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$(n,m)-$Groups in the Light of the Neutral Operations Survey Article

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Abstract. This text is as an attempt to systematize the results about $(n,m)-$groups in the light of the neutral operations. (The case $m = 1$ is the monograph [23].)

1. Notion and examples

1.1. Definition *[1]: Let $n \geq m+1$ $(n,m \in N)$ and $(Q;A)$ be an $(n,m)-$groupoid $(A : Q^n \to Q^m)$. We say that $(Q;A)$ is an $(n,m)-$group iff the following statements hold:*

($|$) *For every $i,j \in \{1,\ldots,n-m+1\}$, $i<j$, the following law holds*
$A(x_1^{i-1},A(x_i^{i+n-1}),x_{i+n}^{2n-m}) = A(x_1^{j-1},A(x_j^{j+n-1}),x_{j+n}^{2n-m})$
[: $<i,j>-$associative law][1]; and

($||$) *For every $i \in \{1,\ldots,n-m+1\}$ and for every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds*

$$A(a_1^{i-1},x_1^m,a_i^{n-m}) = a_{n-m+1}^n.$$

Remark: For $m=1$ $(Q;A)$ is an $n-$group [6]. Cf. Def. 1.1–I in [23].

1.2. Remark: *A notion of an $(n,m)-$group was introduced by Ǵ. Čupona in [1] as a generalization of the notion of a group ($n-$group). The paper [3] is mainly a survey on the know results for wector valued groupoids, semigroups and groups (up to 1988).*

1.3. Example *[3]: Let*

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[1] $(Q;A)$ is an $(n,m)-$semigroup.

$\Phi(x,y)\overset{def}{=}(x+\frac{1}{2}\sin y,\ y+\frac{1}{2}\sin x)$

for all $x, y \in R$, *where* R *is the set of real numbers. Then* Φ *[*$:R^2 \to R^2$*] is a bijection. Further on, let*

$A(x,y,z,u)\overset{def}{=}\Phi^{-1}(x+z+\frac{1}{2}(\sin y+\sin u),\ y+u+\frac{1}{2}(\sin x+\sin z))$

for all $x, y, z, u \in R$. *Then* $(R;A)$ *is a (4,2)-group.*

1.4. Example *[3]: Let*

$A(z_1^5)\overset{def}{=}(z_1+z_4+\frac{1+i\sqrt{3}}{2}z_3,\ z_2+z_5+\frac{1-i\sqrt{3}}{2}z_3)$

for all $z_1^5 \in C$, *where* C *is the set of complex numbers. Then* $(C;A)$ *is a (5,2)-group.*

See, also [2], [3], [4] and [5].

2. $\{1, n-m+1\}-$neutral operations of $(n,m)-$groupoids

2.1. Definition *[14]: Let* $n \geq 2m$ *and let* $(Q;A)$ *be an* $(n,m)-$*groupoid. Also, let* $\mathbf{e}_L, \mathbf{e}_R$ *and* $\mathbf{e}$ *be mappings of the set* Q^{n-2m} *into the set* Q^m. *Then:*

1) $\mathbf{e}_L$ *is a* **left** $\{1, n-m+1\}-$**neutral operation of the** $(n,m)-$**groupoid** $(Q;A)$ *iff for every* $x_1^m \in Q$ *and for every sequence* a_1^{n-2m} *over* Q *the following equality holds*

(*l*) $\qquad A(\mathbf{e}_L(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m$;

2) $\mathbf{e}_R$ *is a* **right** $\{1, n-m+1\}-$**neutral operation of the** $(n,m)-$**groupoid** $(Q;A)$ *iff for every* $x_1^m \in Q$ *and for every sequence* a_1^{n-2m} *over* Q *the following equality holds*

(*r*) $\qquad A(x_1^m, a_1^{n-2m}, \mathbf{e}_R(a_1^{n-2m})) = x_1^m$; *and*

3) $\mathbf{e}$ *is a* $\{1, n-m+1\}-$**neutral operation of the** $(n,m)-$**groupoid** $(Q;A)$ *iff for every* $x_1^m \in Q$ *and for every sequence* a_1^{n-2m} *over* Q *the following equalities hold*

(*n*) $\qquad A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m$ *and* $A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m$.

Remark: For $m = 1$ $\mathbf{e}$ is a $\{1, n\}-$neutral operation of the $n-$groupoid $(Q;A)$ [13]. For $(n,m) = (2,1)$, $\mathbf{e}(a_1^\circ)$/$= \mathbf{e}(\emptyset)$/ is a neutral element of the groupoid $(Q;A)$. Cf. Ch. II in [23].

2.2. Proposition *[14]: Let* $(Q;A)$ *be an* $(n,m)-$*groupoid and* $n \geq 2m$. *Then there is* **at most one** $\{1, n-m+1\}-$*neutral operation of* $(Q;A)$.

Proof. Suppose that $\mathbf{e}_1$ and $\mathbf{e}_2$ are $\{1, n-m+1\}-$neutral operation of an $(n,m)-$groupoid $(Q;A)$. Then, by Def. 2.1, for every sequence a_1^{n-2m} over Q the following equalities hold

$A(\mathbf{e}_1(a_1^{n-2m}), a_1^{n-2m}, \mathbf{e}_2(a_1^{n-2m})) = \mathbf{e}_2(a_1^{n-2m})$ and

$A(\mathbf{e}_1(a_1^{n-2m}), a_1^{n-2m}, \mathbf{e}_2(a_1^{n-2m})) = \mathbf{e}_1(a_1^{n-2m})$,

whence we conclude that $\mathbf{e}_1 = \mathbf{e}_2$. □

2.3. Proposition *[14]: Let $(Q;A)$ be an $(n,m)-$groupoid and $n \geq 2m$. Then: if $\mathbf{e}_L$ is a* **left** *$\{1, n-m+1\}-$neutral operation of $(Q;A)$ and $\mathbf{e}_R$ is a* **right** *$\{1, n-m+1\}-$neutral operation of $(Q;A)$, then $\mathbf{e}_L = \mathbf{e}_R$ and $\mathbf{e} = \mathbf{e}_L = \mathbf{e}_R$ is an $\{1, n-m+1\}-$neutral operation of $(Q;A)$.*

Proof. By Def. 2.1,we conclude that for every sequence a_1^{n-2m} over Q the following equalities hold

$A(\mathbf{e}_L(a_1^{n-2m}), a_1^{n-2m}, \mathbf{e}_R(a_1^{n-2m})) = \mathbf{e}_R(a_1^{n-2m})$ and

$A(\mathbf{e}_L(a_1^{n-2m}), a_1^{n-2m}, \mathbf{e}_R(a_1^{n-2m})) = \mathbf{e}_L(a_1^{n-2m})$,

whence we conclude that $\mathbf{e}_L = \mathbf{e}_R$. □

2.4. Proposition *[19]: Let $(Q;A)$ be an $(n,m)-$groupoid and $n \geq 2m$. Further on, let the following statements hold:*

(i) The $< 1, n-m+1 > -$associative law holds in $(Q;A)$;

(ii) For every $a_1^n \in Q$ there is **at least one** *$x_1^m \in Q^m$ such that the equality $A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n$ holds; and*

(iii) For every $a_1^n \in Q$ there is **at least one** *$y_1^m \in Q^m$ such that the equality $A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n$ holds.*

Then $(Q;A)$ has a $\{1, n-m+1\}-$neutral operation.

Proof. Firstly we prove the following statements:

1° $(Q;A)$ has a left $\{1, n-m+1\}-$neutral operation; and

2° $(Q;A)$ has a right $\{1, n-m+1\}-$neutral operation.

The proof of 1° :

Let b_1^m be an arbitrary (fixed) sequence over Q. Then, by (iii), for every sequence a_1^{n-2m} over Q there is at least one $\widehat{\mathbf{e}}_L(a_1^{n-2m}) \in Q^m$ such that the following equality holds

(a) $\quad A(\widehat{\mathbf{e}}_L(a_1^{n-2m}), a_1^{n-2m}, b_1^m) = b_1^m$.

On the other hand, by (ii), for every $c_1^m \in Q^m$ and for every sequence k_1^{n-2m} over Q there is at least one $t_1^m \in Q^m$ such that the following equality holds

$(b)\qquad c_1^m = A(b_1^m, k_1^{n-2m}, t_1^m).$

By $(a), (b)$ and (i), we conclude that the following series of equalities hold

$$\begin{aligned} A(\widehat{\mathbf{e}}_L(a_1^{n-2m}), a_1^{n-2m}, c_1^m) &\overset{(b)}{=} A(\widehat{\mathbf{e}}_L(a_1^{n-2m}), a_1^{n-2m}, A(b_1^m, k_1^{n-2m}, t_1^m)) \\ &\overset{(i)}{=} A(A(\widehat{\mathbf{e}}_L(a_1^{n-2m}), a_1^{n-2m}, b_1^m), k_1^{n-2m}, t_1^m) \\ &\overset{(a)}{=} A(b_1^m, k_1^{n-2m}, t_1^m) \\ &\overset{(b)}{=} c_1^m, \end{aligned}$$

whence we conclude that for every $c_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following equality holds

$$A(\widehat{\mathbf{e}}_L(a_1^{n-2m}), a_1^{n-2m}, c_1^m) = c_1^m,$$

i.e. that $(Q; A)$ has the left $\{1, n-m+1\}-$neutral operation.

Similarly, it is possible to prove the statement 2°.

Finally, by Prop. 2.3, we conclude that there is a $\{1, n-m+1\}-$neutral operation $\mathbf{e}$*[*$= \widehat{\mathbf{e}}_L = \widehat{\mathbf{e}}_R$*]*. □

By Prop. 2.4 and Def. 1.1, we obtain:

2.5. Theorem *[14]: Every $(n,m)-$group, $n \geq 2m$, has a $\{1, n-m+1\}-$neutral operation.*

By Th. 2.5 and by Prop. 2.2, we have:

2.6. Theorem *[2]: Let $(Q; A)$ be an $(n,m)-$group and $n = 2m$. Then there is exactly one $e_1^m \in Q^m$ such that for all $x_1^m \in Q^m$ the following equalities hold*

$(\widehat{n})\qquad A(x_1^m, e_1^m) = x_1^m$ *and* $A(e_1^m, x_1^m) = x_1^m.$

Remark: For $m = 1$, e_1^m is a neutral element of the group $(Q; A)$.

2.7. Theorem *[2]: Let $(Q; A)$ be a $(2m, m)-$***group** *and let $e_1^m \in Q^m$ satisfying $(\widehat{n})$ [from Th.2.6] for all $x_1^m \in Q^m$. Then, for all $i \in \{0, 1, \ldots, m\}$ and for every $x_1^m \in Q^m$ the following equality holds*

$$A(x_1^i, e_1^m, x_{i+1}^m) = x_1^m.$$

Sketch of the proof. $m > 1$:

$$\begin{aligned} A(x_1^i, e_1^m, x_{i+1}^m) &\overset{(\widehat{n})}{=} A(e_1^m, A(x_1^i, e_1^m, x_{i+1}^m)) \\ &\overset{1.1(|)}{=} A(e_1^i, A(e_{i+1}^m, x_1^i, e_1^m), x_{i+1}^m) \\ &\overset{(\widehat{n})}{=} A(e_1^i, e_{i+1}^m, x_1^i, x_{i+1}^m) \\ &= A(e_1^m, x_1^m) \\ &\overset{(\widehat{n})}{=} x_1^m. \quad \square \end{aligned}$$

By the proof of Th. 2.7, we conclude that the following proposition, also, holds:

2.8. Theorem: *Let* $(Q;A)$ *be a* $(2m,m)-$**semigroup** *and let* $e_1^m \in Q^m$ *satisfying* $(\widehat{n})$ *[from Th. 2.6] for all* $x_1^m \in Q^m$. *Then, for all* $i \in \{0,1,\ldots,m\}$ *and for every* $x_1^m \in Q^m$ *the following equality holds*

$$A(x_1^i, e_1^m, x_{i+1}^m) = x_1^m.$$

2.9. Theorem *[2]: Let* $(Q;A)$ *be a* $(2m,m)-$**group** *and let* $e_1^m \in Q^m$ *satisfying* $(\widehat{n})$ *[from Th. 2.6] for all* $x_1^m \in Q^m$. *Then:* $e_1 = e_2 = \cdots = e_m$.

Sketch of the proof. $m > 1$:

$$A(e_2^m, e_1^m, e_1) \overset{2.7}{=} (e_2^m, e_1) \Rightarrow$$
$$A(e_2^m, e_1, e_2^m, e_1) = (e_2^m, e_1) \overset{(\widehat{n})}{\Rightarrow}$$
$$A(e_2^m, e_1, e_2^m, e_1) = A(e_2^m, e_1, e_1^m) \overset{1.1(|||)}{\Longrightarrow}$$
$$(e_2^m, e_1) = (e_1^m),$$

whence, we obtain $e_1 = e_2 = \cdots = e_m$. □

2.10. Theorem *[9]: Let* $(Q;A)$ *be an* $(n,m)-$*group*, $\mathbf{e}$ *its* $\{1, n-m+1\}-$*neutral operation (2.1) and* $n > 2m$. *Then, for every* $a_1^{n-2m}, x_1^m \in Q$ *and for all* $i \in \{1,\ldots,n-2m+1\}$ *the following equalities hold*

(1) $\quad A(x_1^m, a_i^{n-2m}, \mathbf{e}(a_1^{n-2m}), a_1^{i-1}) = x_1^m$ *and*

(2) $\quad A(a_i^{n-2m}, \mathbf{e}(a_1^{n-2m}), a_1^{i-1}, x_1^m) = x_1^m$.

Remark: Th. 2.10 for $m = 1$ is proved in [16]. Cf. Prop. 1.1-IV in [23].

Proof. Let

(o) $\quad F(x_1^m, b_1^{n-2m}) \overset{def}{=} A(x_1^m, b_i^{n-2m}, \mathbf{e}(b_1^{n-2m}), b_1^{i-1})$

for all $x_1^m, b_1^{n-2m} \in Q$. Whence, we obtain

$$A(F(x_1^m, b_1^{n-2m}), b_i^{n-2m}, \mathbf{e}(b_1^{n-2m}), b_1^{i-1}) =$$
$$A(A(x_1^m, b_i^{n-2m}, \mathbf{e}(b_1^{n-2m}), b_1^{i-1}), b_i^{n-2m}, \mathbf{e}(b_1^{n-2m}), b_1^{i-1})$$

for all $x_1^m, b_1^{n-2m} \in Q$.

Hence, by Def. 1.1 and by Th. 2.5, we have

$$A(F(x_1^m, b_1^{n-2m}), b_i^{n-2m}, \mathbf{e}(b_1^{n-2m}), b_1^{i-1}) =$$
$$A(x_1^m, b_i^{n-2m}, A(\mathbf{e}(b_1^{n-2m}), b_1^{i-1}, b_i^{n-2m}, \mathbf{e}(b_1^{n-2m})), b_1^{i-1}),$$

i.e.

$$A(F(x_1^m, b_1^{n-2m}), b_i^{n-2m}, \mathbf{e}(b_1^{n-2m}), b_1^{i-1}) =$$
$$A(x_1^m, b_i^{n-2m}, \mathbf{e}(b_1^{n-2m}), b_1^{i-1})$$

for every $x_1^m, b_1^{n-2m} \in Q$.

In adition, hence, by Def. 1.1 (cancelation), we obtain

$$F(x_1^m, b_1^{n-2m}) = x_1^m$$

for all $x_1^m, b_1^{n-2m} \in Q$, whence we have (1).

Similarly, we obtain, also, (2). □

2.11. Theorem *[8]: Let $n > 2m$, $m > 1$, $(Q; A)$ be an $(n, m)-$group and* $\mathbf{e}$ *its $\{1, n-m+1\}-$neutral operation. Then for all $i \in \{0, 1, \ldots, m\}$, for all $t \in \{1, \ldots, n-2m+1\}$, for every $x_1^m \in Q^m$ and for every $a_1^{n-2m} \in Q$ the following equality holds*

$$A(x_1^i, a_t^{n-2m}, \mathbf{e}(a_1^{n-2m}), a_1^{t-1}, x_{i+1}^m) = x_1^m.$$

Remark: Th. 2.11 for $n = 2m$ is proved in [2]. See, also [3].

Sketch of the proof. 1) Instead of $\mathbf{e}(a_1^{n-2m})$ we are sometimes going to write

$$\overline{\mathbf{e}_j(a_1^{n-2m})}\Big|_{j=1}^m.$$

2) $A(x_1^i, a_t^{n-2m}, \mathbf{e}(a_1^{n-2m}), a_1^{t-1}, x_{i+1}^m) \overset{(2)i=1}{=\!=}$

$A(a_1^{n-2m}, \mathbf{e}(a_1^{n-2m}), A(x_1^i, a_t^{n-2m}, \mathbf{e}(a_1^{n-2m}), a_1^{t-1}, x_{i+1}^m)) \overset{1)}{=}$

$A(a_1^{n-2m}, \overline{\mathbf{e}_j(a_1^{n-2m})}\Big|_{j=1}^m, A(x_1^i, a_t^{n-2m}, \mathbf{e}(a_1^{n-2m}), a_1^{t-1}, x_{i+1}^m)) =$

$A(a_1^{n-2m}, \overline{\mathbf{e}_j(a_1^{n-2m})}\Big|_{j=1}^i, \overline{\mathbf{e}_j(a_1^{n-2m})}\Big|_{j=i+1}^m, A(x_1^i, a_t^{n-2m}, \mathbf{e}(a_1^{n-2m}), a_1^{t-1}, x_{i+1}^m)) \overset{(|)}{=}$

$A(a_1^{n-2m}, \overline{\mathbf{e}_j(a_1^{n-2m})}\Big|_{j=1}^i, A(\overline{\mathbf{e}_j(a_1^{n-2m})}\Big|_{j=i+1}^m, x_1^i, a_t^{n-2m}, \mathbf{e}(a_1^{n-2m}), a_1^{t-1}), x_{i+1}^m) \overset{(1)}{=}$

$A(a_1^{n-2m}, \overline{\mathbf{e}_j(a_1^{n-2m})}\Big|_{j=1}^i, \overline{\mathbf{e}_j(a_1^{n-2m})}\Big|_{j=i+1}^m, x_1^i, x_{i+1}^m) =$

$A(a_1^{n-2m}, \overline{\mathbf{e}_j(a_1^{n-2m})}\Big|_{j=1}^m, x_1^m) \overset{1)}{=}$

$A(a_1^{n-2m}, \mathbf{e}(a_1^{n-2m}), x_1^m) \overset{(2)i=1}{=\!=} x_1^m;$

$o < i < m$. *[(2) and (1) from Th. 2.10.]* □

3. One generalization of an inverse operation in the group

3.1. Proposition *[19]: Let $(Q; A)$ be an $(n, m)-$groupoid and $n \geq 2m$. Further on, let the statements $(i) - (iii)$ from Prop. 2.4 hold. Then there is mapping $^{-1}$ set Q^{n-m} into the set Q^m such that the following laws*

$$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, x_1^m)) = x_1^m \text{ and}$$
$$A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = x_1^m$$

hold in the algebra $(Q; A, ^{-1})$.

Proof. Firstly we prove the following statements:

°1 The $< 1, 2n-2m+1 > -$associative law holds in $(Q; \overset{2}{A})$, where

$$\overset{2}{A}(x_1^{2n-m}) \overset{def}{=} A(A(x_1^n), x_{n+1}^{2n-m})$$

for all $x_1^{2n-m} \in Q$.

$^\circ 2$ For every $a_1^{2n-m} \in Q$ there is at least one $x_1^m \in Q^m$ such that the following equality holds

$$\overset{2}{A}(a_1^{2n-2m}, x_1^m) = a_{2n-2m+1}^{2n-m}.$$

$^\circ 3$ For every $a_1^{2n-m} \in Q$ there is at least one $y_1^m \in Q^m$ such that the following equality holds

$$\overset{2}{A}(y_1^m, a_1^{2n-2m}) = a_{2n-2m+1}^{2n-m}.$$

$^\circ 4$ $(Q; \overset{2}{A})$ has a $\{1, 2n-2m+1\}-$neutral operation.

Sketch of the proof of $^\circ 1$:

$$\overset{2}{A}(\overset{2}{A}(x_1^n, u_1^{n-2m}, v_1^m), y_{m+1}^{n-m}, y_{n-m+1}^n, y_{n+1}^{2n-m}) =$$
$$A(A(A(A(x_1^n), u_1^{n-2m}, v_1^m), y_{m+1}^{n-m}, y_{n-m+1}^n), y_{n+1}^{2n-m}) =$$
$$A(A(A(x_1^n), u_1^{n-2m}, v_1^m), y_{m+1}^{n-m}, A(y_{n-m+1}^{2n-m})) =$$
$$A(A(x_1^n), u_1^{n-2m}, A(v_1^m, y_{m+1}^{n-m}, A(y_{n-m+1}^{2n-m}))) =$$
$$A(A(x_1^n), u_1^{n-2m}, A(A(v_1^m, y_{m+1}^{n-m}, y_{n-m+1}^n), y_{n+1}^{2n-m})) =$$
$$\overset{2}{A}(x_1^n, u_1^{n-2m}, \overset{2}{A}(v_1^m, y_{m+1}^n, y_{n+1}^{2n-m})).$$

Sketch of the proof of $^\circ 2$:

$$\overset{2}{A}(a_1^{2n-2m}, x_1^m) = a_{2n-2m+1}^{2n-m} \Leftrightarrow$$
$$A(A(a_1^n), a_{n+1}^{2n-2m}, x_1^m) = a_{2n-2m+1}^{2n-m}.$$

Sketch of the proof of $^\circ 3$:

$$\overset{2}{A}(y_1^m, a_1^{2n-2m}) = a_{2n-2m+1}^{2n-m} \Leftrightarrow$$
$$A(y_1^m, a_1^{n-2m}, A(a_{n-2m+1}^{2n-2m})) = a_{2n-2m+1}^{2n-m}.$$

The proof of $^\circ 4$:

By $^\circ 1 - ^\circ 3$ and by Prop. 2.4, we conclude that the $(2n-m, m)-$groupoid $(Q; \overset{2}{A})$ has an $\{1, 2n-2m+1\}-$neutral operation (let it be denoted by) E.

In addition, let

$$(a_1^{n-2m}, b_1^m)^{-1} \overset{def}{=} \mathsf{E}(a_1^{n-2m}, b_1^m, a_1^{n-2m})$$

for all $a_1^{n-2m}, b_1^m \in Q$. Whence, by $^\circ 4$, we conclude that Prop. 3.1 holds. □

3.2. Proposition *[19]: Let $(Q; A)$ be an $(n, m)-$groupoid and $n \geq 2m$. Further on, let the statements $(i)-(iii)$ from Prop. 2.4 hold. Then there are mappings* $\mathbf{e}$ *and* $^{-1}$*, respectively, of the sets Q^{n-2m} and Q^{n-m} into the set Q^m such that the following laws*

$$A(b_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = \mathbf{e}(a_1^{n-2m}) \text{ and}$$
$$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, b_1^m) = \mathbf{e}(a_1^{n-2m})$$

hold in the algebra $(Q;A,^{-1},\mathbf{e})$.

Proof. By Prop. 2.4 and by Prop. 3.1. □

3.3. Theorem *[19]: Let* $(Q;A)$ *be an* $(n,m)-$*group and* $n \geq 2m$. *Then there are mappings* $\mathbf{e}$ *and* $^{-1}$, *respectively, of the sets* Q^{n-2m} *and* Q^{n-m} *into* Q^m *such that the following laws hold in the algebra* $(Q;A,^{-1},\mathbf{e})$

(2_L) $\quad A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m,$

(2_R) $\quad A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m,$

(3_L) $\quad A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, b_1^m) = \mathbf{e}(a_1^{n-2m}),$

(3_R) $\quad A(b_1^m, a_1^{n-2m}, (a_1^{n-2m} b_1^m)^{-1}) = \mathbf{e}(a_1^{n-2m}),$

(4_L) $\quad A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, x_1^m)) = x_1^m$ *and*

(4_R) $\quad A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = x_1^m.$

Proof. By Def. 1.1, Prop. 2.5, Prop. 3.1 and by Prop. 3.2. □

3.4. Remark: *The case* $m = 1$ *was described in [15]. For* $(n,m) = (2,1), a^{-1}$ *[*$= \mathsf{E}(a)$*] is the inverse element of the element a with respect to the neutral element* $\mathbf{e}(\emptyset)$ *of the group* $(Q;A)$. *Cf. III-1 in [23].*

4. AUXILIARY PROPOSITIONS

4.1. Proposition *[3]: Let* $(Q;A)$ *be an* $(n,m)-$*groupoid and* $n \geq m+2$. *Also, let the following statements hold:*

$(\widehat{|})$ $(Q;A)$ *is an* $(n,m)-$*semigroup;*

$(\widehat{||})$ *For every* $a_1^n \in Q$ *there is exactly one* $x_1^m \in Q^m$ *such that the following equality holds*

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n; \text{ and}$$

$(\widehat{|||})$ *For every* $a_1^n \in Q$ *there is exactly one* $y_1^m \in Q^m$ *such that the following equality holds*

$$A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

Then $(Q;A)$ *is an* $(n,m)-$*group.*

Sketch of the proof.

a) $A(a, a_1^{i-1}, x_1^m, a_i^{n-m-2}, b) = A(a, a_1^{i-1}, y_1^m, a_i^{n-m-2}, b)$ [2] $\Rightarrow$

$A(c_{i+1}^{n-m}, A(a, a_1^{i-1}, x_1^m, a_i^{n-m-2}, b), c_1^i) =$

$A(c_{i+1}^{n-m}, A(a, a_1^{i-1}, y_1^m, a_i^{n-m-2}, b), c_1^i) \overset{(\widehat{|})}{\Longrightarrow}$

$A(A(c_{i+1}^{n-m}, a, a_1^{i-1}, x_1^m), a_i^{n-m-2}, b, c_1^i) =$

[2] $i \in \{1, \ldots, n-m-1\}$.

$A(A(c_{i+1}^{n-m}, a, a_1^{i-1}, y_1^m), a_i^{n-m-2}, b, c_1^i) \overset{(\widehat{|||})}{\Longrightarrow}$

$A(c_{i+1}^{n-m}, a, a_1^{i-1}, x_1^m) = A(c_{i+1}^{n-m}, a, a_1^{i-1}, y_1^m) \overset{(\widehat{||})}{\Longrightarrow}$

$x_1^m = y_1^m.$

b) $A(a, a_i^{n-m-2}, x_1^m, a_1^{i-1}, b) = A(a, a_i^{n-m-2}, y_1^m, a_1^{i-1}, b) \Rightarrow$

$A(c_1^i, A(a, a_i^{n-m-2}, x_1^m, a_1^{i-1}, b), c_{i+1}^{n-m}) =$

$A(c_1^i, A(a, a_i^{n-m-2}, y_1^m, a_1^{i-1}, b), c_{i+1}^{n-m}) \overset{(\widehat{|})}{\Longrightarrow}$

$A(c_1^i, a, a_i^{n-m-2}, A(x_1^m, a_1^{i-1}, b, c_{i+1}^{n-m})) =$

$A(c_1^i, a, a_i^{n-m-2}, A(y_1^m, a_1^{i-1}, b, c_{i+1}^{n-m})) \overset{(\widehat{||})}{\Longrightarrow}$

$A(x_1^m, a_1^{i-1}, b, c_{i+1}^{n-m}) = A(y_1^m, a_1^{i-1}, b, c_{i+1}^{n-m}) \overset{(\widehat{|||})}{\Longrightarrow}$

$x_1^m = y_1^m.$

c) $A(a, a_1^{i-1}, x_1^m, a_i^{n-m-2}, b) = b_1^m \overset{b)}{\Longleftrightarrow}$

$A(c_{i+1}^{n-m}, A(a, a_1^{i-1}, x_1^m, a_i^{n-m-2}, b), c_1^i) = A(c_{i+1}^{n-m}, b_1^m, c_1^i) \overset{(\widehat{|})}{\Longleftrightarrow}$

$A(A(c_{i+1}^{n-m}, a, a_1^{i-1}, x_1^m), a_i^{n-m-2}, b, c_1^i) = A(c_{i+1}^{n-m}, b_1^m, c_1^i),$

where c_1^{n-m} is an arbitrary sequence over Q. □

4.2_1. **Proposition** *[19]: Let $n > m+1$ and let $(Q;A)$ be an $(n,m)-$groupoid. Also, let*

(a) The $<1,2>-$associative law holds in $(Q;A)$; and

(b) For every $a_1^{n-m} \in Q$ and for each $x_1^m, y_1^m \in Q^m$ the following implication holds

$A(x_1^m, a_1^{n-m}) = A(y_1^m, a_1^{n-m}) \Rightarrow x_1^m = y_1^m.$

Then $(Q;A)$ is an $(n,m)-$semigroup.

Sketch of the proof.

1) $i = 1:$ (a).

2) $i = s:$

$$A(a_1^{s-1}, A(a_s^{s+n-1}), a_{s+n}^{2n-m}) = A(a_1^s, A(a_{s+1}^{s+n}), a_{s+n+1}^{2n-m}).$$

3) $s \rightarrow s+1:$

$A(a_1^{s-1}, A(a_s^{s+n-1}), a_{s+n}^{2n-m}) = A(a_1^s, A(a_{s+1}^{s+n}), a_{s+n+1}^{2n-m}) \Rightarrow$

$A(b_1, A(a_1^{s-1}, A(a_s^{s+n-1}), a_{s+n}^{2n-m}), b_2^{n-m}) =$

$A(b_1, A(a_1^s, A(a_{s+1}^{s+n}), a_{s+n+1}^{2n-m}), b_2^{n-m}) \overset{(a)}{\Rightarrow}$

$A(A(b_1, a_1^{s-1}, A(a_s^{s+n-1}), a_{s+n}^{2n-m-1}), a_{2n-m}, b_2^{n-m}) =$

$$A(A(b_1, a_1^s, A(a_{s+1}^{s+n}), a_{s+n+1}^{2n-m-1}), a_{2n-m}, b_2^{n-m}) \overset{(b)}{\Rightarrow}$$
$$A(b_1, a_1^{s-1}, A(a_s^{s+n-1}), a_{s+n}^{2n-m-1}) = A(b_1, a_1^s, A(a_{s+1}^{s+n}), a_{s+n+1}^{2n-m-1}). \quad \Box$$

4.2$_2$. **Proposition** *[19]: Let $n > m+1$ and let $(Q;A)$ be an $(n,m)-$groupoid. Also, let*

($\overline{a}$) *The $< n-m, n-m+1 > -$associative law holds in $(Q;A)$; and*

($\overline{b}$) *For every $a_1^{n-m} \in Q$ and for each $x_1^m, y_1^m \in Q^m$ the following implication holds*

$$A(a_1^{n-m}, x_1^m) = A(a_1^{n-m}, y_1^m) \Rightarrow x_1^m = y_1^m.$$

Then $(Q;A)$ is an $(n,m)-$semigroup.

The sketch of a part of the proof.

$$A(a_1^{s-1}, A(a_s^{s+n-1}), a_{s+n}^{2n-m}) = A(a_1^s, A(a_{s+1}^{s+n}), a_{s+n+1}^{2n-m}) \Rightarrow$$
$$A(b_2^{n-m}, A(a_1^{s-1}, A(a_s^{s+n-1}), a_{s+n}^{2n-m}), b_1) =$$
$$A(b_2^{n-m}, A(a_1^s, A(a_{s+1}^{s+n}), a_{s+n+1}^{2n-m}), b_1) \overset{(\overline{a})}{\Rightarrow}$$
$$A(b_2^{n-m}, a_1, A(a_2^{s-1}, A(a_s^{s+n-1}), a_{s+n}^{2n-m}, b_1)) =$$
$$A(b_2^{n-m}, a_1, A(a_2^s, A(a_{s+1}^{s+n}), a_{s+n+1}^{2n-m}, b_1)) \overset{(\overline{b})}{\Rightarrow}$$
$$A(a_2^{s-1}, A(a_s^{s+n-1}), a_{s+n}^{2n-m}, b_1) = A(a_2^s, A(a_{s+1}^{s+n}), a_{s+n+1}^{2n-m}, b_1).$$

(Cf. the proof of Prop. 4.2$_1$. $\Box$

4.2$_3$. **Proposition** *[22]: Let $n \geq m+2$, $i \in \{2, \ldots, n-m\}$ and let $(Q;A)$ be an $(n,m)-$groupoid. Also, let*

(i) *The $< i, i+1 > -$associative law holds in $(Q;A)$;*

(ii) *The $< i-1, i > -$associative law holds in $(Q;A)$; and*

(iii) *For every $a_1^{n-m} \in Q$ and for each $x_1^m, y_1^m \in Q^m$ the following implication holds*

$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = A(a_1^{i-1}, y_1^m, a_i^{n-m}) \Rightarrow x_1^m = y_1^m.$$

Then $(Q;A)$ is an $(n,m)-$semigroup.

The sketch of a part of of the proof. 1) Let $n = m+2$ $(n-m=2,\ i=2)$. Then, by $(i), (ii)$ and by Def. 1.1 $-$ $(|)$, $(Q;A)$ is an $(n,m)-$semigroup.

2) $i < n-m$ $(i \in \{1,\dots,n-m\})$:

$$A(a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-m}) \overset{(i)}{=} A(a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{2n-m}) \Rightarrow$$

$$A(c_1^i, A(a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-m}), c_{i+1}^{n-m}) =$$

$$A(c_1^i, A(a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{2n-m}), c_{i+1}^{n-m}) \overset{(i)}{\Rightarrow}$$

$$A(c_1^{i-1}, A(c_i, a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-m-1}), a_{2n-m}, c_{i+1}^{n-m}) =$$

$$A(c_1^{i-1}, A(c_i, a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{2n-m-1}), a_{2n-m}, c_{i+1}^{n-m}) \overset{(iii)}{\Rightarrow}$$

$$A(c_i, a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-m-1}) = A(c_i, a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{2n-m-1}).$$

3) $i > 2$:

$$A(a_1^{i-2}, A(a_{i-1}^{i+n-2}), a_{i+n-1}^{2n-m}) \overset{(ii)}{=} A(a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-m}) \Rightarrow$$

$$A(c_1^{i-2}, A(a_1^{i-2}, A(a_{i-1}^{i+n-2}), a_{i+n-1}^{2n-m}), c_{i-1}^{n-m}) =$$

$$A(c_1^{i-2}, A(a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-m}), c_{i-1}^{n-m}) \overset{(ii)}{\Rightarrow}$$

$$A(c_1^{i-2}, a_1, A(a_2^{i-2}, A(a_{i-1}^{i+n-2}), a_{i+n-1}^{2n-m}, c_{i-1}), c_i^{n-m}) =$$

$$A(c_1^{i-2}, a_1, A(a_2^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-m}, c_{i-1}), c_i^{n-m}) \overset{(iii)}{\Rightarrow}$$

$$A(a_2^{i-2}, A(a_{i-1}^{i+n-2}), a_{i+n-1}^{2n-m}, c_{i-1}) = A(a_2^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-m}, c_{i-1}). \quad \square$$

4.3. **Definition:** *Let $(Q; A)$ be an $(n,m)-$groupoid; $n > m$. Then:*

(α) $\overset{1}{A} \overset{def}{=} A$; *and*

(β) *For every $s \in N$ and for every $x_1^{(s+1)(n-m)+m} \in Q$*

$$\overset{s+1}{A}(x_1^{(s+1)(n-m)+m}) \overset{def}{=} A(\overset{s}{A}(x_1^{s(n-m)+m}), x_{s(n-m)+m+1}^{(s+1)(n-m)+m}).$$

4.4. **Proposition:** *Let $(Q; A)$ be an $(n,m)-$semigroup, and $s \in N$. Then, for every $x_1^{(s+1)(n-m)+m} \in Q$ and for every $t \in \{1,\dots,s(n-m)+1\}$ the following equality holds*

$$\overset{s+1}{A}(x_1^{(s+1)(n-m)+m}) = \overset{s}{A}(x_1^{t-1}, A(x_t^{t+n-1}), x_{t+n}^{(s+1)(n-m)+m}).$$

Sketch of the proof. 1) $s = 1$: By Def. 1.1 $-$ (|) and by Def. 4.3, we have

$$\overset{1+1}{A}(x_1^{2(n-m)+m}) = \overset{1}{A}(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2(n-m)+m})$$

for every $x_1^{2(n-m)+m} \in Q$ and for every $i \in \{1,\dots,n-m+1\}$.

2) $s = v$: Let for every $x_1^{v(n-m)+m} \in Q$ and for all $t \in \{1,\dots,v(n-m)+1\}$ the following equality holds

$$\overset{v+1}{A}(x_1^{(v+1)(n-m)+m}) = \overset{v}{A}(x_1^{t-1}, A(x_t^{t+n-1}), x_{t+n}^{(v+1)(n-m)+m}).$$

3) $v \to v+1$:

$$\overset{(v+1)+1}{A}(x_1^{(v+2)(n-m)+m}) \overset{(\beta)}{=} A(\overset{v+1}{A}(x_1^{(v+1)(n-m)+m}), x_{(v+1)(n-m)+m+1}^{(v+2)(n-m)+m}) \overset{2)}{=}$$

$$A(\overset{v}{A}(x_1^{t-1}, A(x_t^{t+n-1}), x_{t+n}^{(v+1)(n-m)+m}), x_{(v+1)(n-m)+m+1}^{(v+2)(n-m)+m}) \overset{(\beta)}{=}$$

$$\overset{v+1}{A}(x_1^{t-1}, A(x_t^{t+n-1}), x_{t+n}^{(v+1)(n-m)+m}, x_{(v+1)(n-m)+m+1}^{(v+2)(n-m)+m}) \overset{2)}{=}$$

$$\overset{v}{A}(x_1^{t-1}, A(A(x_t^{t+n-1}), x_{t+n}^{t+2(n-m)+m-1}, x_{t+2(n-m)+m}^{(v+2)(n-m)+m}) \overset{1.1(|)}{=\!=}$$

$$\overset{v}{A}(x_1^{t-1}, A(x_t^{t+i-2}, A(x_{t+i-1}^{t+i+n-2}), x_{t+i+n-1}^{t+2(n-m)+m-1}), x_{t+2(n-m)+m}^{(v+2)(n-m)+m}) \overset{2)}{=}$$

$$\overset{v+1}{A}(x_1^{t-1}, x_t^{t+i-2}, A(x_{t+i-1}^{t+i+n-2}), x_{t+i+n-1}^{t+2(n-m)+m-1}), x_{t+2(n-m)+m}^{(v+2)(n-m)+m}) =$$

$$\overset{v+1}{A}(x_1^{t+i-2}, A(x_{t+i-1}^{t+i+n-2}), x_{t+i+n-1}^{(v+2)(n-m)+m}). \quad \square$$

By Def. 1.1 – (|), Def. 4.3 and Prop. 4.4, we obtain:

4.5. **Proposition** *[1]: Let $(Q; A)$ be an $(n,m)-$semigroup and $(i,j) \in N^2$. Then, for every $x_1^{(i+j)(n-m)+m} \in Q$ and for every $t \in \{1, \dots, i(n-m)+1\}$ the following equality holds*

$$\overset{i+j}{A}(x_1^{(i+j)(n-m)+m}) = \overset{i}{A}(x_1^{t-1}, \overset{j}{A}(x_t^{t+j(n-m)+m-1}), x_{t+j(n-m)+m}^{(i+j)(n-m)+m}).$$

By Prop. 4.5 and by Def. 1.1 – (|), we have:

4.6. **Proposition** *[1]: Let $(Q; A)$ be an $(n,m)-$semigroup and let $s \in N$. Then $(Q; \overset{s}{A})$ is an $(s(n-m)+m, m)-$semigroup.*

Remark: In [1] $\overset{s}{A}$ is written as $[\ ]_s$.

4.7. **Proposition** *[1]: Let $(Q; A)$ be an $(n,m)-$group, $n \geq 2m$ and let $s \in N$. Then $(Q; \overset{s}{A})$ is an $(s(n-m)+m, m)-$group.*

Sketch of the proof. Firstly we prove the following statements:

1° $(Q; \overset{s}{A})$ is an $(s(n-m)+m, m)-$semigroup.

2° For every $a_1^{s(n-m)+m} \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds

$$\overset{s}{A}(a_1^{s(n-m)}, x_1^m) = a_{s(n-m)+1}^{s(n-m)+m}.$$

3° For every $a_1^{s(n-m)+m} \in Q$ there is exactly one $y_1^m \in Q^m$ such that the following equality holds

$$\overset{s}{A}(y_1^m, a_1^{s(n-m)}) = a_{s(n-m)+1}^{s(n-m)+m}.$$

The proof of 1° : By Prop. 4.6.

Sketch of the proof of 2° :

$s \geq 2$:
$\overset{s}{A}(a_1^{s(n-m)}, x_1^m) = a_{s(n-m)+1}^{s(n-m)+m} \overset{4.3}{\Longleftrightarrow}$
$A(\overset{s-1}{A}(a_1^{(s-1)(n-m)+m}), a_{(s-1)(n-m)+m+1}^{s(n-m)}, x_1^m) = a_{s(n-m)+1}^{s(n-m)+m}$

Sketch of the proof of 3° :

$s \geq 2$:
$\overset{s}{A}(y_1^m, a_1^{s(n-m)}) = a_{s(n-m)+1}^{s(n-m)+m} \overset{4.5}{\Longleftrightarrow}$
$A(y_1^m, a_1^{n-2m}, \overset{s-1}{A}(a_{n-2m+1}^{s(n-m)})) = a_{s(n-m)+1}^{s(n-m)+m}$.

Finally, by 1° − 3° and by Prop. 4.1, we conclude that Prop. 4.7 holds. □

5. Some characterizations of (n, m)−groups

5.1_1. **Proposition** *[19]: Let $n \geq 2m$, $m \geq 2$ and let $(Q; A)$ be an (n, m)−groupoid. Then, $(Q; A)$ is an (n, m)−group iff there is a mapping $^{-1}$ of the set Q^{n-m} into the set Q^m such that the laws*

(1_L) $A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m})$,

(4_L) $A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, x_1^m)) = x_1^m$ *and*

(4_R) $A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = x_1^m$

hold in the algebra $(Q; A,^{-1})$.

Remark: For $m = 1$ see IX-1 in [23].

Proof. 1) ⇒: By Def. 1.1 and by Th. 3.3.

2) ⇐: Firstly we prove the following statements:

°1 For every $x_1^m, y_1^m \in Q^m$ and for every sequence a_1^{n-m} over Q the following implication holds

$A(x_1^m, a_1^{n-m}) = A(y_1^m, a_1^{n-m}) \Rightarrow x_1^m = y_1^m$.

°2 $(Q; A)$ is an (n, m)−semigroup.

°3 For every $x_1^m, y_1^m \in Q^m$ and for every sequence a_1^{n-m} over Q the following implication holds

$A(a_1^{n-m}, x_1^m) = A(a_1^{n-m}, y_1^m) \Rightarrow x_1^m = y_1^m$.

°4 For every $x_1^m, y_1^m, b_1^m, c_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following equivalences holds

$A(x_1^m, a_1^{n-2m}, b_1^m) = c_1^m \Leftrightarrow x_1^m = A(c_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1})$ and

$A(b_1^m, a_1^{n-2m}, y_1^m) = c_1^m \Leftrightarrow y_1^m = A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, c_1^m).$

Sketch of the proof of $^\circ 1$:

$A(x_1^m, a_1^{n-2m}, b_1^m) = A(y_1^m, a_1^{n-2m}, b_1^m) \Rightarrow$
$A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) =$
$A(A(y_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{(4_R)}{\Longrightarrow}$
$x_1^m = y_1^m.$

The proof of 2° : By $(1L)$, $^\circ 1$ and by Prop. 4.2_1.

Sketch of the proof of $^\circ 3$:

$A(b_1^m, a_1^{n-2m}, x_1^m) = A(b_1^m, a_1^{n-2m}, y_1^m) \Rightarrow$
$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, x_1^m)) =$
$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, y_1^m)) \overset{(4_L)}{\Longrightarrow}$
$x_1^m = y_1^m.$

Sketch of the proof of $^\circ 4$:

a) $A(x_1^m, a_1^{n-2m}, b_1^m) = c_1^m \overset{^\circ 1, mon.}{\Longleftrightarrow}$

$A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) =$

$A(c_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{(4_R)}{\Longleftrightarrow}$

$x_1^m = A(c_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}).$

b) $A(b_1^m, a_1^{n-2m}, y_1^m) = c_1^m \overset{^\circ 3}{\Longleftrightarrow}$

$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, y_1^m)) =$

$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, c_1^m)) \overset{(4_L)}{\Longleftrightarrow}$

$y_1^m = A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, c_1^m).$

Finally, by $^\circ 1 - ^\circ 4$ and by Prop. 4.1, we conclude that $(Q;A)$ is an $(n,m)-$group. Whence, by " $\Rightarrow$ ", we obtain Th. 5.1_1. □

Similarly, it is posible to prove also the following proposition:

5.1_2. **Theorem** *[19]: Let $n \geq 2m$, $m \geq 2$ and let $(Q;A)$ be an $(n,m)-$groupoid. Then, $(Q;A)$ is an $(n,m)-$group iff there is a mapping $^{-1}$ of the set Q^{n-m} into the set Q^m such that the laws*

(1_R) $A(x_1^{n-m-1}, A(x_{n-m}^{2n-m-1}), x_{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m})),$
(4_L) $A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, x_1^m)) = x_1^m$ *and*
(4_R) $A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = x_1^m$
hold in the algebra $(Q; A, ^{-1})$.

Remark: For $m = 1$ see IX-1 in [23].

5.2_1. **Theorem** *[21]: Let $(Q; A)$ be an $(n, m)-$groupoid, $m \geq 2$ and $n \geq 2m$. Then: $(Q; A)$ is an $(n, m)-$group iff the following statements hold:*

(1) *The $< 1, 2 > -$associative law holds in $(Q; A)$;*

(2) *The $< 1, n - m + 1 > -$associative law holds in $(Q; A)$;*

(3) *For every $a_1^n \in Q$ there is* **at least one** *$x_1^m \in Q^m$ such that the following equality*

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n$$

holds; and

(4) *For every $a_1^n \in Q$ there is* **at least one** *$y_1^m \in Q^m$ such that the following equality*

$$A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n$$

holds.

Remark: For $m = 1$ Prop. 5.1_1 is proved in [18]. See, also Chapter IX in [Ušan 2003]; 3.1–3.3.

Proof. *a)* $\Rightarrow$: By Def. 1.1.

b) $\Leftarrow$: Firstly we prove the following statement:

1° There is mapping $^{-1}$ of the set Q^{n-m} into the set Q^m such that the following laws hold in the algebra $(Q; A, ^{-1})$ *[*of the type $< n, n-1 >$*]*

(a) $A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, x_1^m)) = x_1^m$ and

(b) $A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = x_1^m$.

The proof of 1° : By (2)-(4) and by Prop. 3.1.

Finally, by (1), by 1° and by Th. 5.1_1, we conclude that $(Q; A)$ is an $(n, m)-$group. Whence, by " $\Rightarrow$ ", we obtain Th. 5.2_1. □

Similarly, it is posible to prove also the following proposition:

5.2_2. **Theorem** *[21]: Let $(Q; A)$ be an $(n, m)-$groupoid, $m \geq 2$ and $n \geq 2m$. Then: $(Q; A)$ is an $(n, m)-$group iff the following statements hold:*

$(\bar{1})$ *The $< n - m, n - m + 1 > -$associative law holds in $(Q; A)$;*

$(\bar{2})$ *The $< 1, n - m + 1 > -$associative law holds in $(Q; A)$;*

$(\bar{3})$ *For every $a_1^n \in Q$ there is* **at least one** *$x_1^m \in Q^m$ such that the following equality*

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n$$

holds; and

$(\overline{4})$ *For every* $a_1^n \in Q$ *there is* **at least one** $y_1^m \in Q^m$ *such that the following equality*

$$A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n$$

holds.

5.3_1. **Theorem** *[19]: Let* $n \geq 2m,\ m \geq 2$ *and let* $(Q; A)$ *be an* $(n, m)-$*groupoid. Then,* $(Q; A)$ *is an* $(n, m)-$*group iff there are mappings* $^{-1}$ *and* $\mathbf{e}$, *respectively, of the sets* Q^{n-m} *and* Q^{n-2m} *into the set* Q^m *such that the laws*

(1_L) $A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m})$,

(2_L) $A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m$ *and*

(4_R) $A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = x_1^m$

hold in the algebra $(Q; A,^{-1}, \mathbf{e})$.

Remark: For $m = 1$ Th. 5.3_1 is proved in [17]. Cf. Chapter III in [23].

Proof. 1) $\Rightarrow$: By Def. 1.1 and by Th.3.3.

2) $\Leftarrow$: Firstly we prove the following statements:

$\overset{\circ}{1}$ For every $x_1^m, y_1^m \in Q^m$ and for every sequence a_1^{n-m} over Q the following implication holds

$A(x_1^m, a_1^{n-m}) = A(y_1^m, a_1^{n-m}) \Rightarrow x_1^m = y_1^m$.

$\overset{\circ}{2}$ $(Q; A)$ is an $(n, m)-$semigroup.

$\overset{\circ}{3}$ Law

(3_R) $A(b_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = \mathbf{e}(a_1^{n-2m})$

holds in the algebra $(Q; A,^{-1}, \mathbf{e})$.

$\overset{\circ}{4}$ Law

(2_R) $A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m$

holds in the algebra $(Q; A,^{-1}, \mathbf{e})$.

$\overset{\circ}{5}$ Law

(3_L) $A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, b_1^m) = \mathbf{e}(a_1^{n-2m})$

holds in the algebra $(Q; A,^{-1}, \mathbf{e})$.

$\overset{\circ}{6}$ Law (4_L) holds in the algebra $(Q; A,^{-1}, \mathbf{e})$.

Sketch of the proof of $\overset{\circ}{1}$:

$A(x_1^m, a_1^{n-2m}, b_1^m) = A(y_1^m, a_1^{n-2m}, b_1^m) \Rightarrow$

$A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) =$

$A(A(y_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{(4_R)}{\Longrightarrow} x_1^m = y_1^m.$

Sketch of the proof of $\overset{\circ}{2}$: By (1_L), $\overset{\circ}{1}$ and by Prop. 4.2_1.

Sketch of the proof of $\overset{\circ}{3}$:

$A(A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{(4_R)}{=} \mathbf{e}(a_1^{n-2m}) \overset{(2_L)}{\Longrightarrow}$

$A(b_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = \mathbf{e}(a_1^{n-2m}).$

Sketch of the proof of $\overset{\circ}{4}$:

$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = y_1^m \Rightarrow$

$A(A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})), a_1^{n-2m}, b_1^m) =$

$A(y_1^m, a_1^{n-2m}, b_1^m) \overset{\overset{\circ}{2}}{\Longrightarrow}$ [3]

$A(x_1^m, a_1^{n-2m}, A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, b_1^m)) =$

$A(y_1^m, a_1^{n-2m}, b_1^m) \overset{(2_L)}{\Longrightarrow}$

$A(x_1^m, a_1^{n-2m}, b_1^m) =$

$A(y_1^m, a_1^{n-2m}, b_1^m) \overset{\overset{\circ}{1}}{\Longrightarrow} x_1^m = y_1^m.$

Sketch of the proof of $\overset{\circ}{5}$:

$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, b_1^m) = y_1^m \Rightarrow$

$A(A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) =$

$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{\overset{\circ}{2}}{\Longrightarrow}$ [4]

[3] $A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m}))$; (1M) in Th. 5.4_1.

[4] See footnote 3).

$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1})) =$

$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{\overset{\circ}{3}}{\Longrightarrow}$

$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) =$

$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{\overset{\circ}{4},(2L)}{\Longrightarrow}$

$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) =$

$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{\overset{\circ}{1}}{\Longrightarrow} y_1^m = \mathbf{e}(a_1^{n-2m}).$

Sketch of the proof of $\overset{\circ}{6}$:

$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, x_1^m)) \overset{\overset{\circ}{2}}{=}$ [5]

$A(A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, b_1^m), a_1^{n-2m}, x_1^m) \overset{\overset{\circ}{5}}{=}$

$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) \overset{(2_L)}{=} x_1^m.$

Finally, by $(1_L), (4_R), \overset{\circ}{6}$ and by Theorem 5.2_1, we conclude that $(Q; A)$ is an $(n, m)-$group. Whence, by " $\Rightarrow$ ", we obtain Th. 5.3_1. □

Similarly, it is posible to prove also the following proposition:

5.3_2. **Theorem** *[19]: Let $n \geq 2m$, $m \geq 2$ and let $(Q; A)$ be an $(n, m)-$groupoid. Then, $(Q; A)$ is an $(n, m)-$group iff there are mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-m} and Q^{n-2m} into the set Q^m such that the laws*

(1_R) $A(x_1^{n-m-1}, A(x_{n-m}^{2n-m-1}), x_{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m})),$

(2_R) $A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m$ *and*

(4_L) $A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, x_1^m)) = x_1^m$

hold in the algebra $(Q; A, ^{-1}, \mathbf{e})$.

Remark: For $m = 1$ Th. 5.3_2 is proved in [17]. Cf. III-3 in [23].

5.4_1. **Theorem** *[19]: Let $n \geq 2m$, $m \geq 2$ and let $(Q; A)$ be an $(n, m)-$groupoid. Then, $(Q; A)$ is $(n, m)-$group iff there are mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-m} and Q^{n-2m} into the set Q^m such that the laws*

(1_L) $A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m}),$

(1_M) $A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m})),$

(2_R) $A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m$ *and*

[5] See footnote 3).

(3_R) $A(b_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = \mathbf{e}(a_1^{n-2m})$

hold in the algebra $(Q; A,^{-1}, \mathbf{e})$.

Remark: The case $m = 1$ was described in [17]. Cf. III-3 in [23].

Proof. 1) $\Rightarrow$: By Def. 1.1 and by Th. 3.3.

2) $\Leftarrow$: Firstly we prove the following statements:

$\widehat{1}$ For every $x_1^m, y_1^m \in Q^m$ and for every sequence a_1^{n-m} over Q the following implication holds

$A(x_1^m, a_1^{n-m}) = A(y_1^m, a_1^{n-m}) \Rightarrow x_1^m = y_1^m$.

$\widehat{2}$ Law

(2_L) $\qquad A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m$

hold in the algebra $(Q; A,^{-1}, \mathbf{e})$.

$\widehat{3}$ Law

(3_L) $\qquad A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, b_1^m) = \mathbf{e}(a_1^{n-2m})$

hold in the algebra $(Q; A,^{-1}, \mathbf{e})$.

$\widehat{4}$ Laws (4_L) and (4_R) hold in the algebra $(Q; A,^{-1}, \mathbf{e})$.

Sketch of the proof of $\widehat{1}$:

$$A(x_1^m, a_1^{n-2m}, b_1^m) = A(y_1^m, a_1^{n-2m}, b_1^m) \Rightarrow$$

$$A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) =$$

$$A(A(y_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{(1_M)}{\Rightarrow}$$

$$A(x_1^m, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1})) =$$

$$A(y_1^m, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1})) \overset{(3_R)}{\Rightarrow}$$

$$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) =$$

$$A(y_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) \overset{(2_R)}{\Rightarrow} x_1^m = y_1^m.$$

Sketch of the proof of $\widehat{2}$:

$$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = y_1^m \Rightarrow$$

$$A(A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m), a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) =$$

$$A(y_1^m, a_1^{n-2m}, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) \overset{(1_M)}{\Rightarrow}$$

$$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, A(x_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1})) =$$

$$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) \overset{(3_R)}{\Rightarrow}$$

$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) =$

$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) \overset{(2_R)}{\Rightarrow}$

$\mathbf{e}(a_1^{n-2m}) = A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) \overset{(3_R)}{\Rightarrow}$

$A(x_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) =$

$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) \overset{\widehat{1}}{\Rightarrow} x_1^m = y_1^m.$

Sketch of the proof of $\widehat{3}$:

$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, b_1^m) = y_1^m \Rightarrow$

$A(A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) =$

$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{(1_M)}{\Rightarrow}$

$A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1})) =$

$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{(3_R)}{\Rightarrow}$

$A(a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) =$

$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{(2_R)\widehat{2}}{\Longrightarrow}$

$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) =$

$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{\widehat{1}}{\Longrightarrow} y_1^m = \mathbf{e}(a_1^{n-2m}).$

Sketch of the proof of $\widehat{4}$:

$a)$ $A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, x_1^m) \overset{(1_M)}{=}$

$A(A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, b_1^m), a_1^{n-2m}, x_1^m) \overset{\widehat{3}}{=}$

$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) \overset{\widehat{(2)}}{=} x_1^m.$

$b)$ $A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{(1_M)}{=}$

$A(x_1^m, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) \overset{(3_R)}{=}$

$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m}) \overset{(2_R)}{=} x_1^m.$

Finally, by $(1L), \widehat{4}$ and by Th. 5.1_1, we conclude that $(Q;A)$ is an $(n,m)-$group. Whence, by " $\Rightarrow$ ", we obtain Th. 5.4_1. □

Similarly, it is posible to proved also the following proposition:

5.4_2. **Theorem** *[19]: Let $n \geq 2m$, $m \geq 2$ and let $(Q;A)$ be an $(n,m)-$groupoid. Then, $(Q;A)$ is an $(n,m)-$group iff there are mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-m} and Q^{n-2m} into the set Q^m such that the laws*

(1_R) $A(x_1^{n-m-1}, A(x_{n-m}^{2n-m-1}), x_{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m}))$,

(1_M) $A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m}))$,

(2_L) $A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m$ *and*

(3_L) $A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, b_1^m) = \mathbf{e}(a_1^{n-2m})$

hold in the algebra $(Q; A,^{-1}, \mathbf{e})$.

Remark: The case $m = 1$ was described in [17]. Cf. III-3 in [23].

5.5_1. **Theorem** *[26]: Let $n \geq 3m$, $m \geq 2$ and let $(Q;A)$ be an $(n,m)-$groupoid. Then, $(Q;A)$ is an $(n,m)-$group iff there is a mapping $\mathbf{e}$ of the set Q^{n-2m} into the set Q^m such that the laws*

(1_L) $A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m})$,

(1_{Lm}) $A(A(a_1^m, b_1^{n-m}), c_1^m, d_1^{n-2m}) = A(a_1^m, A(b_1^{n-m}, c_1^m), d_1^{n-2m})$,

(2_L) $A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m$ *and*

(2_R) $A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m$

hold in the algebra $(Q; A, \mathbf{e})$.

Remarks: a) For $m = 1 :$ $(1L) = (1Lm)$. b) For $m = 1$ Th. 5.5_1 is proved in [17]. Cf. IX-2 in [23].

Proof. 1) $\Rightarrow$: By Def. 1.1 and by Th. 3.3.

2) $\Leftarrow$: Firstly we prove the following statements:

$\overline{1}$ For every $x_1^m, y_1^m, b_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following implication holds

$A(x_1^m, b_1^m, a_1^{n-2m}) = A(y_1^m, b_1^m, a_1^{n-2m}) \Rightarrow x_1^m = y_1^m$.

$\overline{2}$ $(Q;A)$ is an $(n,m)-$semigroup.

$\overline{3}$ For every $x_1^m, y_1^m, b_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following implication holds

$A(a_1^{n-m}, b_1^m, x_1^m) = A(a_1^{n-m}, b_1^m, y_1^m) \Rightarrow x_1^m = y_1^m$.

$\overline{4}$ For every $a_1^n \in Q$ there is exactly one sequence x_1^m over Q and exactly one sequence y_1^m over Q such that the following equalities hold

$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n$ and $A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n$.

Sketch of the proof of $\overline{1}$:

$A(x_1^m, b_1^m, a_1^{n-2m}) = A(y_1^m, b_1^m, a_1^{n-2m}) \overset{n \geq 3m}{\Longrightarrow}$

$A(A(x_1^m, b_1^m, a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) =$
$A(A(y_1^m, b_1^m, a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) \overset{(1Lm)}{\Longrightarrow}$
$A(x_1^m, A(b_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) =$
$A(y_1^m, A(b_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) \overset{(2R)}{\Longrightarrow}$
$A(x_1^m, b_1^m, c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) =$
$A(x_1^m, b_1^m, c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) \overset{(2R)}{\Longrightarrow} x_1^m = y_1^m.$

The proof of $\overline{2}$: By $\overline{1}, (1_L)$ and by Prop. 4.2_1.

Sketch of the proof of $\overline{3}$:

$A(a_1^{n-2m}, b_1^m, x_1^m) = A(a_1^{n-2m}, b_1^m, y_1^m) \overset{n \geq 3m}{\Longrightarrow}$
$A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, \mathbf{e}(a_1^{n-2m}), A(a_1^{n-2m}, b_1^m, x_1^m)) =$
$A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, \mathbf{e}(a_1^{n-2m}), A(a_1^{n-2m}, b_1^m, y_1^m)) \overset{\overline{2}}{\Longrightarrow}$
$A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, b_1^m), x_1^m)) =$
$A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, b_1^m), y_1^m)) \overset{(2L)}{\Longrightarrow}$
$A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, b_1^m, x_1^m) =$
$A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, b_1^m, y_1^m) \overset{(2L)}{\Longrightarrow} x_1^m = y_1^m.$

Sketch of the proof of $\overline{4}$:

a) $A(a_1^{n-2m}, b_1^m, x_1^m) = d_1^m \overset{\overline{3}}{\Longleftrightarrow}$ [6]
$A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, \mathbf{e}(a_1^{n-2m}), A(a_1^{n-2m}, b_1^m, x_1^m)) =$
$A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, \mathbf{e}(a_1^{n-2m}), d_1^m) \overset{\overline{2},(2L)}{\Longleftrightarrow}$
$x_1^m = A(\mathbf{e}(c_1^{n-3m}, b_1^m), c_1^{n-3m}, \mathbf{e}(a_1^{n-2m}), d_1^m).$

b) $A(y_1^m, b_1^m, a_1^{n-2m}) = d_1^m \overset{\overline{1}}{\Longleftrightarrow}$
$A(A(y_1^m, b_1^m, a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) =$
$A(d_1^m, \mathbf{e}(a_1^{n-2m}), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})) \overset{\overline{2},(2R)}{\Longleftrightarrow}$
$y_1^m = A(d_1^m, \mathbf{e}(a_1^{n-2m}), c_1^{n-3m}, \mathbf{e}(b_1^m, c_1^{n-3m})).$

c) By *a)* and $\overline{1}$ and by *b)* and $\overline{3}$, we obtain $\overline{4}$.

Finally, by $\overline{2}, \overline{4}$ and by Prop. 4.1, we conclude that $(Q; A)$ is an $(n, m)-$group. Whence, by " $\Rightarrow$ ", we obtain Th 5.5_1. □

Similarly, one could prove also the following proposition:

5.5_2. **Theorem** *[26]: Let $n \geq 3m$, $m \geq 2$ and let $(Q; A)$ be an $(n, m)-$groupoid. Then, $(Q; A)$ is an $(n, m)-$group iff there is a mapping* $\mathbf{e}$ *of the set Q^{n-2m} into the set Q^m such that the laws*

[6] $\overset{\overline{3}}{\Leftarrow}$. $\Rightarrow$: monotony.

(1_R) $A(x_1^{n-m-1}, A(x_{n-m}^{2n-m-1}), x_{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m}))$,
(1_{Rm}) $A(a_1^{n-2m}, A(b_1^m, c_1^{n-m}), d_1^m) = A(a_1^{n-2m}, b_1^m, A(c_1^{n-m}, d_1^m))$,
(2_L) $A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m$ *and*
(2_R) $A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m$
hold in the algebra $(Q; A, \mathbf{e})$.

Remarks: a) For $m = 1$: $(1R) = (1Rm)$. b) For $m = 1$ Th. 5.5_2 is proved in [17]. Cf. IX-2 in [23].

5.6_1. **Theorem** *[19]: Let $n \geq 3m$, $m \geq 2$ and let $(Q; A)$ be an $(n,m)-$groupoid. Then, $(Q; A)$ is an $(n,m)-$group iff there are mappings $^{-1}$ and* $\mathbf{e}$, *respectively, of the sets Q^{n-m} and Q^{n-2m} into the set Q^m such that the laws*
(1_L) $A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m})$,
(1_{Lm}) $A(A(a_1^m, b_1^{n-m}), c_1^m, d_1^{n-2m}) = A(a_1^m, A(b_1^{n-m}, c_1^m), d_1^{n-2m})$,
(2_R) $A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m$
(3_R) $A(b_1^m, a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = \mathbf{e}(a_1^{n-2m})$
hold in the algebra $(Q; A, ^{-1}, \mathbf{e})$.

Remarks: a) For $m = 1$: $(1Lm) = (1L)$. b) For $m = 1$ see III-3 in [23].

Proof. 1) $\Rightarrow$: By Def. 1.1 and by Th. 3.3.

2) $\Leftarrow$: Firstly we prove the following statements:

$\bar{\bar{1}}$ For every $x_1^m, y_1^m, b_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following implication holds

$A(x_1^m, b_1^m, a_1^{n-2m}) = A(y_1^m, b_1^m, a_1^{n-2m}) \Rightarrow x_1^m = y_1^m$.

$\bar{\bar{2}}$ $(Q; A)$ is an $(n,m)-$semigroup.

$\bar{\bar{3}}$ Law

(2_L) $A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m$

hold in the algebra $(Q; A, ^{-1}, \mathbf{e})$.

Sketch of the proof of $\bar{\bar{1}}$: Sketch of the proof of $\bar{1}$ in the proof of Th. 5.5_1.

The proof of $\bar{\bar{2}}$: By $\bar{\bar{1}}, (1L)$ and by Prop. 4.2_1.

Sketch of the proof of $\bar{\bar{3}}$:

$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = y_1^m \Rightarrow$
$A(A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m), a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) =$
$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) \overset{\bar{\bar{2}}}{\Longrightarrow}$
$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, A(x_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1})) =$
$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) \overset{(3R)}{\Longrightarrow}$

$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) =$
$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1})\overset{(2R)}{\Longrightarrow}$
$\mathbf{e}(a_1^{n-2m}) =$
$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1})\overset{(3R)}{\Longrightarrow}$
$A(x_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) =$
$A(y_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1})\overset{\overline{\overline{1}}}{\Longrightarrow} x_1^m = y_1^m.$
*[*Cf. the proof of Th. 5.4_1.*]*

Finally, by $(1L), (1Lm), (2R), \overline{\overline{3}}$ and by Th. 5.5_1, we conclude that $(Q; A)$ is an $(n,m)-$group. Whence, by " $\Rightarrow$ ", we obtain Th. 5.6_1. □

Similarly, it is posible to prove also the following proposition:

5.6_2. **Theorem** *[19]: Let $n \geq 3m,\ m \geq 2$ and let $(Q; A)$ be an $(n,m)-$groupoid. Then, $(Q; A)$ is an $(n,m)-$group iff there are mappings $^{-1}$ and* **e**, *respectively, of the sets Q^{n-m} and Q^{n-2m} into the set Q^m such that the laws*

$(1_R)\quad A(x_1^{n-m-1}, A(x_{n-m}^{2n-m-1}), x_{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m})),$
$(1_{Rm})\quad A(a_1^{n-2m}, A(b_1^m, c_1^{n-m}), d_1^m) = A(a_1^{n-2m}, b_1^m, A(c_1^{n-m}, d_1^m)),$
$(2_L)\quad A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m$ *and*
$(3_L)\quad A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, b_1^m) = \mathbf{e}(a_1^{n-2m})$
hold in the algebra $(Q; A,^{-1}, \mathbf{e})$.

Remarks: a) For $m = 1:\ (1Rm) = (1R)$. b) For $m = 1$ see III-3 in [23].

5.7. **Theorem** *[22]: Let $n \geq 3m,\ m \geq 2$ and let $(Q; A)$ be an $(n,m)-$groupoid. Then, $(Q; A)$ is an $(n,m)-$group iff there is $i \in \{m+1, \ldots, n-2m+1\}$ such that the following statements hold:*

(a) The $< i-1, i > -$associative law holds in $(Q; A)$;

(b) The $< i, i+1 > -$associative law holds in $(Q; A)$; and

(c) For every $a_1^n \in Q$ there is **exactly one** *$x_1^m \in Q^m$ such that the following equality holds*

$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n.$$

Remark: For $m = 1$ Th. 5.7 is proved in [20]. Cf. IX-3 in [23].

Proof. 1) $(c) \Leftrightarrow (c_1) \wedge (c_2)$, where

(c_1) For every $a_1^{n-m}, x_1^m, y_1^m \in Q$ the implication holds

$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = A(a_1^{i-1}, y_1^m, a_i^{n-m}) \Rightarrow x_1^m = y_1^m; \text{ and}$$

(c_2) For every $a_1^n \in Q$ there is at least one $x_1^m \in Q^m$ such that the following equality holds $A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n$.

2) $\Rightarrow$: By Def. 1.1.

3) $\Leftarrow$: Firstly we prove the following statements:

$1'$ $(Q;A)$ is an $(n,m)-$semigroup.

$2'$ For every $a_1^n \in Q$ there is at least one $x_1^m \in Q^m$ such that the following equality holds

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n.$$

$3'$ For every $a_1^n \in Q$ there is at least one $y_1^m \in Q^m$ such that the following equality holds

$$A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

The proof of $1'$: By $(a), (b), (c_1)$ and by Prop. 4.2_3.

Sketch of the proof of $2'$:

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n \overset{(c_1)}{\Longleftrightarrow} {}^{7}$$
$$A(c_1^{i-1}, A(a_1^{n-m}, x_1^m), c_i^{n-m}) = A(c_1^{i-1}, a_{n-m+1}^n, c_i^{n-m}) \overset{1'}{\Longleftrightarrow}$$
$$A(c_1^{i-1-m}, A(c_{i-m}^{i-1}, a_1^{n-m}), x_1^m, c_i^{n-m}) = A(c_1^{i-1}, a_{n-m+1}^n, c_i^{n-m}),$$

i. e. that

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n \Longleftrightarrow$$
$$A(c_1^{i-1-m}, A(c_{i-m}^{i-1}, a_1^{n-m}), x_1^m, c_i^{n-m}) = A(c_1^{i-1}, a_{n-m+1}^n, c_i^{n-m}),$$

where c_1^{n-m} is an arbitrary sequence over Q.

Whence, by (c), we conclude that the statement $2'$ holds.

Remark: Since $n \geq 3m$ and $i \in \{m+1, \ldots, n-2m+1\}$, we have $|c_1^{i-1}| \geq m$ and $|c_i^{n-m}| \geq m$.

Similarly, it is possible that the statement $3'$ holds.

Finally, by $1' - 3'$ and Th. 5.2_1 (or Th. 5.2_2), we conclude that $(Q;A)$ is an $(n,m)-$group. Whence, by " $\Rightarrow$ ", we obtain Th. 5.7. □

5.8_1. **Theorem** *[27]: Let $n \geq 3m$, $m \geq 2$ and let $(Q;A)$ be an $(n,m)-$groupoid. Then, $(Q;A)$ is an $(n,m)-$group iff there is an mapping E of the set Q^{n-2m} into the set Q^m such that the laws*

(1_L) $A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m}),$

(1_{Lm}) $A(A(a_1^m, b_1^{n-m}), c_1^m, d_1^{n-2m}) = A(a_1^m, A(b_1^{n-m}, c_1^m), d_1^{n-2m}),$

$(\widehat{2_L})$ $A(a_1^{n-2m}, \mathsf{E}(a_1^{n-2m}), x_1^m) = x_1^m$ *and*

(2_R) $A(x_1^m, a_1^{n-2m}, \mathsf{E}(a_1^{n-2m})) = x_1^m$

hold in the algebra $(Q;A,\mathsf{E})$.

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[7] $\overset{(c_1)}{\Longleftarrow}$. $\Rightarrow$: monotony.

Remark: For $m=1$: $(1Lm)=(1L)$. The case $m=1$ is described in [7]. See, also XII-1 in [23].

Proof. 1) $\Rightarrow$: By Def. 1.1, Th. 3.3 and by Th. 2.10.

2) $\Leftarrow$: Firstly we prove the following statements:

1" For every $x_1^m, y_1^m, b_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following implication holds

$A(x_1^m, b_1^m, a_1^{n-2m}) = A(y_1^m, b_1^m, a_1^{n-2m}) \Rightarrow x_1^m = y_1^m$.

2" $(Q;A)$ is an $(n,m)-$semigroup.

3" $(\forall b_1^m \in Q^m)(\forall c_i \in Q)_1^{n-3m} b_1^m = \mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}))$.

4" For every $x_1^m, y_1^m, b_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following implication

$A(b_1^m, x_1^m, a_1^{n-2m}) = A(b_1^m, y_1^m, a_1^{n-2m}) \Rightarrow x_1^m = y_1^m$

holds.

5" For every $x_1^m, y_1^m, b_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following implication

$A(a_1^{n-2m}, x_1^m, b_1^m) = A(a_1^{n-2m}, y_1^m, b_1^m) \Rightarrow x_1^m = y_1^m$

holds.

6" For every $x_1^m, b_1^m, d_1^m \in Q^m$ and for every sequence a_1^{n-2m} over Q the following equivalence holds

$A(b_1^m, x_1^m, a_1^{n-2m}) = d_1^m \Leftrightarrow$
$x_1^m = A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), d_1^m, \mathsf{E}(a_1^{n-2m}))$,
where c_1^{n-3m} arbitrary sequence over Q.

Sketch of the proof of 1" :

$A(x_1^m, b_1^m, a_1^{n-2m}) = A(y_1^m, b_1^m, a_1^{n-2m}) \overset{n\geq 3m}{\Longrightarrow}$
$A(A(x_1^m, b_1^m, a_1^{n-2m}), \mathsf{E}(a_1^{n-2m}), c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})) =$
$A(A(y_1^m, b_1^m, a_1^{n-2m}), \mathsf{E}(a_1^{n-2m}), c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})) \overset{(1Lm)}{\Longrightarrow}$
$A(x_1^m, A(b_1^m, a_1^{n-2m}, \mathsf{E}(a_1^{n-2m})), c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})) =$
$A(y_1^m, A(b_1^m, a_1^{n-2m}, \mathsf{E}(a_1^{n-2m})), c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})) \overset{(2R)}{\Longrightarrow}$
$A(x_1^m, b_1^m, c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})) =$
$A(y_1^m, b_1^m, c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})) \overset{(2R)}{\Longrightarrow} x_1^m = y_1^m$.

The proof of 2" : By 1", $(1L)$ and by Prop. 4.2_1.

Sketch of the proof of 3" :

$A(b_1^m, c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), \mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}))) \overset{(\widehat{2L})}{=}$

$$\mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})),$$
$A(b_1^m, c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), \mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})))\overset{(2R)}{=} b_1^m.$

Sketch of the proof of 4" :

$A(b_1^m, x_1^m, a_1^{n-2m}) = A(b_1^m, y_1^m, a_1^{n-2m})\overset{n\geq 3m}{\Longrightarrow}$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), A(b_1^m, x_1^m, a_1^{n-2m}), \mathsf{E}(a_1^{n-2m})) =$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), A(b_1^m, y_1^m, a_1^{n-2m}), \mathsf{E}(a_1^{n-2m}))\overset{2"}{\Longrightarrow}$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), b_1^m, A(x_1^m, a_1^{n-2m}, \mathsf{E}(a_1^{n-2m}))) =$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), b_1^m, A(y_1^m, a_1^{n-2m}, \mathsf{E}(a_1^{n-2m})))\overset{2R}{\Longrightarrow}$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), b_1^m, x_1^m) =$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), b_1^m, y_1^m)\overset{3"}{\Longrightarrow}$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), \mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})), x_1^m) =$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), \mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})), y_1^m)\overset{\widehat{2L}}{\Longrightarrow}$
$x_1^m = y_1^m.$

Sketch of the proof of 5" :

$A(a_1^{n-2m}, x_1^m, b_1^m) = A(a_1^{n-2m}, y_1^m, b_1^m)\overset{n\geq 3m}{\Longrightarrow}$
$A(c_1^{2m}, A(a_1^{n-2m}, x_1^m, b_1^m), d_1^{n-3m}) =$
$A(c_1^{2m}, A(a_1^{n-2m}, y_1^m, b_1^m), d_1^{n-3m})\overset{2"}{\Longrightarrow}$
$A(A(c_1^{2m}, a_1^{n-2m}), x_1^m, b_1^m, d_1^{n-3m}) =$
$A(A(c_1^{2m}, a_1^{n-2m}), y_1^m, b_1^m, d_1^{n-3m})\overset{4"}{\Longrightarrow} x_1^m = y_1^m.$

Sketch of the proof of 6" :

$A(b_1^m, x_1^m, a_1^{n-2m}) = d_1^m \overset{5"}{\Longleftrightarrow}$[8]
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), A(b_1^m, x_1^m, a_1^{n-2m}), \mathsf{E}(a_1^{n-2m})) =$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), d_1^m, \mathsf{E}(a_1^{n-2m}))\overset{2"}{\Longleftrightarrow}$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), b_1^m, A(x_1^m, a_1^{n-2m}, \mathsf{E}(a_1^{n-2m}))) =$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), d_1^m, \mathsf{E}(a_1^{n-2m}))\overset{(2R)}{\Longleftrightarrow}$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), b_1^m, x_1^m) =$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), d_1^m, \mathsf{E}(a_1^{n-2m}))\overset{3"}{\Longleftrightarrow}$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), \mathsf{E}(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m})), x_1^m) =$
$A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), d_1^m, \mathsf{E}(a_1^{n-2m}))\overset{(\widehat{2L})}{\Longleftrightarrow}$
$x_1^m = A(c_1^{n-3m}, \mathsf{E}(b_1^m, c_1^{n-3m}), d_1^m, \mathsf{E}(a_1^{n-2m})).$

Finally, by 2", 4", 6" and by Th. 5.7, we conclude that $(Q; A)$ is an $(n, m)-$group. Whence, by " $\Rightarrow$ ", we obtain Th. 5.8_1 □

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[8] $\overset{5"}{\Longleftarrow}$. $\Rightarrow$: monotony.

Similarly, one could prove also the following proposition:

5.8$_2$. **Theorem** *[27]: Let $n \geq 3m,\ m \geq 2$ and let $(Q;A)$ be an (n,m)−groupoid. Then, $(Q;A)$ is an (n,m)−group iff there is an mapping* E *of the set Q^{n-2m} into the set Q^m such that the laws*

$(1_R)\ \ A(x_1^{n-m-1}, A(x_{n-m}^{2n-m-1}), x_{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m})),$

$(1_{Rm})\ \ A(a_1^{n-2m}, A(b_1^m, c_1^{n-m}), d_1^m) = A(a_1^{n-2m}, b_1^m, A(c_1^{n-m}, d_1^m)),$

$(2_L)\ \ A(\mathsf{E}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m$ *and*

$(\widehat{2_R})\ \ A(x_1^m, \mathsf{E}(a_1^{n-2m}), a_1^{n-2m}) = x_1^m$

hold in the algebra $(Q;A,\mathsf{E})$.

Remark: For $m = 1:\ (1Rm) = (1R)$. The case $m = 1$ is described in [7]. See, also XII-1 in [23].

6. About (km,m)−groups for $k > 2$ and $m \geq 2$

6.1. Theorem *[24]: Let $k > 2,\ m \geq 2, n = k \cdot m,\ (Q;A)\ (n,m)$−group and* $\mathbf{e}$ *its $\{1, n-m+1\}$−neutral operation. Also let there exist a sequence a_1^{n-2m} over Q such that for all $i \in \{0,1,\ldots,2m-1\}$, and for every $x_1^{2m} \in Q$ the following equality holds*

$(0)\ \ A(x_1^i, a_1^{n-2m}, x_{i+1}^{2m}) = A(x_1^{2m}, a_1^{n-2m}).$

Further on, let

$(1)\ \ B(x_1^{2m}) \overset{def}{=} A(x_1^m, a_1^{n-2m}, x_{m+1}^{2m})$ *and*

$(2)\ \ c_1^m \overset{def}{=} A(\overline{\mathbf{e}(a_1^{n-2m})}^{k}\big|)$

for all $x_1^{2m} \in Q$. Then the following statements hold

(i) $(Q;B)$ is a $(2m,m)$−group;

(ii) For all $x_1^{k\cdot m} \in Q$

$$A(x_1^{k\cdot m}) = \overset{k}{B}(x_1^{k\cdot m}, c_1^m);\ \text{and}$$

(iii) For all $j \in \{0,\ldots,m-1\}$ and for every $x_1^m \in Q$ the following equality holds

$$B(x_1^j, c_1^m, x_{j+1}^m) = B(x_1^m, c_1^m).$$

Proof. Firstly we prove the following statements:

1° For all $x_1^{3m} \in Q$ the following equality holds

$$B(B(x_1^{2m}), x_{2m+1}^{3m}) = B(x_1, B(x_2^{2m+1}), x_{2m+2}^{3m}).$$

2° For all $b_1^{2m} \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds

$$B(x_1^m, b_1^m) = b_{m+1}^{2m}.$$

3° $(Q; B)$ is a $(2m, m)-$semigroup.

4° For all $b_1^{2m} \in Q$ there is exactly one $y_1^m \in Q^m$ such that the following equality holds

$$B(b_1^m, y_1^m) = b_{m+1}^{2m}.$$

Sketch of the proof of 1° :

$$B(B(x_1^{2m}), x_{2m+1}^{3m}) \overset{(1)}{=}$$
$$A(A(x_1^m, a_1^{n-2m}, x_{m+1}^{2m}), a_1^{n-2m}, x_{2m+1}^{3m}) \overset{(0)}{=}$$
$$A(A(x_1^m, a_1^{n-2m}, x_{m+1}^{2m}), x_{2m+1}, a_1^{n-2m}, x_{2m+2}^{3m}) \overset{1.1(I)}{=}$$
$$A(x_1, A(x_2^m, a_1^{n-2m}, x_{m+1}^{2m}, x_{2m+1}), a_1^{n-2m}, x_{2m+2}^{3m}) \overset{(0)(1)}{=}$$
$$B(x_1, B(x_2^{2m+1}), x_{2m+2}^{3m}).$$

Sketch of the proof of 2° :

$$B(x_1^m, b_1^m) = b_{m+1}^{2m} \overset{(1)}{\Leftrightarrow}$$
$$A(x_1^m, a_1^{n-2m}, b_1^m) = b_{m+1}^{2m},$$

whence, by Def. 1.1-(II), we obtain 2°.

Sketch of the proof of 3° :

By 2° and by Prop. 2.1.

Sketch of the proof of 4° :

$$B(b_1^m, x_1^m) = b_{m+1}^{2m} \overset{(1)}{\Leftrightarrow}$$
$$A(b_1^m, a_1^{n-2m}, y_1^m) = b_{m+1}^{2m},$$

whence, by Def. 1.1-(II), we have 4°.

The proof of (i) :

By $2°, 3°, 4°$ and by Prop.2.2.

Sketch of the proof of (ii) *[*to the case $k = 4$*]*:

$$A(x_1^m, y_1^m, z_1^m, u_1^m) \overset{2.5}{=}$$
$$A(x_1^m, y_1^m, z_1^m, A(u_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m}))) \overset{1.1(I)}{=}$$
$$A(x_1^m, y_1^m, A(z_1^m, u_1^m, a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{(0)}{=}$$
$$A(x_1^m, y_1^m, A(z_1^m, a_1^{n-2m}, u_1^m), \mathbf{e}(a_1^{n-2m})) \overset{(1)}{=}$$
$$A(x_1^m, y_1^m, B(z_1^m, u_1^m), \mathbf{e}(a_1^{n-2m})) \overset{2.5}{=}$$
$$A(x_1^m, y_1^m, A(B(z_1^m, u_1^m), a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})), \mathbf{e}(a_1^{n-2m})) \overset{1.1(I)}{=}$$

$A(x_1^m, A(y_1^m, B(z_1^m, u_1^m), a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{(0)}{=}$
$A(x_1^m, A(y_1^m, a_1^{n-2m}, B(z_1^m, u_1^m)), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{(1)}{=}$
$A(x_1^m, B(y_1^m, B(z_1^m, u_1^m)), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{2.5}{=}$
$A(x_1^m, A(B(y_1^m, B(z_1^m, u_1^m)), a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{1.1(I)}{=}$
$A(A(x_1^m, B(y_1^m, B(z_1^m, u_1^m)), a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{(0)}{=}$
$A(A(x_1^m, a_1^{n-2m}, B(y_1^m, B(z_1^m, u_1^m))), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{(1)}{=}$
$A(B(x_1^m, B(y_1^m, B(z_1^m, u_1^m)), \mathbf{e}(a_1^{n-2m}), (\overset{2}{\overline{\mathbf{e}(a_1^{n-2m})}}\big|) \overset{4.4}{=}$
$A(\overset{3}{B}(x_1^m, y_1^m, z_1^m, u_1^m)), A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, \mathbf{e}(a_1^{n-2m})), (\overset{2}{\overline{\mathbf{e}(a_1^{n-2m})}}\big|) \overset{(0)}{=}$
$A(\overset{3}{B}(x_1^m, y_1^m, z_1^m, u_1^m), A(a_1^{n-2m}, \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})), \overset{2}{\overline{\mathbf{e}(a_1^{n-2m})}}\big|) \overset{1.1(I)}{=}$
$A(\overset{3}{B}(x_1^m, y_1^m, z_1^m, u_1^m), a_1^{n-2m}, A(\overset{4}{\overline{\mathbf{e}(a_1^{n-2m})}}\big|)) \overset{(1)}{=}$
$B(\overset{3}{B}(x_1^m, y_1^m, z_1^m, u_1^m), A(\overset{4}{\overline{\mathbf{e}(a_1^{n-2m})}}\big|)) \overset{(2)}{=}$
$B(\overset{3}{B}(x_1^m, y_1^m, z_1^m, u_1^m), c_1^m) \overset{4.3}{=}$
$\overset{4}{B}(x_1^m, y_1^m, z_1^m, u_1^m, c_1^m).$

Sketch of a part of the proof of (iii) :

By (ii) and by

$$A(A(x_1^{k\cdot m}), x_{k\cdot m+1}^{2km-m}) = A(x_1, A(x_2^{k\cdot m+1}), x_{k\cdot m+2}^{2km-m}),$$

we have
$\overset{k}{B}(x_1, \overset{k}{B}(x_2^{k\cdot m}, c_1^m, x_{k\cdot m+1}), x_{k\cdot m+2}^{2km-m}, c_1^m) =$
$\overset{k}{B}(x_1, \overset{k}{B}(x_2^{k\cdot m}, c_1^m), x_{k\cdot m+2}^{2km-m}, c_1^m),$
and by Def.1.1-(II), we have
$\overset{k}{B}(x_2^{k\cdot m}, c_1^m, x_{k\cdot m+1}) = \overset{k}{B}(x_2^{k\cdot m+1}, c_1^m),$
i.e., by Prop. 4.4,
$\overset{k-1}{B}(x_2^{(k-1)\cdot m+1}, B(x_{(k-1)\cdot m+2}^{k\cdot m}, c_1^m, x_{k\cdot m+1})) =$
$\overset{k-1}{B}(x_2^{(k-1)\cdot m+1}, B(x_{(k-1)\cdot m+2}^{k\cdot m+1}, c_1^m)).$
Finaly, hence we obtain

$$B(x_{(k-1)\cdot m+2}^{k\cdot m}, c_1^m, x_{k\cdot m+1}) = B(x_{(k-1)\cdot m+2}^{k\cdot m+1}, c_1^m),$$

i.e., we obtain (iii) for $j = m - 1$. $\square$

6.2. Theorem *[24]: Let $m \geq 2$, $(Q;B)$ be a $(2m,m)-$group, and let $\overset{m}{e} \in Q^m$ its neutral element (cf. Prop. 2.9). Also let c_1^m be an element of the set Q^m such that for every $i \in \{0,1,\ldots,m-1\}$ and for every $x_1^m \in Q^m$ the following equality holds*

(a) $B(x_1^i, c_1^m, x_{i+1}^m) = B(x_1^m, c_1^m)$

(cf. Prop. 2.6 and Prop. 2.7). Further on, let $k > 2$ and

(b) $A(x_1^{k\cdot m}) = \overset{k}{B}(x_1^{k\cdot m}, c_1^m)$

for all $x_1^{k\cdot m} \in Q$. Then $(Q;A)$ is a $(km,m)-$group with condition:

(c) There exists a sequence $a_1^{(k-2)\cdot m}$ over Q such that for all $j \in \{0,\ldots,2m-1\}$ and for every $x_1^{2m} \in Q$ the following equality holds

$A(x_1^j, a_1^{(k-2)\cdot m}, x_{j+1}^{2m}) = A(x_1^{2m}, a_1^{(k-2)\cdot m})$.

Proof. Firstly we prove the following statements:

$\overset{\circ}{1}$ For all $x_1^{2km-m} \in Q$ the following equality holds

$A(A(x_1^{k\cdot m}), x_{k\cdot m+1}^{2km-m}) = A(x_1, A(x_2^{k\cdot m+1}), x_{k\cdot m+2}^{2km-m})$

[$<1,2>$ $-$associative law*]*.

$\overset{\circ}{2}$ For all $b_1^{2km} \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds

$A(x_1^m, b_1^{k\cdot m-m}) = b_{k\cdot m-m+1}^{k\cdot m}$.

$\overset{\circ}{3}$ $(Q;A)$ is a $(km,m)-$semigroup.

$\overset{\circ}{4}$ For all $b_1^{2km} \in Q$ there is exactly one $y_1^m \in Q^m$ such that the following equality holds

$A(b_1^{k\cdot m-m}, y_1^m) = b_{k\cdot m-m+1}^{k\cdot m}$.

$\overset{\circ}{5}$ For all $j \in \{0,\ldots,2m-1\}$ and for every $x_1^{2km} \in Q$ the following equality holds

$A(x_1^j, \overset{(k-3)\cdot m}{e}, (c_1^m)^{-1}, x_{j+1}^{2m}) = A(x_1^{2m}, \overset{(k-3)\cdot m}{e}, (c_1^m)^{-1})$,

where

(d) $B((c_1^m)^{-1}, c_1^m) = \overset{m}{e}$ *[*cf. Prop. 1.5 and Prop. 2.8*]*.

Sketch of the proof of $\overset{\circ}{1}$:

$A(A(x_1^{k\cdot m}), x_{k\cdot m+1}^{2km-m}) \overset{(b)}{=}$

$\overset{k}{B}(\overset{k}{B}(x_1^{k\cdot m}, c_1^m), x_{k\cdot m+1}^{2km-m}, c_1^m) \overset{4.5}{=}$

$\overset{k}{B}(x_1, \overset{k}{B}(x_2^{k\cdot m}, c_1^m, x_{k\cdot m+1}), x_{k\cdot m+2}^{2km-m}, c_1^m) \overset{4.4}{=}$

$\overset{k}{B}(x_1, \overset{k-1}{B}(x_2^{(k-1)\cdot m+1}, B(x_{(k-1)\cdot m+2}^{k\cdot m}, c_1^m, x_{k\cdot m+1})), x_{k\cdot m+2}^{2km-m}, c_1^m) \overset{(a)}{=}$

$\overset{k}{B}(x_1, \overset{k-1}{B}(x_2^{(k-1)\cdot m+1}, B(x_{(k-1)\cdot m+2}^{k\cdot m+1}, c_1^m)), x_{k\cdot m+2}^{2km-m}, c_1^m) \overset{4.4}{=}$

$\overset{k}{B}(x_1, \overset{k}{B}(x_2^{k\cdot m+1}, c_1^m), x_{k\cdot m+2}^{2km-m}, c_1^m) \overset{(b)}{=}$

$A(x_1, A(x_2^{k\cdot m+1}), x_{k\cdot m+2}^{2km-m})$.

Sketch of the proof of $\overset{\circ}{2}$:

$A(x_1^m, b_1^{(k-1)\cdot m}) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{(b)}{\Leftrightarrow}$

$\overset{k}{B}(x_1^m, b_1^{(k-1)\cdot m}, c_1^m) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{4.5}{\Leftrightarrow}$

$B(x_1^m, (\overset{k-1}{B} b_1^{(k-1)\cdot m}, c_1^m)) = b_{(k-1)\cdot m+1}^{k\cdot m}$.

Sketch of the proof of $\overset{\circ}{3}$:

By $\overset{\circ}{1}, \overset{\circ}{2}$ and by Prop. 2.1.

Sketch of the proof of $\overset{\circ}{4}$:

$A(b_1^{(k-1)\cdot m}, y_1^m) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{(b)}{\Leftrightarrow}$

$\overset{k}{B}(b_1^{(k-1)\cdot m}, y_1^m, c_1^m) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{4.4}{\Leftrightarrow}$

$\overset{k-1}{B}(b_1^{(k-1)\cdot m}, B(y_1^m, c_1^m)) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{(a) i=0}{\Longleftrightarrow}$

$\overset{k-1}{B}(b_1^{(k-1)\cdot m}, B(c_1^m, y_1^m)) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{4.4}{\Longleftrightarrow}$

$\overset{k}{B}(b_1^{(k-1)\cdot m}, c_1^m, y_1^m) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{4.3}{\Longleftrightarrow}$

$B(\overset{k-1}{B}(b_1^{(k-1)\cdot m}, c_1^m), y_1^m) = b_{(k-1)\cdot m+1}^{k\cdot m}$.

Sketch of a part of the proof of $\overset{\circ}{5}$ [to case $k = 4$]:

$A(x_1^{2m}, \overset{m}{e}, (c_1^m)^{-1}) \overset{(b)}{=}$

$\overset{4}{B}(x_1^{2m}, \overset{m}{e}, (c_1^m)^{-1}, c_1^m) \overset{4.4}{=}$

$\overset{3}{B}(x_1^{2m}, \overset{m}{e}, B((c_1^m)^{-1}, c_1^m)) \overset{(d)}{=}$

$\overset{3}{B}(x_1^{2m}, \overset{m}{e}, \overset{m}{e}) \overset{4.4}{=}$

$\overset{2}{B}(x_1^m, B(x_{m+1}^{2m}, \overset{m}{e}), \overset{m}{e}) \overset{(a)}{=}$

$\overset{2}{B}(x_1^m, B(x_{m+1}^{2m-1}, \overset{m}{e}, x_{2m}), \overset{m}{e}) \overset{4.4}{=}$

$\overset{2}{B}(x_1^{2m-1}, e, B(\overset{m-1}{e}, x_{2m}, \overset{m}{e})) \overset{(a)}{=}$

$\overset{2}{B}(x_1^{2m-1}, e, B(\overset{m-1}{e}, \overset{m}{e}, x_{2m})) \overset{4.4}{=}$

$\overset{3}{B}(x_1^{2m-1}, \overset{m}{e}, \overset{m}{e}, x_{2m}) \overset{(d)}{=}$

$\overset{3}{B}(x_1^{2m-1}, \overset{m}{e}, B((c_1^m)^{-1}, c_1^m), x_{2m}) =$ [9]
$\overset{3}{B}(x_1^{2m-1}, \overset{m}{e}, B((\bar{c}_1^m, c_1^m), x_{2m}) \overset{4.4}{=}$
$\overset{3}{B}(x_1^{2m-1}, \overset{m}{e}, \bar{c}_1, B((\bar{c}_2^m, c_1^m, x_{2m})) \overset{(a)}{=}$
$\overset{3}{B}(x_1^{2m-1}, \overset{m}{e}, \bar{c}_1, B(\bar{c}_2^m, x_{2m}, c_1^m)) \overset{4.4}{=}$
$\overset{4}{B}(x_1^{2m-1}, \overset{m}{e}, \bar{c}_1^m, \bar{c}_2^m, x_{2m}, c_1^m) =$
$\overset{4}{B}(x_1^{2m-1}, \overset{m}{e}, (c_1^m)^{-1}, x_{2m}, c_1^m) \overset{(b)}{=}$
$A(x_1^{2m-1}, \overset{m}{e}, (c_1^m)^{-1}, x_{2m})$.

By $\overset{\circ}{2} - \overset{\circ}{4}$, Prop. 4.1 and by $\overset{\circ}{5}$, we obtain $(Q; A)$ is a $(km, m)-$group with condition (c). $\square$

6.3. Remarks: *a) In [3] the following proposition is proved. Let $(Q; A)$ be a $(km, m)-$group, $m \geq 2$, $k \geq 3$ and let*

$$\mathbf{A}(x_1^m, x_{m+1}^{2m}, \dots, x_{(k-1)\cdot m+1}^{k\cdot m}) \overset{def}{=} A(x_1^{k\cdot m})$$

for all $x_1^{k\cdot m} \in Q$. Then there exist binary group $(Q^m, \mathbf{B})$, an element $c_1^m \in Q^m$ and an automorphism φ of this group, such that for each $x_1^m, x_{m+1}^{2m}, \dots, x_{(k-1)\cdot m+1}^{k\cdot m} \in Q^m$

$$\mathbf{A}(x_1^m, x_{m+1}^{2m}, \dots, x_{(k-1)\cdot m+1}^{k\cdot m}) =$$
$$\overset{k}{\mathbf{B}}(x_1^m, \varphi(x_{m+1}^{2m}), \dots, \varphi^{k-1}(x_{(k-1)\cdot m+1}^{k\cdot m}), c_1^m),$$
$$\varphi(c_1^m) = c_1^m \text{ and}$$
$$\mathbf{B}(\varphi^{k-1}(x_1^m), c_1^m) = \mathbf{B}(c_1^m, x_1^m).$$

b) $\mathbf{B}, \varphi$ and c_1^m from a), according to [16], are defined in the following way

$$\mathbf{B}(x_1^m, y_1^m) \overset{def}{=} A(x_1^m, a_1^{(k-2)\cdot m}, y_1^m),$$
$$\varphi(x_1^m) \overset{def}{=} A(\mathbf{e}(a_1^{(k-2)\cdot m}), x_1^m, a_1^{(k-2)\cdot m}) \text{ and}$$
$$c_1^m \overset{def}{=} A(\overbrace{\mathbf{e}(a_1^{(k-2)\cdot m})}^{k}\big|)$$

for all $x_1^m, y_1^m \in Q^m$, where $(Q; A)$ is a $(km, m)-$group, $\mathbf{e}$ its $\{1, n-m+1\}-$neutral operation and $k \geq 3$. [Cf. Th. 3.1-IV in [23].

c) If condition (c) from Th. 3.2 in $(Q; A)$ holds, then $\varphi(x_1^m) = x_1^m$ for all $x_1^m \in Q^m$.

d) $(km, m)-$groups $(k \geq 3, m \geq 2)$ with condition (0) from Th.6.1 exist, because $(2m, m)-$groups exist and Th.6.2 holds. However, we do not know if $(km, m)-$groups $(k \geq 3, m \geq 2)$ without condition (0) from Th.6.1 exist.

[9] $\bar{c}_1^m = (c_1^m)^{-1}$

7. On (n, m)−groups for $n > 2m$ and $n \neq km$

7.1. Theorem *[25]: Let $m \geq 2,\ s \geq 2,\ 0 < r < m,\ n = s\cdot m + r$ and let $(Q; A)$ be an (n, m)−group. Also, let there exist a sequence $a_1^{k\cdot m-2m}$, where $k = r - m + 1$, such that for all $i \in \{0, 1, \dots, 2m-1\}$, and for every $x_1^{2m} \in Q$ the following equality holds*

(0) $\overset{m}{A}(x_1^i, a_1^{k\cdot m-2m}, x_{i+1}^{2m}) = \overset{m}{A}(x_1^{2m}, a_1^{k\cdot m-2m})$.

Then there are mapping B of the set Q^{2m} into the set Q^m, $c_1^m \in Q^m$ and the sequence $\varepsilon_1^{(m-1)(n-m)}$ over Q such that the following statements hold

(1) *$(Q; B)$ is a $(2m, m)$−group;*

(2) *For all $j \in \{0, \dots, m-1\}$ and for every $x_1^m \in Q$ the following equality holds*

$$B(x_1^j, c_1^m, x_{j+1}^m) = B(x_1^m, c_1^m);$$

(3) *For all $x_1^m \in Q$ the following equality holds*

$$A(x_1^m) = B(\overset{n-m}{B}(x_1^n, \varepsilon_1^{(m-1)(n-m)}), c_1^m).$$

(4) *For all $t \in \{0, \dots, m-1\}$ and for every $y_1^r, z_1^m \in Q$ the following equality holds*

$$\overset{n-m-s+1}{B}(y_1^r, z_1^t, \varepsilon_1^{(m-1)(n-m)}, z_{t+1}^m) = \overset{n-m-s+1}{B}(y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)}).$$

Proof. Firstly we prove the following statements:

1° $(Q, \overset{m}{A})$ is a (km, m)−group, where $k = n - m + 1$.

2° Let E be a $\{1, km - m + 1\}$−neutral operation of (km, m)−group $(Q; \overset{m}{A})$. Also let

$$a)\quad B(x_1^m, y_1^m) \overset{def}{=} \overset{m}{A}(x_1^m, a_1^{km-2m}, y_1^m)$$

for all $x_1^m, y_1^m \in Q^m$, where a_1^{km-2m} from (0); and

$$b)\quad c_1^m \overset{def}{=} \overset{m}{A}(\overline{\mathsf{E}(a_1^{km-2m})}^{\,k}\Big|).$$

Then:

1) $(Q; B)$ is a $(2m, m)$−group;

2) For all $x_1^m \in Q^m$ and for all $j \in \{0, \dots, m-1\}$ the following equality holds

$$B(x_1^i, c_1^m, x_{i+1}^m) = B(x_1^m, c_1^m);\ \text{and}$$

3) For all $x_1^{km} \in Q$ the following equality holds

$$\overset{m}{A}(x_1^{km}) = \overset{k}{B}(x_1^{km}, c_1^m).$$

3° Let $\mathbf{e}$ be a $\{1, n-m+1\}$−neutral operation of (n, m)−group $(Q; A)$. Then for all $x_1^m \in Q$ and for every $\overset{(i)}{b}{}_1^{n-2m}, i \in \{1, \dots, m-1\}$, the following equality holds

$$A(x_1^n) = \overset{m}{A}(x_1^n, \overline{\overset{(i)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(i)}{b}{}_1^{n-2m})}\Big|_{i=1}^{m-1}).$$

4° Let $\overset{(i)}{b}{}_1^{n-2m}$, $i \in \{1, \ldots, m-1\}$, be an arbitrary sequence over Q. Also, let $\varepsilon_1^{(m-1)(n-m)} \overset{def}{=} \overline{\overset{(i)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(i)}{b}{}_1^{n-2m})}\Big|_{i=1}^{m-1}$.

Then for all $x_1^{(s-1)m}, y_1^r, z_1^m \in Q$ and for all $j \in \{0, \ldots, m-1\}$ the following equality holds

$$\overset{m}{A}(x_1^{(s-1)m}, y_1^r, z_1, \varepsilon_1^{(m-1)(n-m)}, z_{j+1}) = \overset{m}{A}(x_1^{(s-1)m}, y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)}).$$

The proof of 1° : By Prop. 4.7.

The proof of 2° : By Th.6.1

Sketch of the proof of 3° :

$a)$ $m = 2$:

$$\overset{2}{A}(x_1^n, b_1^{n-2m}, \mathbf{e}(b_1^{n-2m})) \overset{4.3}{=}$$
$$A(A(x_1^n), b_1^{n-2m}, \mathbf{e}(b_1^{n-2m})) \overset{2.1}{=} A(x_1^n)$$

$b)$ $m > 2$:

$$\overset{m}{A}(x_1^n, \overline{\overset{(i)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(i)}{b}{}_1^{n-2m})}\Big|_{i=1}^{m-2}, \overset{(m-1)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(m-1)}{b}{}_1^{n-2m}) \overset{4.3}{=}$$
$$A(\overset{m-1}{A}(x_1^n, \overline{\overset{(i)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(i)}{b}{}_1^{n-2m})}\Big|_{i=1}^{m-2}), \overset{(m-1)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(m-1)}{b}{}_1^{n-2m}) \overset{2.1}{=}$$
$$\overset{m-1}{A}(x_1^n, \overline{\overset{(i)}{b}{}_1^{n-2m}, \mathbf{e}(\overset{(i)}{b}{}_1^{n-2m})}\Big|_{i=1}^{m-2}) = \ldots \overset{2.1}{=} A(x_1^n).$$

Sketch of the proof of 4° [to the case $m = 3$, $n = 7$]:

$$\overset{3}{A}(x_1^3, y, z_1^3, b, \mathbf{e}(b), c, \mathbf{e}(c)) \overset{4.4}{=}$$
$$\overset{2}{A}(x_1^3, y, A(z_1^3, b, \mathbf{e}(b)), c, \mathbf{e}(c)) \overset{2.1, 2.11}{=}$$
$$\overset{2}{A}(x_1^3, y, A(z_1^i, b, \mathbf{e}(b), z_{i+1}^3), c, \mathbf{e}(c)) =$$
$$\overset{2}{A}(x_1^3, y, A(z_1^i, b, \overline{\mathbf{e}_j(b)}\Big|_{j=1}^{3}, z_{i+1}^3), c, \mathbf{e}(c)) =$$
$$\overset{2}{A}(x_1^3, y, A(z_1^i, b, \overline{\mathbf{e}_j(b)}\Big|_{j=1}^{3-i}, \overline{\mathbf{e}_j(b)}\Big|_{j=3-i+1}^{3}, z_{i+1}^3), c, \mathbf{e}(c)) \overset{4.4}{=}$$
$$\overset{2}{A}(x_1^3, y, z_1^i, b, \overline{\mathbf{e}_j(b)}\Big|_{j=1}^{3-i}, A(\overline{\mathbf{e}_j(b)}\Big|_{j=3-i+1}^{3}, z_{i+1}^3, c, \mathbf{e}(c))) \overset{2.1, 2.11}{=}$$
$$\overset{2}{A}(x_1^3, y, z_1^i, b, \overline{\mathbf{e}_j(b)}\Big|_{j=1}^{3-i}, A(\overline{\mathbf{e}_j(b)}\Big|_{j=3-i+1}^{3}, c, \mathbf{e}(c), z_{i+1}^3)) \overset{4.4}{=}$$
$$\overset{3}{A}(x_1^3, y, z_1^i, b, \overline{\mathbf{e}_j(b)}\Big|_{j=1}^{3-i}, \overline{\mathbf{e}_j(b)}\Big|_{j=3-i+1}^{3}, c, \mathbf{e}(c), z_{i+1}^3) =$$
$$\overset{3}{A}(x_1^3, y, z_1^i, b, \mathbf{e}(b), c, \mathbf{e}(c), z_{i+1}^3).$$

By 1° and 2°, we have (1) and (2).

Sketch of the proof of (3): By 2° [:3)] and by 3°.
$(k = n - m + 1, \varepsilon_1^{(m-1)(n-m)} \overset{def}{=} \overline{\overset{(i)}{b}\,{}_1^{n-2m}, \mathbf{e}(\overset{(i)}{b}\,{}_1^{n-2m})}\Big|_{i=1}^{m-1}.)$

Sketch of the proof of (4):

$\overset{m}{A}(x_1^{(s-1)m}, y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)}) \overset{4°}{=}$

$\overset{m}{A}(x_1^{(s-1)m}, y_1^r, z_1^j, \varepsilon_1^{(m-1)(n-m)}, z_{j+1}^m) \overset{4°-3)}{\Longrightarrow}$

$\overset{k}{B}(x_1^{(s-1)m}, y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)}, c_1^m) =$

$\overset{k}{B}(x_1^{(s-1)m}, y_1^r, z_1^j, \varepsilon_1^{(m-1)(n-m)}, z_{j+1}^m, c_1^m) \overset{1°,4.5}{\Longrightarrow}$

$\overset{s}{B}(x_1^{(s-1)m}, \overset{n-m-s+1}{B}(y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)}), c_1^m) =$

$\overset{s}{B}(x_1^{(s-1)m}, \overset{n-m-s+1}{B}(y_1^r, z_1^j, \varepsilon_1^{(m-1)(n-m)}, z_{j+1}^m), c_1^m) \overset{1°,4.7}{\Longrightarrow}$

$\overset{n-m-s+1}{B}(y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)}) = \overset{n-m-s+1}{B}(y_1^r, z_1^j, \varepsilon_1^{(m-1)(n-m)}, z_{j+1}^m).$

The proof of Th. 7.1 is completed. □

7.2. Theorem *[25]: Let $(Q; B)$ be a $(2m, m)-$group and $m \geq 2$. Also let:*

(a) c_1^m be an element of the set Q^m such that for every $i \in \{0, \dots, m-1\}$, and for every $x_1^m \in Q$ the following equality holds

$B(x_1^i, c_1^m, x_{i+1}^m) = B(x_1^m, c_1^m)$; and

(b) $\varepsilon_1^{(m-1)(n-m)}$ be a sequence over Q such that for all $j \in \{0, \dots, m-1\}$, and for every $y_1^r, z_1^m \in Q$ the following equality holds

$\overset{n-m-s+1}{B}(y_1^r, z_1^j, \varepsilon_1^{(m-1)(n-m)}, z_{j+1}^m) = \overset{n-m-s+1}{B}(y_1^r, z_1^m, \varepsilon_1^{(m-1)(n-m)}),$

where $s \geq 2$, $0 < r < m$ and $n = s \cdot m + r$.

Further on, let

(c) $A(x_1^m) \overset{def}{=} B(\overset{n-m}{B}(x_1^n, \varepsilon_1^{(m-1)(n-m)}), c_1^m)$

for all $x_1^n \in Q$.

Then $(Q; A)$ is an $(n, m)-$group.

Proof. Firstly we prove the following statements:

$\overset{\circ}{1}$ The $< 1, 2 > -$associative law holds in $(Q; A)$.

$\overset{\circ}{2}$ For every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds

$A(x_1^m, a_1^{n-m}) = a_{n-m+1}^n.$

$\overset{\circ}{3}$ $(Q; A)$ is an $(n, m)-$group.

$\overset{\circ}{4}$ For every $a_1^n \in Q$ there is exactly one $y_1^m \in Q^m$ such that the following equality holds

$A(a_1^{n-m}, y_1^m) = a_{n-m+1}^n$.

Sketch of the proof of $\overset{\circ}{1}$:

$a)\ A(A(x_1^n), x_{n+1}^{2n-m}) \overset{(c)}{=}$
$\overset{n-m+1}{B}(\overset{n-m+1}{B}(x_1^n, \varepsilon_1^{(m-1)(n-m)}, c_1^m), x_{n+1}, x_{n+2}^{2n-m}, \varepsilon_1^{(m-1)(n-m)}, c_1^m) \overset{4.5}{=}$
$\overset{n-m+1}{B}(x_1, \overset{n-m+1}{B}(x_2^n, \varepsilon_1^{(m-1)(n-m)}, c_1^m, x_{n+1}), x_{n+2}^{2n-m}, \varepsilon_1^{(m-1)(n-m)}, c_1^m)$.

$b)\ \overset{n-m+1}{B}(x_2^n, \varepsilon_1^{(m-1)(n-m)}, c_1^m, x_{n+1}) \overset{4.4}{=}$
$\overset{n-m}{B}(x_2^n, \varepsilon_1^{(m-1)(n-m-1)}, B(\varepsilon_{(m-1)(n-m+1)+1}^{(m-1)(n-m)}, c_1^m, x_{n+1})) \overset{(a)}{=}$
$\overset{n-m}{B}(x_2^n, \varepsilon_1^{(m-1)(n-m-1)}, B(\varepsilon_{(m-1)(n-m+1)+1}^{(m-1)(n-m)}, x_{n+1}, c_1^m)) \overset{4.4}{=}$
$\overset{n-m+1}{B}(x_2^n, \varepsilon_1^{(m-1)(n-m-1)}, \varepsilon_{(m-1)(n-m+1)+1}^{(m-1)(n-m)}, x_{n+1}, c_1^m) =$
$\overset{n-m+1}{B}(x_2^n, \varepsilon_1^{(m-1)(n-m-1)}, x_{n+1}, c_1^m) \overset{4.5}{=}$
$\overset{s}{B}(x_2^{(s-1)m+1}, \overset{n-m-s+1}{B}(x_{(s-1)m+2}^n, {}^{10}\ \varepsilon_1^{(m-1)(n-m)}, x_{n+1}), c_1^m) \overset{(b)}{=}$
$\overset{s}{B}(x_2^{(s-1)m+1}, \overset{n-m-s+1}{B}(x_{(s-1)m+2}^n, x_{n+1}, \varepsilon_1^{(m-1)(n-m)}), c_1^m) =$
$\overset{s}{B}(x_2^{(s-1)m+1}, \overset{n-m-s+1}{B}(x_{(s-1)m+2}^{n+1}, \varepsilon_1^{(m-1)(n-m)}), c_1^m) \overset{4.5}{=}$
$\overset{n-m+1}{B}(x_2^{(s-1)m+2}, x_{(s-1)m+2}^{n+1}, \varepsilon_1^{(m-1)(n-m)}, c_1^m) =$
$\overset{n-m+1}{B}(x_2^{n+1}, \varepsilon_1^{(m-1)(n-m)}, c_1^m) \overset{(c)}{=} A(x_2^{n+1})$.

Finally, by $a)$, $b)$ and by (c), we obtain $\overset{\circ}{1}$.

Sketch of the proof of $\overset{\circ}{2}$:

$A(x_1^m, a_1^{n-m}) = a_{n-m+1}^n \overset{(c)}{\Longleftrightarrow}$
$\overset{n-m+1}{B}(x_1^m, a_1^{n-m}, \varepsilon_1^{(m-1)(n-m)}, c_1^m) = a_{n-m+1}^n \overset{4.5}{\Longleftrightarrow}$
$B(x_1^m, \overset{n-m}{B}(a_1^{n-m}, \varepsilon_1^{(m-1)(n-m)}, c_1^m)) = a_{n-m+1}^n$.

The proof of $\overset{\circ}{3}$: By $\overset{\circ}{1}, \overset{\circ}{2}$ and Prop. 4.2_1.

Sketch of the proof of $\overset{\circ}{4}$:

$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n \overset{(c)}{\Longleftrightarrow}$
$\overset{n-m+1}{B}(a_1^{n-m}, y_1^m, \varepsilon_1^{(m-1)(n-m)}, c_1^m) = a_{n-m+1}^n$.

Whence, by Prop. 4.7 and by Def. 1.1, we obtain $\overset{\circ}{4}$.

Finally, by $\overset{\circ}{2} - \overset{\circ}{4}$ and by Prop. 4.1, we conclude that Th. 7.2 holds. □

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[10] $n = sm + r$.

8. Skew operation on (n, m)−groups

8.1. Definition *[28]: Let $(Q;A)$ be an (n,m)−group and $n \geq 2m+1$. Further on, let $^-$ be a mapping of the set Q into the set Q^m. Then, we shall say that mapping $^-$ is a* **skew operation** *of the (n,m)−group $(Q;A)$ iff for each $a \in Q$ there is (exactly one) $\overline{a} \in Q^m$ such that the following equality holds*

(0) $$A(\overset{n-m}{a}, \overline{a}) = \overset{m}{a}\ ^{11}.$$

Remark: For $m = 1$ skew operation is introduced in [6].

8.2. Proposition: *Let $(Q;A)$ be an (n,m)−group and $n \geq 2m+1$. Then for all $i \in \{1, \ldots, n-m+1\}$ and for every $a \in Q$ the following equality holds*

$$A(\overset{i-1}{a}, \overline{a}, \overset{n-(i-1+m)}{a}) = \overset{m}{a}.$$

Sketch of the proof.

$A(\overset{n-m}{a}, \overline{a}) \overset{(0)}{=} \overset{m}{a} \Rightarrow$

$A(\overset{i-1}{a}, A(\overset{n-m}{a}, \overline{a}), \overset{n-(i-1+m)}{a}) = A(\overset{i-1}{a}, \overset{m}{a}, \overset{n-(i-1+m)}{a}) \Rightarrow$

$A(\overset{i-1}{a}, A(\overset{n-m}{a}, \overline{a}), \overset{n-(i-1+m)}{a}) = A(\overset{n}{a}) \overset{1.1(|)}{\Longrightarrow}$

$A(\overset{i-1}{a}, \overset{n-(i-1+m)}{a}, A(\overset{i-1}{a}, \overline{a}, \overset{n-(i-1+m)}{a})) = A(\overset{n}{a}) \Rightarrow$

$A(\overset{n-m}{a}, A(\overset{i-1}{a}, \overline{a}, \overset{n-(i-1+m)}{a})) = A(\overset{n-m}{a}, \overset{m}{a}) \overset{1.1(||)}{\Longrightarrow}$

$A(\overset{i-1}{a}, \overline{a}, \overset{n-(i-1+m)}{a}) = \overset{m}{a}.$ $\square$

8.3. Proposition *[28]: Let $(Q;A)$ be an (n,m)−group and $n \geq 2m+1$. Then for all $a, x_1^m \in Q$ the equality*

$$A(x_1^m, \overset{n-2m}{a}, \overline{a}) = x_1^m$$

holds.

Sketch of the proof.

$A(x_1^m, \overset{n-2m}{a}, \overline{a}) = y_1^m \Rightarrow$

$A(A(x_1^m, \overset{n-2m}{a}, \overline{a}), \overset{n-m}{a}) = A(y_1^m, \overset{n-m}{a}) \overset{1.1(|)}{\Longrightarrow}$

$A(x_1^m, \overset{n-2m}{a}, A(\overline{a}, \overset{n-m}{a})) = A(y_1^m, \overset{n-m}{a}) \overset{8.2, i=1}{\Longrightarrow}$

$A(x_1^m, \overset{n-2m}{a}, \overset{m}{a}) = A(y_1^m, \overset{n-m}{a}) \Rightarrow$

$A(x_1^m, \overset{n-m}{a}) = A(y_1^m, \overset{n-m}{a}) \overset{1.1(||)}{\Longrightarrow} x_1^m = y_1^m.$ $\square$

8.4. Theorem *[28]: Let $n \geq 2m+1$, $(Q;A)$ be an (n,m)−group,* $\mathbf{e}$ *its $\{1, n-m+1\}$−neutral operation and $^-$ its skew operation. Then for all $a \in Q$ the following equality holds*

$$\overline{a} = \mathbf{e}(\overset{n-2m}{a}).$$

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[11] See Def. 1.1.-(||).

Sketch of the proof.

$A(x_1^m, \overset{n-2m}{a}, \overline{a}) \overset{8.3}{=} x_1^m \ \wedge\ A(x_1^m, \overset{n-2m}{a}, \mathbf{e}(\overset{n-2m}{a})) \overset{2.5}{=} x_1^m \Rightarrow$

$A(x_1^m, \overset{n-2m}{a}, \overline{a}) = A(x_1^m, \overset{n-2m}{a}, \mathbf{e}(\overset{n-2m}{a})) \overset{1.1(|\,|)}{\Longrightarrow} \overline{a} = \mathbf{e}(\overset{n-2m}{a})$. □

8.5. Theorem *[28]: Let $(Q;A)$ be an $(n,m)-$group,* $\mathbf{e}$ *its $\{1, n-m+1\}-$neutral operation, $^{-}$ its skew operation and $n > 3m$. Then for every sequence a_1^{n-m+1} over Q the following equality holds*

$$\mathsf{E}(\overset{m}{a}_1, \ldots, \overset{m}{a}_{n-m+1}) = \overset{n-2m-1}{A}(\overline{a}_{n-m-1}, \overset{n-3m}{a}_{n-m+1}, \ldots, \overline{a}_1, \overset{n-3m}{a_1})\ ^{12},$$

where E *is the $\{1, m(n-m)+1\}-$neutral operation of $(m(n-m)+m, m)-$group $(Q; \overset{m}{A})$.*

Sketch of the proof.

$\overset{m}{A}(\overset{n-2m-1}{A}(\overline{a}_{n-m-1}, \overset{n-3m}{a}_{n-m+1}, \ldots, \overline{a}_1, \overset{n-3m}{a_1}), \overset{m}{a}_1, \ldots, \overset{m}{a}_{n-m-1}, x_1^m) \overset{8.4}{=}$

$\overset{m}{A}(\overset{n-2m-1}{A}(\mathbf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-3m}{a}_{n-m-1}, \ldots, \mathbf{e}(\overset{n-2m}{a_1}), \overset{n-3m}{a_1}), \overset{m}{a}_1, \ldots, \overset{m}{a}_{n-m-1}, x_1^m) \overset{4.5}{=}$

$\overset{n-m-1}{A}(\mathbf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-3m}{a}_{n-m-1}, \ldots, \mathbf{e}(\overset{n-2m}{a_1}), \overset{n-3m}{a_1}, \overset{m}{a}_1, \ldots, \overset{m}{a}_{n-m-1}, x_1^m) \overset{4.4}{=}$

$\overset{n-m-2}{A}(\mathbf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-3m}{a}_{n-m-1}, \ldots, A(\mathbf{e}(\overset{n-2m}{a_1}), \overset{n-3m}{a_1}, \overset{m}{a}_1, \overset{m}{a}_2), \overset{m}{a}_3, \ldots, \overset{m}{a}_{n-m-1}, x_1^m) \overset{2.1}{=}$

$\overset{n-m-2}{A}(\mathbf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-3m}{a}_{n-m-1}, \ldots, \overset{m}{a}_2, \overset{m}{a}_3, \ldots, \overset{m}{a}_{n-m-1}, x_1^m) =$

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$A(\mathbf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-3m}{a}_{n-m-1}, \overset{m}{a}_{n-m-1}, x_1^m) =$

$A(\mathbf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-2m}{a}_{n-m-1}, x_1^m) \overset{2.1}{=} x_1^m$.

Hence, by Prop. 4.7 and Th.2.5, we conclude that for every sequence a_1^{n-m-1} over Q and for all $x_1^m \in Q^m$ the following equality holds

$\overset{m}{A}(\overset{n-2m-1}{A}(\mathbf{e}(\overset{n-2m}{a}_{n-m-1}), \overset{n-3m}{a}_{n-m-1}, \ldots, \mathbf{e}(\overset{n-2m}{a_1}), \overset{n-3m}{a_1}), \overset{m}{a}_1, \ldots, \overset{m}{a}_{n-m-1}, x_1^m) =$

$\overset{m}{A}(\mathsf{E}(\overset{m}{a}_1, \ldots, \overset{m}{a}_{n-m-1}), \overset{m}{a}_1, \ldots, \overset{m}{a}_{n-m-1}, x_1^m),$

where E is the $\{1, m(n-m)+1\}-$neutral operation of $(m(n-m)+m, m)-$group $(Q; \overset{m}{A})$.

Finaly, whence, by Def. 1.1, we conclude that the proposition holds. □

For $m = 1$ Th.8.5 is reduced to:

8.6. Theorem *[30]: Let $(Q;A)$ be an $n-$group,* $\mathbf{e}$ *its $\{1,n\}-$neutral operation, $^{-}$ its skew operation and $n > 3$. Then for every sequence a_1^{n-2} over Q the following equality holds*

$$\mathbf{e}(a_1^{n-2}) = \overset{n-3}{A}(\overline{a}_{n-2}, \overset{n-3}{a}_{n-2}, \ldots, \overline{a}_1, \overset{n-3}{a_1}).$$

[12] $n > 3m \Rightarrow \overset{n-3m}{a}_i \neq \emptyset \ \ (i \in \{1, \ldots, n-m-1\})$.

Remark: See, also VIII-2.9 and Appendix 2 in [23].

8.7. Remark: *In [Ušan 1998] topological* $n-$*groups for* $n \geq 2$ *are defined on* $n-$*groups as algebras* $(Q; A, ^{-1})$ *of the type* $< n, n-1 >$ *[15], [17]; cf. Ch. III and Ch. IX in [23]]. In [29] topological* $n-$*groups for* $n \geq 3$ *are considered on* $n-$*groups as algebras* $(Q; A, ^{-})$ *of the type* $< n, 1 >$ *[[10]]. In [Ušan 1998] it is proved that for* $n \geq 3$ *these definitions are mutually equivalent. The key roole in the proof had Theorem 8.6. About topological* $n-$*groups see, also, Chapter VIII in [23]. Topological* $(n,m)-$*groups are not defined.*

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