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Cardinal Invariants for Commutative Group Algebras

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Abstract. A new kind of major structural invariants for commutative group algebras and pairs of commutative group algebras are here obtained. The present statements are a sequel to our recent results published in *Ricerche Math.* (Napoli, 2001 and 2003) plus *Rend. Circolo Mat. Palermo* (2002).

1. Introduction

This paper is a natural supplement to previous results in this aspect due to the author [3-9]. It is to be understood throughout that all groups considered in the current work are Abelian. Following the notions from [12], if among the pure subgroups of a group G which contain A there exists a minimal one, we say that A is contained in, or is imbedded in, a minimal pure subgroup of G. We emphasize that the subgroup A of G is said to be purifiable if, among the pure subgroups of G containing A, there is a minimal one. Such a minimal pure subgroup of G is called a pure hull of A in G. The terminology, notations and other material on Abelian groups not expressly introduced here follow the usage of [10] and [7]. For F an arbitrary field of $\mathrm{char}F = p$, FG will denote the group algebra of G over F. For an arbitrary subgroup A in G, (FG, FA) designates a pair of F-group algebras. Recall that $V(RG)$ is the normalized group of units with p-component $S(RG)$, and $I(RG; A)$ denotes the relative augmentation ideal of RG with respect to A, whenever R is a commutative ring with identity. For the basic background on group rings see [15] and [16].

In the theory of commutative group algebras a central problem is that of deducing information about G from the F-group algebra FG as well as about the group pair (G, A) from the F-pair (FG, FA). The principal known results in this direction may be found in [16], [2], [15], [3-9]. Moreover, of some importance are also the following other invariants of G and (G, A), which are in the focus of our interest, namely:

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(a) For every ordinal α, the α-th defect of A in G is the vector space over the field F_p of p-elements (see [17])

$$D_\alpha(G,A) = (G/A)^{p^\alpha}[p]/(G^{p^\alpha}[p]A)/A.$$

These invariants play a key role in the intersection problem, and shed an information on the purity and isotypity as well.

(b) Let A be purifiable in G. For every integer $n \geq 0$, we define the dimension of $(P/A)^{p^n}[p]$ for the pure hull P of A in G as a vector F_p-space by (cf. [17])

$$\mathrm{Cov}_n(G,A) = \dim_{F_p}(P/A)^{p^n}[p].$$

This cardinal number is called the n-th covering dimension of A in G. It is a relative invariant of A in G. We also set $Def(A) = \dim_{F_p}(N/A)[p]$, where N is a neat hull of A in G and A is neatable in G. If N is pure in G, then N is vertical in $G \Rightarrow Def(A) = \mathrm{Cov}_0(G,A)$.

(c) Let A be purifiable in G. For each natural $n \geq 0$, we put the dimension of $G^{p^n}[p]/P^{p^n}[p]$ by (see [17])

$$\mathrm{Com}_n(G,A) = \dim_{F_p}(G^{p^n}[p]/P^{p^n}[p]).$$

The last cardinal number is called the n-th complemental dimension of A in G. It is a relative invariant of A in G. Besides, since P is pure in G, we trivially detect that $G^{p^n}[p]/P^{p^n}[p] \cong (G/P)^{p^n}[p]$, and therefore

$$\mathrm{Com}_n(G,A) = \dim_{F_p}(G/P)^{p^n}[p].$$

(d) P. Hill in [11] has introduced the following cardinal functions (called Hill numbers or Hill invariants): For μ a limit ordinal not cofinal with $\omega = \omega_0$, set

$$E_\mu = \bigcap_{\substack{\lambda<\mu,\\ \lambda+\sigma=\mu}} (G^{p^\lambda}A/A)^{p^\sigma}/(G^{p^\mu}A/A).$$

Then

$$H_\mu(\lambda) = \begin{cases} \dim(G^{p^\alpha}[p]/G^{p^{\alpha+1}}[p]), & \text{if } \mu = 0 \text{ and } \alpha < \infty \\ \dim(E_\mu^{p^\alpha}[p]/E_\mu^{p^{\alpha+1}}[p]), & \text{if } \mu \neq 0 \text{ and } \alpha < \infty \\ \dim E_\mu^{p^\alpha}[p], & \text{if } \mu \neq 0 \text{ and } \alpha = \infty. \end{cases}$$

(e) We select the relative p-Warfield invariants of A in G with respect to the ordinal α as follows

$$W_{\alpha,p}(G,A) = \dim_{F_p}\big(G^{p^\alpha}/([(G^{p^{\alpha+1}}A) \cap G^{p^\alpha}](G^{p^\alpha})_t)\big).$$

This construction strengthens the classical long-known definition of the ordinary Warfield p-invariants.

We continue with the statement of the major assertions.

2. Main Results

Now we are in position to formulate and prove the attainments on functional invariants for abelian group algebras, motivated this article. Some of them were previously announced in [9]. And so, we start with

Theorem 1 (Invariants). *The following claims are valid:*

(*) *For any ordinal* $\alpha, W_{\alpha,p}(G,A)$ *is an isomorphic cardinal invariant of* (FG,FA).

(**) *For each ordinal* $\alpha, D_\alpha(G,A)$ *and* $H_\mu(\alpha)$ *are structural cardinal invariants for* (FG,FA).

(***) *For every purifiable subgroup* A *of* p*-primary* G, $\mathrm{Cov}_n(G,A)$ *and* $\mathrm{Com}_n(G,A)$ *are functional cardinal invariants of* (FG,FA).

Begin further with a statement consequence.

Proposition 1 (Properties). *Suppose* $(FG,FA) \cong (FH,FB)$ *as pair of* F*-algebras. Then the following hold:*

($^\circ$) *If* A *is pure (isotype) in* G_p, *then* B *is pure (isotype) in* H_p.

($^{\circ\circ}$) *If* A *is purifiable in* G_p, *then* B *is purifiable in* H_p.

($^{\circ\circ\circ}$) *If* A *is an intersection of pure (isotype) subgroups in* G_p, *then* B *is an intersection of pure (isotype) subgroups in* H_p.

We can now attack their proofs, which are demonstrated in the next paragraph.

Proofs of Preliminary and Central Affirmations. First and foremost we list (cf. [3, 4, 6]) a lemma needed for our presentation, namely:

Lemma 1. *Let* $T \leq A \leq G$ *and* $M \leq G$. *Then*

$$I(FG;AM) = I(FG;A) + I(FG;M).$$

Besides for $1 \in P \leq R$, *the following intersection ratio holds true*

$$I(FA;T) \cap PM = I(P(A \cap M); T \cap M).$$

Now, we are ready to begin with the proofs. In fact, we proceed
Proof of: (*)

Since

$$(G^{p^\alpha})_t = (G_t)^{p^\alpha} = (G_p)^{p^\alpha}\left(\coprod_{q\neq p} G_q\right),$$

we observe that

$$[(G^{p^{\alpha+1}}A)\cap G^{p^\alpha}](G^{p^\alpha})_t = [(G^{p^{\alpha+1}}A)\cap G^{p^\alpha}](G^{p^\alpha})_p.$$

Henceforth, we apply the methods from [3, 4, 6] together with the Lemma to conclude that the fundamental ideals $I(FG;G_p^{p^\alpha})$ along with

$$I(FG;G^{p^{\alpha+1}}A) = I(FG;G^{p^{\alpha+1}}) + I(FG;A) \quad \text{and} \quad I(FG;(G^{p^{\alpha+1}}A)\cap G^{p^\alpha})$$

may be recovered by (FG, FA). As a finish, exploiting a result due to Karpilovsky [15], we have that the explored relative p-invariants of Warfield can be recaptured from the F-pair (FG, FA), as wanted.
Proof of: (**)

The fact that $D_\alpha(G, A)$ is an invariant of (FG, FA) follows thus. As we have seen in [3, 4], $I(FG; G^{p^\alpha}[p])$ can be retrieved from FG. On the other hand

$$I(FG; G^{p^\alpha}[p]A) = I(FG; G^{p^\alpha}[p]) + I(FG; A) = I(FG; G^{p^\alpha}[p]) + FG \cdot I(FA; A)$$

may be obtained from (FG, FA) using the Lemma. Furthermore by [15]

$$\dim_{F_p}(G/A)^{p^\alpha}[p]/(G^{p^\alpha}[p]A)/A = \dim_F(I(F(G/A); (G/A)^{p^\alpha}[p])/ \\ /(I(F(G/A); G/A) \cdot I(F(G/A); (G/A)^{p^\alpha}[p]) + I(F(G/A); G^{p^\alpha}[p]A/A))).$$

Since $F(G/A) \cong FG/I(FG; A) = FG/FG \cdot I(FA; A)$ may be gotten by (FG, FA) and moreover

$$I(FG; G^{p^\alpha}[p]A)/I(FG; A) \cong I(F(G/A); G^{p^\alpha}[p]A/A)$$

can be determined also from this pair, the result holds directly by virtue of [3, 4] or [15].

Now, we shall apply the same procedure to get that $H_\mu(\alpha)$ are invariants for the pair (FG, FA). For this purpose it is enough to establish only that $I(F(G/A/G^{p^\mu}A/A); E_\mu^{p^\tau}[p])$ is an invariant of (FG, FA), i.e in other words it is sufficient to verify via [3,4] and [15] that $I(FE_\mu; E_\mu)$ can be recovered from (FG, FA). Indeed, we consider the F-algebra FE_μ. Evidently

$$FE_\mu = \bigcap_{\substack{\lambda<\mu \\ \lambda+\sigma=\mu}} F[(G^{p^\lambda}A/A)^{p^\sigma}/(G^{p^\mu}A/A)].$$

After this, we shall check that $F[(G^{p^\lambda}A/A)^{p^\sigma}/(G^{p^\mu}A/A)]$ may be determined by (FG, FA). Indeed, this follows from noticing that the factor-algebra is isomorphic to

$$F[(G^{p^\lambda}A/A)^{p^\sigma}]/I(F(G^{p^\lambda}A/A)^{p^\sigma}; (G^{p^\mu}A/A)).$$

But,

$$F(G^{p^\lambda}A/A)^{p^\sigma} = [F(G^{p^\lambda}A/A)]^{p^\sigma},$$

and

$$F(G^{p^\lambda}A/A) \cong F(G^{p^\lambda}A)/I(F(G^{p^\lambda}A); A),$$

where

$$F(G^{p^\lambda}A) = FG^{p^\lambda} \cdot FA = (FG)^{p^\lambda} \cdot FA$$

and

$$I(F(G^{p^\lambda}A); A) = F(G^{p^\lambda}A) \cdot I(FA; A).$$

On the other hand

$$F(G^{p^\mu}A/A) \cong F(G^{p^\mu}A)/I(F(G^{p^\mu}A); A),$$

where as above

$$F(G^{p^\mu}A) = FG^{p^\mu} \cdot FA \quad \text{and} \quad I(F(G^{p^\mu}A); A) = FG^{p^\mu} \cdot I(FA; A).$$

So, our claim is substantiated.

Proof of: (***)

Since $FG = FH$, $FA = FB$ and $FP = FM$ for some pure hulls P of A in G and M of B in H respectively (see the constructions below), we detect that the algebras $F(P/A)$ and $F(G/P)$ can be extracted from (FG, FA) and (FH, FB).

The theorem is proved in general after all. □

We now concentrate on the verification of the corollary.

($^\circ$) Since A is isotype in G, we deduce $V(FB) = V(FA)$ is p-isotype in $V(FG) = V(FH)$. Thereby, B as p-isotype in $V(FB)$ must be p-isotype in $V(FH)$ whence it is isotype in H_p.

We give an independent approach to confirm once again ($^\circ$). Exploiting [17] and [18], A is balanced (nice and isotype) in G_p if and only if $D_\alpha(G, A) = 0$ for each ordinal α. But, as we have argued in the Theorem, $D_\alpha(G, A)$ can be gotten from (FG, FA). Besides, A is pure in G_p if and only if $D_n(G, A) = 0$ for all naturals n.

($^{\circ\circ}$) Assume $A \subseteq P$ where P is a minimal pure subgroup of G_p, i.e. P is a pure hull of A in G_p; in other words there is no proper subgroup of P that is pure in G_p. After this, we may presume that F is perfect. By hypothesis, $FG = FH$ and $FA = FB$ for some subgroup $B \leq H_p$. Given $B \subseteq M \subseteq H_p$ so that M is pure in H_p. We search such a minimal group M with this property. Since $A \subseteq S(FA) = S(FB) \subseteq S(FM)$ and since $S(FM)$ is pure in $S(FH) = S(FG)$, it follows at once that $P \subseteq S(FM)$. Henceforth, we choose $M \leq H_p$ on which $FM = FP$. Furthermore,

$$\begin{aligned} M \cap H_p^{p^n} &\subseteq S(FM) \cap S^{p^n}(FH) = S(FP) \cap S^{p^n}(FG) = \\ &= S(FP) \cap S(FG^{p^n}) = S(F(P \cap G^{p^n})) = \\ &= S(FP^{p^n}) = S^{p^n}(FP) = S^{p^n}(FM), \end{aligned}$$

hence

$$M \cap H_p^{p^n} \subseteq S^{p^n}(FM) \cap M = S(FM^{p^n}) \cap M = M^{p^n},$$

for each natural number n, that is M is pure in H_p. Next, if there is $N \subset M$ such that N is pure in H_p, we select $T \leq G_p$ with $FN = FT$. As above, we may infer that T is pure in G_p. Moreover, $T \subseteq FN \subset FM = FP$ whence $T \subset P$, because if $T = P$ we have that $FM = FN$ jointly with $N \subset M$ force $M = N$, which is the desired contradiction. Thereby, M is a minimal pure subgroup of H_p containing B. So, M is a pure hull of B in H_p and consequently B is purifiable in H_p, as expected.

($^{\circ\circ\circ}$) Utilizing [17], for each ordinal number α, $(G^{p^\alpha}[p]A)/A = 1$ yields $(G/A)^{p^\alpha}[p] = 1$. But, owing to our method described above, the two factor-groups may be retrieved from the couple (FG, FA). So, again invoking to [17], the proof of this point is fulfilled.

The proof of the corollary is completed. □

Claim 1. *Assume $P \leq G_p$. Then P is minimal pure in $G_p \Leftrightarrow P$ is minimal pure in $S(FG)$.*

Proof. If there exists a pure subgroup K of $S(FG)$ so that $K \subset P$, we obtain that K must be pure in G_p which contradicts the minimality of P in G_p.

Conversely, if L is a pure subgroup of G_p and is contained in P, the purity of G_p in $S(FG)$ and its transitivity imply that L is pure in $S(FG)$. But this fails owing to the minimality of P in $S(FG)$. □

Corollary 1. *Assume $A \leq G_p$. Then A is purifiable in $G_p \Leftrightarrow A$ is purifiable in $S(FG)$.*

We end the investigation with

Problems. *What are the divisible hull and the pure hull for the group $S(FG)$?*

In [4] we have asked whether or not FG determines G/B_u, where B_u is an upper basic subgroup of G. We now precise this as turn our attention to the question for the existence of invariance of $I(FG; B_u)$ from FG. In that aspect, does it follow that $FG = FH$ implies

$$F(G/H^{(G_p)}) \cong F(H/H^{(H_p)}) \quad \text{and} \quad I(FG; H^{(G_p)}) = I(FH; H^{(H_p)})$$

whenever $H^{(G_p)}$ and $H^{(H_p)}$ are G_p-high and H_p-high subgroups of G and H, respectively. For the convenience of the reader, we emphasize that a subgroup K of G is G_p-high if it is maximal with respect to $\cap G_p = 1$, that is $K[p] = 1$ and K is pure in G (see, for instance, [13] or [14]). Thus if $FG \cong FH$ and G being p-splitting (G_p is a direct factor of G) yield that H is p-splitting, then $FG \cong FH$ assures $FG_p \cong FH_p$.

Let $\mathbb{N}$ be the set of nonnegative integers, and let $B = \oplus_{i \in I} \langle b_i \rangle$ be the direct sum of cyclic groups with order $(b_i) = p^{i+1}$. Denote by B^- the torsion-completion of B. If G is a pure subgroup of B^-, let

$$I(G) = \{i \in \mathbb{N} \mid i^{th}\text{Ulm invariant of } G \text{ is nonzero}\}.$$

Beaumont and Pierce introduced a further invariant for G, archived in [1] (see [19] too), namely

$$U(G) = \{I(A) \mid A \text{ is a pure torsion-complete subgroup of } G\}.$$

Clearly, $U(G)$ is a (boolean) ideal in $P(\mathbb{N})$, the power set of $\mathbb{N}$.

A problem of central interest is whether $U(G)$ is isomorphically retrieved from the F-algebra FG.

However, these are a problem of some other study.

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