly we prove the following statements:

$\widehat{1}$ For all $x,y,a_1^{n-1}\in Q$ the following implication holds
$A(x,a_1^{n-1})=A(y,a_1^{n-1})\Rightarrow x=y.$

$\widehat{2}$ The statement 1° from Def. 1.1 holds.

$\widehat{3}$ For all $x, y, a_1^{n-1} \in Q$ the following implication holds
$A(a_1^{n-1}, x) = A(a_1^{n-1}, y) \Rightarrow x = y.$

$\widehat{4}$ For every $a_1^n \in Q$ there is exactly one $x \in Q$ and exactly one $y \in Q$ such that the following equalities hold
$A(a_1^{n-1}, x) = a_n$ and $A(a_1^{n-1}, y) = a_n.$

Sketch of the proof $\widehat{1}$: Sketch of the proof of $\overset{\circ}{1}$ from the proof of Th. 3.3_1.

The proof of $\widehat{2}$: By $(1Ls)$, $\widehat{1}$ and by Prop. 2.2_1.

Sketch of the proof of $\widehat{3}$:

$$A(a_s^{n-2}, a, a_1^{s-1}, x) = A(a_s^{n-2}, a, a_1^{s-1}, y) \Rightarrow$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a}, a_1^{s-1}), \overset{n-2-s}{a}, \mathbf{e}(a_1^{n-2}), a_1^{s-1}, A(a_s^{n-2}, a, a_1^{s-1}, x) =$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a}, a_1^{s-1}), \overset{n-2-s}{a}, \mathbf{e}(a_1^{n-2}), a_1^{s-1}, A(a_s^{n-2}, a, a_1^{s-1}, y) \overset{\widehat{2}}{\Rightarrow}$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a}, a_1^{s-1}), \overset{n-2-s}{a}, A(\mathbf{e}(a_1^{n-2}), a_1^{s-1}, a_s^{n-2}, a), a_1^{s-1}, x) =$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a}, a_1^{s-1}), \overset{n-2-s}{a}, A(\mathbf{e}(a_1^{n-2}), a_1^{s-1}, a_s^{n-2}, a), a_1^{s-1}, y) \overset{(2L)}{\Rightarrow}$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a}, a_1^{s-1}), \overset{n-2-s}{a}, a, a_1^{s-1}, x) =$$
$$A(\mathbf{e}(\overset{n-2-s+1}{a}, a_1^{s-1}), \overset{n-2-s}{a}, a, a_1^{s-1}, y) \overset{(2L)}{\Rightarrow} x = y.$$

Sketch of the proof of $\widehat{4}$:

$a)$ $A(x, a_1^{s-1}, a, a_s^{n-2}) = b \overset{\widehat{1}}{\Leftrightarrow}$
$$A(A(x, a_1^{s-1}, a, a_s^{n-2}), a_1^{s-1}, \mathbf{e}(a_s^{n-2}, a_1^{s-1}), c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s})) =$$
$$A(b, a_1^{s-1}, \mathbf{e}(a_s^{n-2}, a_1^{s-1}), c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s})) \overset{(1Ls)}{\Leftrightarrow}$$
$$A(x, a_1^{s-1}, A(a, a_s^{n-2}, a_1^{s-1}, \mathbf{e}(a_s^{n-2}, a_1^{s-1})), c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s})) =$$
$$A(b, a_1^{s-1}, \mathbf{e}(a_s^{n-2}, a_1^{s-1}), c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s})) \overset{(2R)}{\Leftrightarrow}$$
$$A(x, a_1^{s-1}, a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s})) =$$
$$A(b, a_1^{s-1}, \mathbf{e}(a_s^{n-2}, a_1^{s-1}), c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s})) \overset{(2R)}{\Leftrightarrow}$$
$$x = A(b, a_1^{s-1}, \mathbf{e}(a_s^{n-2}, a_1^{s-1}), c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s})),$$
where c_1^{n-2-s} is an arbitrary sequence over Q.

$b)$ $A(a_s^{n-2}, a, a_1^{s-1}, x) = b \overset{\widehat{3}}{\Leftrightarrow}$
$$A(\mathbf{e}(c_1^{n-2-s}, a, a_1^{s-1}), c_1^{n-2-s}, \mathbf{e}(a_1^{n-2}), a_1^{s-1}, A(a_s^{n-2}, a, a_1^{s-1}, x)) =$$
$$A(\mathbf{e}(c_1^{n-2-s}, a, a_1^{s-1}), c_1^{n-2-s}, \mathbf{e}(a_1^{n-2}), a_1^{s-1}, b) \overset{\widehat{2}}{\Leftrightarrow}$$
$$A(\mathbf{e}(c_1^{n-2-s}, a, a_1^{s-1}), c_1^{n-2-s}, A(\mathbf{e}(a_1^{n-2}), a_1^{s-1}, a_s^{n-2}, a), a_1^{s-1}, x) =$$
$$A(\mathbf{e}(c_1^{n-2-s}, a, a_1^{s-1}), c_1^{n-2-s}, \mathbf{e}(a_1^{n-2}), a_1^{s-1}, b) \overset{(2L)}{\Leftrightarrow}$$
$$A(\mathbf{e}(c_1^{n-2-s}, a, a_1^{s-1}), c_1^{n-2-s}, a, a_1^{s-1}, x) =$$
$$A(\mathbf{e}(c_1^{n-2-s}, a, a_1^{s-1}), c_1^{n-2-s}, \mathbf{e}(a_1^{n-2}), a_1^{s-1}, b) \overset{(2L)}{\Leftrightarrow}$$

$x = A(\mathbf{e}(c_1^{n-2-s}, a, a_1^{s-1}), c_1^{n-2-s}, \mathbf{e}(a_1^{n-2}), a_1^{s-1}, b)$,
where c_1^{n-2-s} is an arbitrary sequence over Q.

$c)$ By $a)$ and $\widehat{1}$ and by $b)$ and $\widehat{3}$, we obtain $\widehat{4}$.

Finally, by $\widehat{2}, \widehat{4}$ and by Prop. 2.1, we conclude that $(Q; A)$ is an $NP-$polyagroup of the type $(s, n-1)$. Whence, by " $\Rightarrow$ ", we obtain Th. 3.4_1. □

Similarly, it is posible to prove also the following proposition:

3.4_2. **Theorem** *[15]: Let* $k > 1, s \geq 1, n = k \cdot s + 1$ *and let* $(Q; A)$ *be an* $n-$*groupoid. Then,* $(Q; A)$ *is an* $NP-$*polyagroup of the type* $(s, n-1)$ *iff there is mapping* $\mathbf{e}$ *of the set* Q^{n-2} *into the set* Q *such that the laws*

$(1Rs)$ $A(x_1^{(k-1)\cdot s}, A(x_{(k-1)\cdot s+1}^{(k-1)\cdot s+n}), x_{(k-1)\cdot s+n+1}^{2n-1}) = A(x_1^{k\cdot s}, A(x_{k\cdot s+1}^{2n-1}))$,

$(2L)$ $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*

$(2R)$ $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$

hold in the algebra $(Q; A, \mathbf{e})$ *of the type* $< n, n-2 >$.

Remark: For $s = 1$ *Th.* 3.4_2 *is proved in [7]. Cf. 3-III in [10].*

The following proposition, also, holds:

3.5. Theorem *[12]: Let* $k > 1, s \geq 1, n = k \cdot s + 1$ *and let* $(Q; A)$ *be an* $n-$*groupoid. Then,* $(Q; A)$ *is an* $NP-$*polyagroup of the type* $(s, n-1)$ *iff there are mappings* $^{-1}$ *and* $\mathbf{e}$, *respectively, of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra* $(Q; A, ^{-1}, \mathbf{e})$ *[of the type* $< n, n-1, n-2 >$*]:*

1) $(1Ls), (2R),$ *and* $(4L)$; *or*

2) $(1Rs), (2R),$ *and* $(4L)$; *or*

3) $(1Ls), (2L),$ *and* $(4R)$; *or*

4) $(1Rs), (2L),$ *and* $(4R)$.

Remarks: a) For $s = 1$ *is proved in [7]. Cf. 2-IX in [10]. b) Cf. the proof of Th.*3.3_1 *and the proof of Th.*3.4_1.

4. SOME MORE PROPOSITIONS

4.1. Proposition *[15]: Let* $k > 1, s \geq 1, n = k \cdot s + 1$ *and let* $(Q; A)$ *be an* $NP-$*polyagroup of the type* $(s, n-1)$ *and* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation. Then the following laws*

$A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, x) = x$ *and*

$A(x, a_1^{s-1}, \mathbf{e}(c_1^{n-2-s}, a, a_1^{s-1}), c_1^{n-2-s}, a) = x$

hold in the algebra $(Q; A, \mathbf{e})$.

Remark: For $s = 1$ *see Prop. 1.1-IV in [10].*

Sketch of a part of the proof.

$F(x, a_1^{s-1}, a, c_1^{n-2-s}) \overset{def}{=} A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, x) \Rightarrow$
$A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, F(x, a_1^{s-1}, a, c_1^{n-2-s})) =$
$A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, x)) \overset{1.1,1^\circ}{\Longleftrightarrow}$
$A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, F(x, a_1^{s-1}, a, c_1^{n-2-s})) =$
$A(a, c_1^{n-2-s}, A(\mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s})), a_1^{s-1}, x) \overset{2.4}{\Longleftrightarrow}$
$A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, F(x, a_1^{s-1}, a, c_1^{n-2-s})) =$
$A(a, c_1^{n-2-s}, \mathbf{e}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, x) \overset{1.1,2^\circ}{\Longleftrightarrow} F(x, a_1^{s-1}, a, c_1^{n-2-s})) = x.$ □

4.2. Theorem *[15]: Let* $k > 1, s > 1, n = k \cdot s + 1$ *and let* $(Q; A)$ *be an* $n-$*groupoid and let* E *be an* $(n-2)-$*ary operation in* Q. *Also, let the following laws*

(o) $\mathsf{E}(c_1^{n-2-s}, b, a_1^{s-1}) = \mathsf{E}(a_1^{s-1}, c_1^{n-2-s}, b)$,

$(1Ls)$ $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^s, A(x_{s+1}^{s+n}), x_{s+n+1}^{2n-1})$,

$(2R)$ $A(x, a_1^{n-2}, \mathsf{E}(a_1^{n-2})) = x$ *and*

$\widehat{(2L)}$ $A(a, c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, x) = x$

hold in the algebra $(Q; A, \mathsf{E})$. *Then* $(Q; A)$ *is an* $NP-$*polyagroup of the type* $(s, n-1)$.

Remarks: a) For $s = 1$ (o) *is reduced to:* $\mathsf{E}(c_1^{n-3}, b) = \mathsf{E}(c_1^{n-3}, b)$. *b) Cf. Th. 4.3. c) For* $s = 1$ *([2]) see 1.1-XII in [10].*

Proof. Firstly we prove the following statements:

$\overline{1}$ For all $x, y, a_1^{n-1} \in Q$ the implication holds

$A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y$.

$\overline{2}$ Statement 1° from Def. 1.1 holds.

$\overline{3}$ For all $a_1^{s-1}, a, c_1^{n-2-s} \in Q$ the following equality holds

$a = \mathsf{E}(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1})$.

$\overline{4}$ For every $a_1^{s-1}, a, c_1^{n-2-s+1}, x, y \in Q$ the implication holds

$A(a, a_1^{s-1}, x, c_1^{n-2-s+1}) = A(a, a_1^{s-1}, y, c_1^{n-2-s+1}) \Rightarrow x = y$.

$\overline{5}$ For every $a_1^{s-1}, a, c_1^{n-2-s+1}, x, y \in Q$ the implication holds

$A(c_1^{n-2-s+1}, x, a_1^{s-1}, a) = A(c_1^{n-2-s+1}, y, a_1^{s-1}, a) \Rightarrow x = y$.

$\overline{6}$ For every $x, a, a_1^{s-1}, a, c_1^{n-2-s+1} \in Q$
$A(a, a_1^{s-1}, x, c_1^{n-2-s+1}) = b \Leftrightarrow$
$x = A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1}))$.

Sketch of the proof of $\overline{1}$: Sketch of the proof of $\overset{\circ}{1}$ from the proof of Th. 3.3_1.

Sketch of the proof of $\overline{2}$: By $\overline{1}$ and by Prop. 2.2_1.

Sketch of the proof of $\overline{3}$:

$$A(a, c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, \mathsf{E}(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1})) \overset{(\widehat{2L})}{=} \mathsf{E}(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1})),$$

$$A(a, c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}, \mathsf{E}(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1})) \overset{(2R)}{=} a.$$

Sketch of the proof of $\overline{4}$:

$$A(a, a_1^{s-1}, x, c_1^{n-2-s+1}) = A(a, a_1^{s-1}, y, c_1^{n-2-s+1}) \Rightarrow$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), A(a, a_1^{s-1}, x, c_1^{n-2-s+1}), a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), A(a, a_1^{s-1}, y, c_1^{n-2-s+1}), a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) \overset{\overline{2}}{\Rightarrow}$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a, a_1^{s-1}, A(x, c_1^{n-2-s+1}, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1}))) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a, a_1^{s-1}, A(y, c_1^{n-2-s+1}, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1}))) \overset{(2R)}{\Rightarrow}$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a, a_1^{s-1}, x) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a, a_1^{s-1}, y) \overset{\overline{3}}{\Rightarrow}$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), \mathsf{E}(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}), a_1^{s-1}, x) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), \mathsf{E}(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}), a_1^{s-1}, y) \overset{(o)}{\Rightarrow}$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), \mathsf{E}(a_1^{s-1}, c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s})), a_1^{s-1}, x) =$$
$$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), \mathsf{E}(a_1^{s-1}, c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s})), a_1^{s-1}, y) \overset{(\widehat{2L})}{\Rightarrow}$$
$$x = y.$$

Sketch of the proof of $\overline{5}$:

$$A(c_1^{n-1-s}, x, a_1^{s-1}, a) = A(c_1^{n-1-s}, y, a_1^{s-1}, a) \Rightarrow$$
$$A(d_1^{2s}, A(c_1^{n-1-s}, x, a_1^{s-1}, a), d_{2s+1}^{n-1}) =$$
$$A(d_1^{2s}, A(c_1^{n-1-s}, y, a_1^{s-1}, a), d_{2s+1}^{n-1}) \overset{\overline{2}}{\Rightarrow}$$
$$A(A(d_1^{2s}, c_1^{n-2s}), c_{n-2s+1}^{n-1-s}, x, a_1^{s-1}, a, d_{2s+1}^{n-1}) =$$
$$A(A(d_1^{2s}, c_1^{n-2s}), c_{n-2s+1}^{n-1-s}, y, a_1^{s-1}, a, d_{2s+1}^{n-1}) \overset{\overline{4}}{\Rightarrow}$$
$$x = y.$$

Sketch of the proof of $\overline{6}$:

$$A(a, a_1^{s-1}, x, c_1^{n-2-s+1}) = b \overset{\overline{5}}{\Leftrightarrow}$$

$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), A(a, a_1^{s-1}, x, c_1^{n-2-s+1}), a_1^{s-1},$
$\mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) =$
$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) \overset{\overline{2}}{\Leftrightarrow}$
$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a, a_1^{s-1}, A(x, c_1^{n-2-s+1}, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) =$
$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) \overset{(2R)}{\Leftrightarrow}$
$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a, a_1^{s-1}, x) =$
$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) \overset{\overline{3}}{\Leftrightarrow}$
$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), \mathsf{E}(c_1^{n-2-s},$
$\mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), a_1^{s-1}), a_1^{s-1}, x) =$
$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) \overset{(o)}{\Leftrightarrow}$
$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s},$
$\mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s})), a_1^{s-1}, x) =$
$A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1})) \overset{(\widehat{2L})}{\Leftrightarrow}$
$x = A(c_1^{n-2-s}, \mathsf{E}(a_1^{s-1}, a, c_1^{n-2-s}), b, a_1^{s-1}, \mathsf{E}(c_1^{n-2-s+1}, a_1^{s-1}))$.

Finally, considering $\overline{2}, \overline{4}, \overline{6}$ and by Prop. 2.2_3, we conclude that $(Q; A)$ is an near$-P-$polyagroup of the type $(s, n-1)$. □

4.3. Theorem *[15]: Let $(Q; \cdot)$ be a group, let α be a mapping of the set Q^{s-1} into the set $Q, k > 1, s > 1$ and let $n = k \cdot s + 1$. Also, let*

$$A(x_1, \overset{(1)}{y}{}_1^{s-1}, \ldots, x_k, \overset{(k)}{y}{}_1^{s-1}, x_{k+1}) \overset{def}{=} x_1 \cdot \alpha(\overset{(1)}{y}{}_1^{s-1}) \cdot \cdots \cdot x_k \cdot \alpha(\overset{(k)}{y}{}_1^{s-1}) \cdot x_{k+1}$$

for all $x_1^{k+1}, \overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1} \in Q$. Further on, let

$$\mathsf{E}(\overset{(1)}{y}{}_1^{s-1}, b_1, \ldots, b_{k-1}, \overset{(k)}{y}{}_1^{s-1}) \overset{def}{=} (\alpha(\overset{(1)}{y}{}_1^{s-1}) \cdot b_1 \cdot \cdots \cdot b_{k-1} \cdot \alpha(\overset{(k)}{y}{}_1^{s-1}))^{-1}$$

where $^{-1}$ is an inverse operation in $(Q; \cdot)$. Then the following statements hold:

(α) $(Q; A)$ is an $NP-$polyagroup of the type $(s, n-1)$;

(β) E is an $\{1, n\}-$neutral operation of the $(Q; A)$;

(γ) If $(Q; \cdot)$ commutative group, then (o) from 4.2 holds in $(Q; A)$; and

(δ) If $(Q; \cdot)$ is no commutative and $(Q; \alpha)$ is a $(s-1)-$quasigroup, then the condition (o) [from 4.2] in $(Q; A)$ does not holds.

Proof. Firstly we prove the following statements:

$\widehat{1}$ The $< 1, s+1 > -$associative law holds in the $(Q; A)$; and

$\widehat{2} = (b)$.

Sketch of the proof of $\widehat{1}$:

$$A(A(x_1, \overset{(1)}{y}{}_1^{s-1}, x_2, \overset{(2)}{y}{}_1^{s-1}, \ldots, x_k, \overset{(k)}{y}{}_1^{s-1}, x_{k+1}), \overset{(k+1)}{y}{}_1^{s-1}, x_{k+2}, \ldots, \overset{(2k)}{y}{}_1^{s-1}, x_{2k+1}) =$$

$$(x_1 \cdot \alpha(\overset{(1)}{y_1^{s-1}}) \cdot x_2 \cdot \alpha(\overset{(2)}{y_1^{s-1}}) \cdot \dots \cdot x_k \cdot \alpha(\overset{(k)}{y_1^{s-1}}) \cdot x_{k+1}) \cdot \alpha(\overset{(k+1)}{y_1^{s-1}}) \cdot x_{k+2} \cdot \alpha(\overset{(k+2)}{y_1^{s-1}}) \cdot \dots \cdot \alpha(\overset{(2k)}{y_1^{s-1}}) \cdot x_{2k+1}) =$$

$$x_1 \cdot \alpha(\overset{(1)}{y_1^{s-1}}) \cdot (x_2 \cdot \alpha(\overset{(2)}{y_1^{s-1}}) \cdot \dots \cdot x_k \cdot \alpha(\overset{(k)}{y_1^{s-1}}) \cdot x_{k+1} \cdot \alpha(\overset{(k+1)}{y_1^{s-1}}) \cdot x_{k+2}) \cdot \alpha(\overset{(k+2)}{y_1^{s-1}}) \cdot \dots \cdot \alpha(\overset{(2k)}{y_1^{s-1}}) \cdot x_{2k+1}) =$$

$$A(x_1, \overset{(1)}{y_1^{s-1}}, A(x_2, \overset{(2)}{y_1^{s-1}}, \dots, \overset{(k+1)}{y_1^{s-1}}, x_{k+2}), \overset{(k+2)}{y_1^{s-1}}, \dots, \overset{(2k)}{y_1^{s-1}}, x_{2k+1}).$$

Sketch of the proof of $\widehat{2}$:

$$x \cdot \alpha(\overset{(1)}{y_1^{s-1}}) \cdot b_1 \cdot \dots \cdot b_{k-1} \cdot \alpha(\overset{(k)}{y_1^{s-1}}) \cdot (\alpha(\overset{(1)}{y_1^{s-1}}) \cdot b_1 \cdot \dots \cdot b_{k-1} \cdot \alpha(\overset{(k)}{y_1^{s-1}}))^{-1} =$$

$$(\alpha(\overset{(1)}{y_1^{s-1}}) \cdot b_1 \cdot \dots \cdot b_{k-1} \cdot \alpha(\overset{(k)}{y_1^{s-1}}))^{-1} \cdot \alpha(\overset{(1)}{y_1^{s-1}}) \cdot b_1 \cdot \dots \cdot b_{k-1} \cdot \alpha(\overset{(k)}{y_1^{s-1}}) \cdot x = x.$$

By $\widehat{1}, \widehat{2}$ and by Th. 3.1, we conclude that the statement (α) holds.

Sketch of the proof of (γ) :

$$(\alpha(\overset{(1)}{y_1^{s-1}}) \cdot b_1 \cdot \dots \cdot b_k \cdot \alpha(\overset{(k)}{y_1^{s-1}}))^{-1} = (\alpha(\overset{(k)}{y_1^{s-1}}) \cdot \alpha(\overset{(1)}{y_1^{s-1}}) \cdot b_1 \cdot \dots \cdot b_k)^{-1}.$$

Sketch of the proof of (δ) : By definition of no commutative group and by definition of $m-$ary quasigroup. □

4.4. Theorem *[15]: Let $k > 1, s \geq 1, n = k \cdot s + 1$ and let $(Q; A)$ be an $n-$groupoid. Also, let E be an $(n-2)-$ary operation in Q such that the following law*

(o) $\mathsf{E}(c_1^{n-2-s}, b, a_1^{s-1}) = \mathsf{E}(a_1^{s-1}, c_1^{n-2-s}, b)$

holds in the $(n-2)-$groupoid $(Q; \mathsf{E})$. Then, $(Q; A)$ is an $NP-$polyagroup iff the laws $(1Ls), (2R)$ and $(\widehat{2L})$ from Th. 4.2 hold in the algebra $(Q; A, \mathsf{E})$.

Remark: For $s = 1$ law (o) holds. In addition, for $s = 1$ $(Q; A, \mathsf{E})$ is a characterization of $n-$group [2]. See, also XII-1 in [10].

Proof. By Prop. 4.1 and by Th. 4.2.

4.5. Remark*: Similarly, we obtain generalization the following proposition [2]: Let $(Q; A)$ be an $n-$groupoid and let $n \geq 3$. Then: $(Q; A)$ is an $n-$group iff there is a mapping E of the set Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q; A, \mathsf{E})$ [of the type $< n, n-2 >$].*

$A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_{n-1}^{2n-2})),$

$A(\mathsf{E}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*

$A(x, \mathsf{E}(a_1^{n-2}), a_1^{n-2}) = x.$

(See, also XII-1 in [10].)

4.6. Theorem *[15]: Let* $k > 1, s \geq 1$ [3], $n = k \cdot s + 1$, *$(Q; A)$ be an $NP-$polyagroup of the type $(s, n-1)$,* $\mathbf{e}$ *its $\{1, n\}-$neutral operation and let*

$$(\widehat{o})\ A(\overline{x_j, \overset{(j)}{y}{}_1^{s-1}}\Big|_{j=1}^{k}, x_{k+1}) = A(x_1, \overline{\overset{(j)}{y}{}_1^{s-1}, x_j}\Big|_{j=2}^{k}, \overset{(1)}{y}{}_1^{s-1}, x_{k+1})$$

for every $x_1^{k+1}, \overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1} \in Q$. *Also, let*

$$c_1^{k-1}, \overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1}$$

arbitrary sequence over Q,

$$Y \overset{def}{=} \overset{(1)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1},$$

and let

$(a)\ B_Y(x, y) \overset{def}{=} A(x, \overset{(1)}{y}{}_1^{s-1}, c_1, \ldots, c_{k-1}, \overset{(k)}{y}{}_1^{s-1}, y),$

$(b)\ \varphi_Y(x) \overset{def}{=} A(\mathbf{e}(\overline{\overset{(i)}{y}{}_1^{s-1}, c_i}\Big|_{i=1}^{k-1}, \overset{(k)}{y}{}_1^{s-1}), \overset{(k)}{y}{}_1^{s-1}, x, \overset{(1)}{y}{}_1^{s-1}, c_1, \ldots, \overset{(k-1)}{y}{}_1^{s-1}, c_{k-1})$ *and*

$(c)\ b_Y \overset{def}{=} A(\mathbf{e}(a_1^{n-2})$[4]$, \overset{(1)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(2)}{y}{}_1^{s-1}, \ldots, \overset{(k)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))$

for all $x, y \in Q$. Then the following statements hold:

(1) $(Q; B_y)$ is a group;

(2) $\varphi_Y \in Aut(Q; B_Y)$;

(3) $\varphi_Y(b_Y) = b_Y$;

(4) For all $x \in Q$, $B_Y(b_Y, x) = B_Y(\varphi_Y^k(x), b_Y)$; and

(5) $A(x_1, \overset{(1)}{y}{}_1^{s-1}, \ldots, x_k, \overset{(k)}{y}{}_1^{s-1}, x_{k+1}) = \overset{k+1}{B}_Y(x_1, \varphi_Y(x_2), \ldots, \varphi_Y^k(x_{k+1}), b_Y)$

for all $x_1^{k+1} \in Q$ and for every sequence Y over Q.

Remark: For $s = 1$ see IV-3 in [10]. See, also Th.4.3.

Proof. Firstly, let

$$x \cdot y \overset{def}{=} B_Y(x, y), \varphi(x) \overset{def}{=} \varphi_Y(x),\ b \overset{def}{=} b_Y.$$

The proof of (1): By (a) and by Def. 1.1.

---

[3] For $s = 1$ $(Q; A)$ is a $(k+1)-$group.

[4] $a_1^{n-2} \overset{def}{=} \overset{(1)}{y}{}_1^{s-1}, c_1, \ldots, c_{k-1}, \overset{(k)}{y}{}_1^{s-1}.$

Sketch of the proof of (2):

$$\begin{aligned}
\varphi(x\cdot y) &= A(\mathbf{e}(a_1^{n-2}), \overset{(k)}{y_1^{s-1}}, A(x, \overset{(1)}{y_1^{s-1}}, c_1, \ldots, c_{k-1}, \overset{(k)}{y_1^{s-1}}, y),\\
&\qquad \overset{(1)}{y_1^{s-1}}, c_1, \ldots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1})\\
&\overset{1.1,1^\circ}{=} A(A(\mathbf{e}(a_1^{n-2}), \overset{(k)}{y_1^{s-1}}, x, \overset{(1)}{y_1^{s-1}}, c_1, \ldots, c_{k-1}), \overset{(k)}{y_1^{s-1}}, y,\\
&\qquad \overset{(1)}{y_1^{s-1}}, c_1, \ldots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1})\\
&\overset{(b)}{=} A(\varphi(x), \overset{(k-1)}{y_1^{s-1}}, y, \overset{(1)}{y_1^{s-1}}, c_1, \ldots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1})\\
&\overset{2.4}{=} A(A(\varphi(x), \overset{(1)}{y_1^{s-1}}, c_1, \ldots, c_{k-1}, \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2})), \overset{(k)}{y_1^{s-1}}, y,\\
&\qquad \overset{(1)}{y_1^{s-1}}, c_1, \ldots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1})\\
&\overset{1.1,1^\circ}{=} A(\varphi(x), \overset{(1)}{y_1^{s-1}}, c_1, \ldots, c_{k-1}, \overset{(k)}{y_1^{s-1}}, A(\mathbf{e}(a_1^{n-2}), \overset{(k)}{y_1^{s-1}}, y,\\
&\qquad \overset{(1)}{y_1^{s-1}}, c_1, \ldots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1}))\\
&\overset{(b)}{=} A(\varphi(x), \overset{(1)}{y_1^{s-1}}, c_1, \ldots, c_{k-1}, \overset{(k)}{y_1^{s-1}}, \varphi(y))\\
&\overset{(a)}{=} \varphi(x)\cdot\varphi(y).
\end{aligned}$$

Sketch of the proof of (3):

$$\begin{aligned}
\varphi(b) &\overset{(b),(c)}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(k)}{y_1^{s-1}}, A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, \ldots, \overset{(k-1)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(k)}{y_1^{s-1}},\\
&\qquad \mathbf{e}(a_1^{n-2})), \overset{(1)}{y_1^{s-1}}, c_1, \ldots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1})\\
&\overset{1.1,1^\circ}{=} A(A(\mathbf{e}(a_1^{n-2}), \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, \ldots, \overset{(k-1)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2})), \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}),\\
&\qquad \overset{(1)}{y_1^{s-1}}, c_1, \ldots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1})\\
&\overset{(\widehat{o})}{=} A(A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \ldots, \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2})), \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}),\\
&\qquad \overset{(1)}{y_1^{s-1}}, c_1, \ldots, \overset{(k-1)}{y_1^{s-1}}, c_{k-1})\\
&\overset{4.1}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \ldots, \overset{(k)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}))\\
&\overset{(c)}{=} b.
\end{aligned}$$

Sketch of the proof of (4) *[for the case $k = 3,\ s > 1$]:*

$$
\begin{aligned}
b\cdot x &\overset{(a)}{=} A(b,a_1^{n-2},x)\\
&\overset{(c)}{=} A(A(\mathbf{e}(a_1^{n-2}),\overset{(1)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}),\overset{(2)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}),\overset{(3)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2})),a_1^{n-2},x)\\
&\overset{1.1,1^\circ}{=} A(\mathbf{e}(a_1^{n-2}),\overset{(1)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}),\overset{(2)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}),\overset{(3)}{y_1^{s-1}},A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},x))\\
&\overset{2.4}{=} A(\mathbf{e}(a_1^{n-2}),\overset{(1)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}),\overset{(2)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}),\overset{(3)}{y_1^{s-1}},A(x,a_1^{n-2},\mathbf{e}(a_1^{n-2})))\\
&\overset{fn4}{=} A(\mathbf{e}(a_1^{n-2}),\overset{(1)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}),\overset{(2)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}),\overset{(3)}{y_1^{s-1}},\\
&\qquad\qquad A(x,\overset{(1)}{y_1^{s-1}},c_1,\overset{(2)}{y_1^{s-1}},c_2,\overset{(3)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2})))\\
&\overset{1.1,1^\circ}{=} A(\mathbf{e}(a_1^{n-2}),\overset{(1)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}),\overset{(2)}{y_1^{s-1}},A(\mathbf{e}(a_1^{n-2}),\overset{(3)}{y_1^{s-1}},\\
&\qquad\qquad x,\overset{(1)}{y_1^{s-1}},c_1,\overset{(2)}{y_1^{s-1}},c_2),\overset{(3)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2})))\\
&\overset{(b)}{=} A(\mathbf{e}(a_1^{n-2}),\overset{(1)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}),\overset{(2)}{y_1^{s-1}},\varphi(x),\overset{(3)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}))\\
&\overset{1.1,4.1}{=} A(\left.\overline{\mathbf{e}(a_1^{n-2}),\overset{(i)}{y_1^{s-1}}}\right|_{i=1}^{2},A(c_1,\overset{(2)}{y_1^{s-1}},c_2,\overset{(3)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2})\overset{(1)}{y_1^{s-1}},\\
&\qquad\qquad A(\varphi(x),\overset{(1)}{y_1^{s-1}},c_1,\overset{(2)}{y_1^{s-1}},c_2,\overset{(3)}{y_1^{s-1}},\\
&\qquad\qquad \mathbf{e}(a_1^{n-2}))),\overset{(3)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}))\\
&\overset{(\hat{o})}{=} A(\left.\overline{\mathbf{e}(a_1^{n-2}),\overset{(i)}{y_1^{s-1}}}\right|_{i=1}^{2},A(c_1,\overset{(1)}{y_1^{s-1}},c_2,\overset{(2)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}),\overset{(3)}{y_1^{s-1}},\\
&\qquad\qquad A(\varphi(x),\overset{(1)}{y_1^{s-1}},c_1,\overset{(2)}{y_1^{s-1}},c_2,\overset{(3)}{y_1^{s-1}},\\
&\qquad\qquad \mathbf{e}(a_1^{n-2}))),\overset{(3)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}))\\
&\overset{1.1,1^\circ}{=} A(\left.\overline{\mathbf{e}(a_1^{n-2}),\overset{(i)}{y_1^{s-1}}}\right|_{i=1}^{2},A(c_1,\overset{(1)}{y_1^{s-1}},c_2,\overset{(2)}{y_1^{s-1}},A(\mathbf{e}(a_1^{n-2}),\overset{(3)}{y_1^{s-1}},\\
&\qquad\qquad \varphi(x),\overset{(1)}{y_1^{s-1}},c_1,\overset{(2)}{y_1^{s-1}},c_2),\overset{(3)}{y_1^{s-1}},\\
&\qquad\qquad \mathbf{e}(a_1^{n-2})),\overset{(3)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}))\\
&\overset{(b)}{=} A(\left.\overline{\mathbf{e}(a_1^{n-2}),\overset{(i)}{y_1^{s-1}}}\right|_{i=1}^{2},A(c_1,\overset{(1)}{y_1^{s-1}},c_2,\overset{(2)}{y_1^{s-1}},\varphi(\varphi(x)),\overset{(3)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2})),\\
&\qquad\qquad \overset{(3)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}))\\
&\overset{(\hat{o})}{=} A(\left.\overline{\mathbf{e}(a_1^{n-2}),\overset{(i)}{y_1^{s-1}}}\right|_{i=1}^{2},A(c_1,\overset{(3)}{y_1^{s-1}},c_2,\overset{(1)}{y_1^{s-1}},\varphi(\varphi(x)),\overset{(2)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2})),\\
&\qquad\qquad \overset{(3)}{y_1^{s-1}},\mathbf{e}(a_1^{n-2}))
\end{aligned}
$$

$$
\begin{aligned}
&\overset{1.1,1^\circ}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(\mathbf{e}(a_1^{n-2}), \overset{(2)}{y}{}_1^{s-1}, c_1, \overset{(3)}{y}{}_1^{s-1}, c_2, \overset{(1)}{y}{}_1^{s-1}, \varphi(\varphi(x))),\\
&\qquad\qquad \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{(\hat{o})}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \varphi(\varphi(x))),\\
&\qquad\qquad \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{2.4}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, \varphi(\varphi(x)), \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{1.1,4.1}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1},\\
&\qquad\qquad A(\varphi(\varphi(x)), \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))),\\
&\qquad\qquad \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{(\hat{o})}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(c_1, \overset{(1)}{y}{}_1^{s-1}, c_2, \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1},\\
&\qquad\qquad A(\varphi(\varphi(x)), \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))),\\
&\qquad\qquad \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{1.1,1^\circ}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(c_1, \overset{(1)}{y}{}_1^{s-1}, c_2, \overset{(2)}{y}{}_1^{s-1}, A(\mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1},\\
&\qquad\qquad \varphi(\varphi(x)), \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})),\\
&\qquad\qquad \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{(\hat{o})}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, A(\mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1},\\
&\qquad\qquad \varphi(\varphi(x)), \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2), \overset{(1)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})),\\
&\qquad\qquad \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{(b)}{=} A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, A(c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \varphi(\varphi(\varphi(x))), \overset{(1)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})),\\
&\qquad\qquad \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{1.1,1^\circ}{=} A(A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y}{}_1^{s-1}, c_1, \overset{(2)}{y}{}_1^{s-1}, c_2, \overset{(3)}{y}{}_1^{s-1}, \varphi(\varphi(\varphi(x)))), \overset{(1)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2})),\\
&\qquad\qquad \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}))\\
&\overset{2.4}{=} A(A(\varphi(\varphi(\varphi(x))), a_1^{n-2}, \mathbf{e}(a_1^{n-2})), \overset{(1)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(2)}{y}{}_1^{s-1}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y}{}_1^{s-1},\\
&\qquad\qquad \mathbf{e}(a_1^{n-2}))
\end{aligned}
$$

$$\overset{1.1,1^\circ}{=} A(\varphi(\varphi(\varphi(x))), a_1^{n-2}, A(\mathbf{e}(a_1^{n-2}), \overset{(1)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(2)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2}), \overset{(3)}{y_1^{s-1}}, \mathbf{e}(a_1^{n-2})))$$

$$\overset{(a),(c)}{=} \varphi(\varphi(\varphi(x))) \cdot b.$$

The proof of (5): By 2.4, 4.1, $1^\circ, (\widehat{o}), (a), (b)$ and (c). Cf. sketch of the proof of (4) and IV-3 in [10].

5. On polyagroups

5.1. Definition *[3]*[5]*: Let $k > 1$, $s \geq 1$, $n = k \cdot s + 1$ and let $(Q; A)$ be an $n-$groupoid. Then: we say that $(Q; A)$ is a* **polyagroup of the type** $(s, n-1)$ *iff the following statements hold:*

$\overset{\circ}{1}$ *For all $i, j \in \{1, \ldots, n\}$ $(i < j)$ if $i \equiv j(mod\ s)$, then the $< i, j > -$associative law holds in $(Q; A)$; and*

$\overset{\circ}{2}$ *$(Q; A)$ is an $n-$quasigroup.*

Remark: For $s = 1$ $(Q; A)$ is a $(k+1)-$group; $k > 1$.

5.2. Proposition*: Every polyagroup of the type $(s, n-1)$ is an $NP-$polyagroup of the type $(s, n-1)$.*

Proof. By Def. 1.1 and by Def. 5.1.

5.3. Proposition*: Every polyagroup of the type $(s, n-1)$ has $\{1, n\}-$neutral operation.*

Proof. By Proop. 5.2 and by Prop. 2.4. (Cf. II-2 in [10].)

5.4. Definition*: Let $k > 1$, $s \geq 1$, $n = k \cdot s + 1$ and let $(Q; A)$ be an $n-$groupoid. Then: we say that $(Q; A)$ is an iPs-***associative** $n-$**groupoid***, $i \in \{1, \ldots, s\}$, iff it is*

(a) If $i = 1$, then $< i, t \cdot s + i > -$associative law holds in $(Q; A)$ for all $t \in \{1, \ldots, k\}$; and

(b) If $i > 1$, then $< i, t \cdot s + i > -$associative law holds in $(Q; A)$ for all $t \in \{1, \ldots, k-1\}$.

5.5. Proposition*: Let $k > 1$, $s \geq 1$, $n = k \cdot s + 1$ and let $(Q; A)$ be an $n-$groupoid. Then, $(Q; A)$ is a* **polyagroup of the type** $(s, n-1)$ *iff the following statements hold:*

$\overline{1}$ *$(Q; A)$ is an iPs-associative $n-$groupoid for all $i \in \{1 \ldots, s\}$; and*

[5]See, also [4].

$\overline{2}$ $(Q;A)$ *is an* $n-$*quasigroup.*

Remark: If $(Q;A)$ *is an* $n-$*group and* $n=k\cdot s+1$, *then* $(Q;A)$ *is a polyagroup of the type* $(s,n-1)$.

Proof. By Def. 5.1 and by Def. 5.4.

5.6. Proposition*: Let* $k>1,\ s>1,\ n=k\cdot s+1,\ i\in\{1,\dots,s\}$ *and let* $(Q;A)$ *be an* $n-$*groupoid. Also, let*

(α) *The* $<i,s+i>-$*associative law holds in the* $(Q;A)$; *and*

(β) *For every* $x,y,a_1^{n-1}\in Q$ *the following implication holds*

$A(a_1^{i-1},x,a_i^{n-1})=A(a_1^{i-1},y,a_i^{n-1})\Rightarrow x=y.$

Then $(Q;A)$ *is an iPs-associative* $n-$*groupoid.*

Remark: For $k=2$ *and* $i\in\{2,\dots,s\}$, $(Q;A)$ *is an iPs-associative* $n-$*groupoid iff* (α).

Proof. See the proof of Prop. 2.2_1.

5.7. Proposition*: Let* $k>1,\ s>1,\ n=k\cdot s+1,\ i\in\{1,\dots,s\}$ *and let* $(Q;A)$ *be an* $n-$*groupoid. Also, let*

(1) $(Q;A)$ *is an iPs-associative* $n-$*groupoid;*

(2) *For every* $a_1^n\in Q$ *there is exactly one* $x\in Q$ *such that the following equality holds*

$$A(a_1^{i-1},x,a_i^{n-1})=a_n;\ \textit{and}$$

(3) *For every* $a_1^n\in Q$ *there is exactly one* $y\in Q$ *such that the following equality holds*

$$A(a_1^{(k-1)s+i-1},y,a_{(k-1)s+i}^{ks})=a_{ks+1}.$$

Then for every $a_1^{ks+1}\in Q$ *and for all* $t\in\{1,\dots,k-2\}$ *there is exactly one* $z\in Q$ *such that the following equality holds*

$$A(a_1^{ts+i-1},z,a_{ts+i}^{ks})=a_{ks+1}.$$

Sketch of the proof.

a) $A(a_1^{ts+i-1},x,b_1^{(k-t)s-i+1})=A(a_1^{ts+i-1},y,b_1^{(k-t)s-i+1})\Rightarrow$
$A(c_1^{i-1},d_1^{(k-t-1)\cdot s},A(a_1^{ts+i-1},x,b_1^{(k-t)s-i+1}),c_i^{ts+i-1},d_{(k-t-1)s+1}^{(k-t)s-i+1})=$
$A(c_1^{i-1},d_1^{(k-t-1)\cdot s},A(a_1^{ts+i-1},y,b_1^{(k-t)s-i+1}),c_i^{ts+i-1},d_{(k-t-1)s+1}^{(k-t)s-i+1})\overset{(1)}{\Rightarrow}$
$A(c_1^{i-1},A(d_1^{(k-t-1)s},a_1^{ts+i-1},x,b_1^{s-i+1}),b_{s-i+2}^{(k-t)s-i+1},c_i^{ts+i-1},d_{(k-t-1)s+1}^{(k-t)s-i+1})=$
$A(c_1^{i-1},A(d_1^{(k-t-1)s},a_1^{ts+i-1},y,b_1^{s-i+1}),b_{s-i+2}^{(k-t)s-i+1},c_i^{ts+i-1},d_{(k-t-1)s+1}^{(k-t)s-i+1})\overset{(2)}{\Rightarrow}$

$A(d_1^{(k-t-1)s}, a_1^{ts+i-1}, x, b_1^{s-i+1}) =$
$A(d_1^{(k-t-1)s}, a_1^{ts+i-1}, y, b_1^{s-i+1}) \overset{(3)}{\Rightarrow} x = y.$
b) $A(a_1^{(k-t-1)s}, b_1^{i-1}, x, c_1^{(t+1)s-i+1}) = A(a_1^{(k-t-1)s}, b_1^{i-1}, y, c_1^{(t+1)s-i+1}) \Rightarrow$
$A(d_1^{ts+i-1}, A(a_1^{(k-t-1)s}, b_1^{i-1}, x, c_1^{(t+1)s-i+1}), e_1^{(k-t)s-i+1}) =$
$A(d_1^{ts+i-1}, A(a_1^{(k-t-1)s}, b_1^{i-1}, y, c_1^{(t+1)s-i+1}), e_1^{(k-t)s-i+1}) \overset{(1)}{\Rightarrow}$
$A(d_1^{ts+i-1}, a_1^{(k-t-1)s}, A(b_1^{i-1}, x, c_1^{(t+1)s-i+1}, e_1^{ks-(t+1)s}), e_{ks-(t+1)s+1}^{(k-t)s-i+1} =$
$A(d_1^{ts+i-1}, a_1^{(k-t-1)s}, A(b_1^{i-1}, y, c_1^{(t+1)s-i+1}, e_1^{ks-(t+1)s}), e_{ks-(t+1)s+1}^{(k-t)s-i+1} \overset{(3)}{\Rightarrow}$
$(b_1^{i-1}, x, c_1^{(t+1)s-i+1}, e_1^{ks-(t+1)s}) =$
$(b_1^{i-1}, y, c_1^{(t+1)s-i+1}, e_1^{ks-(t+1)s}) \overset{(2)}{\Rightarrow} x = y.$
c) $A(a_1^{ts+i-1}, z, b_1^{(k-t)s-i+1}) = c \overset{b)}{\Leftrightarrow}$
$A(c_1^{i-1}, d_1^{(k-t-1)s}, A(a_1^{ts+i-1}, z, b_1^{(k-t)s-i+1}), e_1^{(t+1)s-i+1}) =$
$A(c_1^{i-1}, d_1^{(k-t-1)s}, c, e_1^{(t+1)s-i+1}) \overset{(1)}{\Leftrightarrow}$
$A(c_1^{i-1}, A(d_1^{(k-t-1)s}, a_1^{ts+i-1}, z, b_1^{s-i+1}), b_{s-i+2}^{(k-t)s-i+1}, e_1^{(t+1)s-i+1}) =$
$A(c_1^{i-1}, d_1^{(k-t-1)s}, c, e_1^{(t+1)s-i+1}),$
where $c_1^{i-1}, d_1^{(k-t-1)s}$ and $e_1^{(t+1)s-i+1}$ are arbitrary sequence over Q. *[*$(k-t-1)\ s + ts + i - 1 = (k-1)s + i - 1$; (3). *]* $\square$

5.8. Theorem *[14]: Let* $k > 1, s > 1, n = k\cdot s+1$ *and let* $(Q; A)$ *be an* $n-$*groupoid. Then:* $(Q; A)$ *is a* **polyagroup of the type** $(s, n-1)$ *iff the following statements hold*

(i) $(Q; A)$ *is an* $< i, s+i > -$*associative* $n-$*groupoid for all* $i \in \{1, \ldots, s\}$;

(ii) $(Q; A)$ *is an* $< 1, n > -$*associative* $n-$*groupoid;*

(iii) For every $a_1^n \in Q$ *there is* **at least one** $x \in Q$ *and* **at least one** $y \in Q$ *such that the following equality hold*

$A(x, a_1^{n-1}) = a_n$ *and* $A(a_1^{n-1}, y) = a_n$ *and*

(iv) For every $a_1^n \in Q$ *and for all* $j \in \{2, \ldots, s\} \cup \{(k-1)\cdot s+2, \ldots, k\cdot s\}$ *there is* **exactly one** $x_j \in Q$ *such that the following equality hold*

$A(a_1^{j-1}, x, a_j^{n-1}) = a_n.$

Remarks: The case $s = 1$ *(: (i) – (iii)) is described in [8]. See, also Th.* 3.2_1.

Proof. 1) $\Rightarrow$: By Def. 1.1.

2) $\Leftarrow$: Firstly we prove the following statements:

$\widehat{1}$ $(Q; A)$ is an $NP-$polyagroup;

$\widehat{2}$ $(Q; A)$ is an iPs-associative $n-$groupoid for all $i \in \{2, \ldots, s\}$; and

$\widehat{3}$ $(Q;A)$ is an $n-$groupoid.

The proof. of $\widehat{1}$: By (i) for $i=1,(ii),(iii)$ and Th. 3.2_1.

The proof. of $\widehat{2}$:

$a)$ $k=2$: By (i).

$b)$ $k>2$: By (i) for $i\in\{2,\dots,s\}$, (iv) and by Prop. 5.6.

The proof. of $\widehat{3}$:

$a)$ $k=2$: By (iv).

$b)$ $k>2$: By $\widehat{1},\widehat{2},(iv)$ and by Prop. 5.7.

By $\widehat{1}-\widehat{3}$ and by Prop. 5.5, we conclude that the $n-$groupoid $(Q;A)$ is a polyagroup of the type $(s,n-1)$. $\square$

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