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$n-$Groups $(n > 3)$ as Algebras of the Type $\langle n, n-2, n-2\rangle$ With Four Laws

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Abstract. In this article $n-$groups $(n > 3)$ are described as algebras of the type $\langle n, n-2, n-2\rangle$ with four laws.

1. Preliminaries

Definition 1.1 ([2])**.** Let $n \geq 2$ and let $(Q; A)$ be an $n-$groupoid. Then:

1) We say that $(Q; A)$ is an $n-$**semigroup** iff for every $i, j \in \{1, \ldots, n\}$, $i < j$, the following law holds:
$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-1})$$
[:$\langle i, j\rangle$-associative law*]*;
2) We say that $(Q; A)$ is an $n-$**quasigroup** iff for every $i \in \{1, \ldots, n\}$ and for every $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the following equality holds $A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n$; and 3) We say that $(Q; A)$ is an $n-$**group** iff $(Q; A)$ is an $n-$semigroup and an $n-$quasigroup as well.

Definition 1.2 ([3])**.** Let $n \geq 2$ and let $(Q; A)$ be an $n-$groupoid. Further on, let $\mathbf{e}$ be mapping of the set Q^{n-2} into the set Q. Let also $\{i, j\} \subseteq \{1, \ldots, n\}, i < j$. Then: $\mathbf{e}$ is an $\{i, j\}-$**neutral operation of the** $n-$**groupoid** $(Q; A)$ iff the following formula holds
$$(\forall a_t \in Q)_1^{n-2}(\forall x \in Q)(A(a_1^{i-1}, \mathbf{e}(a_1^{n-2}), a_i^{j-2}, x, a_{j-1}^{n-2})) = x \text{ and}$$
$$A(a_1^{i-1}, x, a_i^{j-2}, \mathbf{e}(a_1^{n-2}), a_{j-1}^{n-2}) = x.$$

Proposition 1.1 ([3])**.** *Let $(Q; A)$ be an $n-$groupoid and $n \geq 2$. Also let $\{i, j\} \subseteq \{1, \ldots, n\}$ and $i < j$. Then there is* **at least one** *$\{i, j\}-$neutral operation of $(Q; A)$.*

Proposition 1.2 ([3])**.** *Every $n-$group $(n \geq 2)$ has an $\{1, n\}-$neutral operation. See, also* [7].

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2. Auxiliary part

Proposition 2.1 ([4]). *Let $(Q;A)$ be an $n-$group, $\mathbf{e}$ its $\{1,n\}-$neutral operation and $n \geq 3$. Then for every sequence a_1^{n-2} over Q and for all $i \in \{1,\dots,n-2\}$ there is exactly one $x_i \in Q$ such that the following equality holds:*

$$\mathbf{e}(a_1^{i-1},x_i,a_i^{n-3}) = a_{n-2}.$$

The proof see in [7].

Proposition 2.2 ([4]). *Let $(Q;A)$ be an $n-$group, $\mathbf{e}$ its $\{1,n\}-$neutral operation and $n \geq 3$. Then for every $a_1^{n-2},x \in Q$ and for all $i \in \{1,\dots,n-1\}$ the following equalities hold*

$$A(x,a_i^{n-2},\mathbf{e}(a_1^{n-2}),a_1^{i-1}) = x \quad and$$
$$A(a_i^{n-2},\mathbf{e}(a_1^{n-2}),a_1^{i-1},x) = x.$$

The proof see in [7].

Proposition 2.3 ([4]). *Let $(Q;A)$ be an $n-$groupoid and let $n \geq 3$. Also let the following statements hold:*

(a) *The $\langle 1,2\rangle-$associative law holds in $(Q;A)$; and*
(b) *For every $a_1^{n-1},x,y \in Q$ the following implication holds*

$$A(x,a_1^{n-1}) = A(y,a_1^{n-1}) \;\Rightarrow\; x=y.$$

Then $(Q;A)$ is $n-$semigroup.

The proof see in [7].

Proposition 2.4 ([6]). *Let $n \geq 3$ and let $(Q;A)$ be an $n-$groupoid. Then: $(Q;A)$ is an $n-$group iff there is $i \in \{2,\dots,n-1\}$ such that the following statements hold:*

(α) *The $\langle i-1,i\rangle-$associative law hold in $(Q;A)$;*
(β) *The $\langle i,i+1\rangle-$associative law hold in $(Q;A)$; and*
(γ) *For every $a_1^n \in Q$ there is* **exactly one** *$x \in Q$ such that the following equality holds: $A(a_1^{i-1},x,a_i^{n-1}) = a_n$.*

The proof see in [7].

3. Main part

Theorem 3.1. *Let $(Q;A)$ be an $n-$groupoid and let $n>3$. Also, let $\mathbf{e}$ and $\boldsymbol{\varepsilon}$ be $(n-2)-$ary operations in Q. Then: $(Q;A)$ is an $n-$group iff the following laws hold in the algebra $(Q;A,\mathbf{e},\boldsymbol{\varepsilon})$ [of the type $\langle n,n-2,n-2\rangle$]:*

(i) $A(A(x_1^n),x_{n+1}^{2n-1}) = A(x_1,A(x_2^{n+1}),x_{n+2}^{2n-1})$,
(ii) $A(x,b_1^{n-2},\mathbf{e}(b_1^{n-2})) = x$,
(iii) $A(b_1^{n-2},\boldsymbol{\varepsilon}(b_2^{n-2},b_1),x) = x$ *and*
(iv) $\mathbf{e}(c_1^{n-3},\boldsymbol{\varepsilon}(c_1^{n-3},a)) = \boldsymbol{\varepsilon}(c_1^{n-3},\mathbf{e}(c_1^{n-3},a))$.

Proof.

1) $\Rightarrow$: Let $n \geq 3$, $(Q;A)$ be an $n-$group and $\mathbf{e}$ its $\{1,n\}-$neutral operation.

Firstly we prove the following statements:

$^\circ 1$ For every $a_1^{n-1} \in Q$, there is $(n-2)-$ary operation $\mathbf{e}^{-1}$ such that the following equivalence holds

$$\text{(a)} \qquad \mathbf{e}(a_1^{n-2}) = a_{n-1} \;\Leftrightarrow\; \mathbf{e}^{-1}(a_1^{n-3}, a_{n-1}) = a_{n-2}, \text{[1]}$$

$^\circ 2$ For every $x, b_1^{n-2} \in Q$, the following equality holds

$$\text{(b)} \qquad (b_1^{n-2}, \mathbf{e}^{-1}(b_2^{n-2}, b_1), x) = x; \text{ and}$$

$^\circ 3$ For all $a \in Q$ and for every sequence c_1^{n-3} over Q the following equality holds

$$\mathbf{e}(c_1^{n-3}, \mathbf{e}^{-1}(c_1^{n-3}, a)) = \mathbf{e}^{-1}(c_1^{n-3}, \mathbf{e}(c_1^{n-3}, a)).$$

The proof of $^\circ 1$: By Prop. 2.1.

The proof of $^\circ 2$:

Putting (a) in

$$A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) \overset{1.4}{=} x,$$

we obtain

$$A(a_{n-1}, a_1^{n-3}, \mathbf{e}^{-1}(a_1^{n-3}, a_{n-1}), x) = x.$$

Whence, by the substitutions $a_{n-1} = b_1$ and $a_1^{n-3} = b_2^{n-2}$, we have (b).

Sketch of the proof of $^\circ 3$:

$a)$ $(\mathbf{e}^{-1})^{-1} = \mathbf{e}$.

$b)$ $\mathbf{e}^{-1}(c_1^{n-3}, a) = \mathbf{e}^{-1}(c_1^{n-3}, a) \overset{def}{\Longleftrightarrow} (\mathbf{e}^{-1})^{-1}(c_1^{n-3}, \mathbf{e}^{-1}(c_1^{n-3}, a)) = a$
$\overset{a)}{\Longleftrightarrow} \mathbf{e}(c_1^{n-3}, \mathbf{e}^{-1}(c_1^{n-3}, a)) = a.$

$c)$ $\mathbf{e}(c_1^{n-3}, a) = \mathbf{e}(c_1^{n-3}, a) \overset{a)}{\Longleftrightarrow} (\mathbf{e}^{-1})^{-1}(c_1^{n-3}, a) = \mathbf{e}(c_1^{n-3}, a)$
$\overset{\text{def}}{\Longleftrightarrow} \mathbf{e}^{-1}(c_1^{n-3}, \mathbf{e}(c_1^{n-3}, a)) = a.$

$d)$ By $b)$ and $c)$, we obtain $^\circ 3$.

Finally, by $^\circ 1 - ^\circ 3$, Def. 1.1.-1) and by Prop. 1.4, we conclude that the direction "$\Rightarrow$" holds.

2) $\Leftarrow$: Firstly, we prove the following statements:

$\overset{\circ}{1}$ For every $x, y, a, a_1^{n-2} \in Q$ the implication holds

$$A(x, a, a_1^{n-2}) = A(y, a, a_1^{n-2}) \Rightarrow x = y;$$

$\overset{\circ}{2}$ $(Q;A)$ is an $n-$semigroup;

$\overset{\circ}{3}$ $(\forall a \in Q)(\forall c_i \in Q)_1^{n-3}\; a = \mathbf{e}(c_1^{n-3}, \boldsymbol{\varepsilon}(c_1^{n-3}, a));$

[1]Cf. [Belousov 1972].

$\overset{\circ}{4}$ For every $x, y, a, a_1^{n-2} \in Q$ the implication

$$A(a, x, a_1^{n-2}) = A(a, y, a_1^{n-2}) \quad \Rightarrow \quad x = y$$

holds;

$\overset{\circ}{5}$ For every $x, y, a, a_1^{n-2} \in Q$ the equivalence holds

$$A(a_1^{n-2}, x, a) = A(a_1^{n-2}, y, a) \quad \Leftrightarrow \quad x = y; \quad \text{and}$$

$\overset{\circ}{6}$ For every $x, a, b, a_1^{n-2} \in Q$

$$\begin{aligned} A(a, x, a_1^{n-2}) = b \quad \Leftrightarrow \\ x = & A(\mathbf{e}(c_1^{n-3}, a), c_1^{n-3}, b, \mathbf{e}(a_1^{n-2})). \end{aligned}$$

Sketch of the proof of $\overset{\circ}{1}$:

$$A(x, a, a_1^{n-2}) = A(y, a, a_1^{n-2}) \overset{n\geq 3}{\Rightarrow}$$

$$\begin{aligned} A(A(x, a, a_1^{n-2}), \mathbf{e}(a_1^{n-2}), c_1^{n-3}, \mathbf{e}(a, c_1^{n-3})) = \\ A(A(y, a, a_1^{n-2}), \mathbf{e}(a_1^{n-2}), c_1^{n-3}, \mathbf{e}(a, c_1^{n-3})) \overset{(i)}{\Rightarrow} \end{aligned}$$

$$\begin{aligned} A(x, A(a, a_1^{n-2}, \mathbf{e}(a_1^{n-2})), c_1^{n-3}, \mathbf{e}(a, c_1^{n-3})) = \\ A(y, A(a, a_1^{n-2}, \mathbf{e}(a_1^{n-2})), c_1^{n-3}, \mathbf{e}(a, c_1^{n-3})) \overset{(ii)}{\Rightarrow} \end{aligned}$$

$$A(x, a, c_1^{n-3}, \mathbf{e}(a, c_1^{n-3})) = A(y, a, c_1^{n-3}, \mathbf{e}(a, c_1^{n-3})) \overset{(ii)}{\Rightarrow}$$

$$x = y.$$

The proof of $\overset{\circ}{2}$: By $\overset{\circ}{1}, (i)$ and by Prop. 2.3.

Sketch of the proof of $\overset{\circ}{3}$:

$$\begin{aligned} A(a, c_1^{n-3}, \boldsymbol{\varepsilon}(c_1^{n-3}, a), \mathbf{e}(c_1^{n-3}, \boldsymbol{\varepsilon}(c_1^{n-3}, a))) &\overset{(iii)}{=} \mathbf{e}(c_1^{n-3}, \boldsymbol{\varepsilon}(c_1^{n-3}, a)), \\ A(a, c_1^{n-3}, \boldsymbol{\varepsilon}(c_1^{n-3}, a), \mathbf{e}(c_1^{n-3}, \boldsymbol{\varepsilon}(c_1^{n-3}, a))) &\overset{(ii)}{=} a. \end{aligned}$$

Sketch of the proof of $\overset{\circ}{4}$:

$$A(a, x, a_1^{n-2}) = A(a, y, a_1^{n-2}) \overset{n\geq 3}{\Rightarrow}$$

$$\begin{aligned} A(\mathbf{e}(c_1^{n-3}, a), c_1^{n-3}, A(a, x, a_1^{n-2}), \mathbf{e}(a_1^{n-2})) = \\ A(\mathbf{e}(c_1^{n-3}, a), c_1^{n-3}, A(a, y, a_1^{n-2}), \mathbf{e}(a_1^{n-2})) \overset{\overset{\circ}{2}}{\Rightarrow} \end{aligned}$$

$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},a,A(x,a_1^{n-2},\mathbf{e}(a_1^{n-2})))=$$
$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},a,A(y,a_1^{n-2},\mathbf{e}(a_1^{n-2})))\ \overset{(ii)}{\Longrightarrow}$$

$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},a,x)=$$
$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},a,y)\ \overset{\overset{\circ}{3}}{\Rightarrow}$$

$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},\mathbf{e}(c_1^{n-3},\boldsymbol{\varepsilon}(c_1^{n-3},a)),x)=$$
$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},\mathbf{e}(c_1^{n-3},\boldsymbol{\varepsilon}(c_1^{n-3},a)),y)\ \overset{(iv)}{\Longrightarrow}$$

$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},\boldsymbol{\varepsilon}(c_1^{n-3},\mathbf{e}(c_1^{n-3},a))x)=$$
$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},\boldsymbol{\varepsilon}(c_1^{n-3},\mathbf{e}(c_1^{n-3},a)),y)\ \overset{(iii)}{\Longrightarrow}$$

$$x=y.$$

Sketch of the proof of $\overset{\circ}{5}$:

$$A(a_1^{n-2},x,a)=A(a_1^{n-2},y,a)\ \Rightarrow$$
$$A(d_1^2,A(a_1^{n-2},x,a),d_3^{n-1})=A(d_1^2,A(a_1^{n-2},y,a),d_3^{n-1})\ \overset{\overset{\circ}{2}}{\Rightarrow}$$
$$A(A(d_1^2,a_1^{n-2}),x,a,d_3^{n-1})=A(A(d_1^2,a_1^{n-2}),y,a,d_3^{n-1})\ \overset{\overset{\circ}{4}}{\Rightarrow}$$
$$x=y.$$

Sketch of the proof of $\overset{\circ}{6}$:

$$A(a,x,a_1^{n-2})=b\ \overset{\overset{\circ}{5}}{\Leftrightarrow}$$
$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},A(a,x,a_1^{n-2}),\mathbf{e}(a_1^{n-2}))=$$
$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},b,\mathbf{e}(a_1^{n-2}))\ \overset{\overset{\circ}{2}}{\Leftrightarrow}$$
$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},a,A(x,a_1^{n-2},\mathbf{e}(a_1^{n-2})))=$$
$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},b,\mathbf{e}(a_1^{n-2}))\ \overset{(ii)}{\Leftrightarrow}$$
$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},a,x)=$$
$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},b,\mathbf{e}(a_1^{n-2}))\ \overset{\overset{\circ}{3}}{\Leftrightarrow}$$
$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},\mathbf{e}(c_1^{n-3},\boldsymbol{\varepsilon}(c_1^{n-3},a)),x)=$$
$$A(\mathbf{e}(c_1^{n-3},a),c_1^{n-3},b,\mathbf{e}(a_1^{n-2}))\ \overset{(iv)}{\Leftrightarrow}$$

$$A(\mathbf{e}(c_1^{n-3}, a), c_1^{n-3}, \boldsymbol{\varepsilon}(c_1^{n-3}, \mathbf{e}(c_1^{n-3}, a)), x) =$$
$$A(\mathbf{e}(c_1^{n-3}, a), c_1^{n-3}, b, \mathbf{e}(a_1^{n-2})) \overset{(iii)}{\Leftrightarrow}$$
$$x = A(\mathbf{e}(c_1^{n-3}, a), c_1^{n-3}, b, \mathbf{e}(a_1^{n-2})).$$

Finally, considering $\overset{\circ}{2}, \overset{\circ}{4}$ and $\overset{\circ}{6}$, by Prop. 2.4, we conclude that $(Q; A)$ is $n-$ group. □

Similarly, one could prove also the following theorem:

Theorem 3.2. *Let $(Q; A)$ be an $n-$groupoid and let $n > 3$. Also, let $\mathbf{e}$ and $\boldsymbol{\varepsilon}$ be $(n-2)-$ary operations in Q. Then: $(Q; A)$ is an $n-$group iff the following laws hold in the algebra $(Q; A, \mathbf{e}, \boldsymbol{\varepsilon})$ [of the type $\langle n, n-2, n-2\rangle$]*

$(\bar{i})$ $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$,
$(\overline{ii})$ $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$,
$(\overline{iii})$ $A(x, \boldsymbol{\varepsilon}(b_{n-2}, b_1^{n-3}), b_1^{n-2}) = x$ *and*
$(\overline{iv})$ $\mathbf{e}(\boldsymbol{\varepsilon}(a, c_1^{n-3}), c_1^{n-3}) = \boldsymbol{\varepsilon}(\mathbf{e}(a, c_1^{n-3}), c_1^{n-3})$.

Remark 3.1. For $n = 3:\ \mathbf{e}^{-1} = \mathbf{e}$.
(See °2 in the Proof of Th. 3.1, Prop. 2.2 and Def. 1.1-2).)

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