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On a Method for Obtaining Iterative Formulas of Higher Order

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Abstract. In this paper a method for obtaining iterative formulas of higher order for finding roots of equations is obtained. These formulas include several already known results.

1. Introduction

Let

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots \tag{1}$$

be an iterative method for finding the root $x = \alpha$ of the real or complex equation $F(x) = 0$.

For the iterative method (1) which converges to $x = \alpha$, we say it is of order k if

$$|x_{n+1} - \alpha| = O(|x_n - \alpha|^k), \quad n \to \infty. \tag{2}$$

If the function $f(x)$ is k times differentiable in a neighborhood of the limit point $x = \alpha$, then the iterative method (1) is of order k if and only if

$$f(\alpha) = \alpha, \quad f'(\alpha) = f''(\alpha) = \cdots = f^{(k-1)}(\alpha) = 0, \quad f^{(k)}(\alpha) \neq 0. \tag{3}$$

This paper deals with a general method for obtaining iterative formulas of higher order.

2. A Theorem for Iterative Formulas of Higher Order

Starting from an iterative method of order $k \geq 1$ for finding the root $x = \alpha$ of the real or complex equation $F(x) = 0$, we give, in this paper, a method for obtaining iterative formulas of order $\geq k + 1$. In this connection the following theorem is proved here.

Theorem 1. *Let* (1) *be an iterative method of order* $k \geq 1$. *Let the function* $f(x)$ *be* $k + 1$ *times differentiable in a neighborhood of the limit point* $x = \alpha$ *and let*

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$f'(\alpha) \neq 1$. Then for the function $h(x)$ k times differentiable in the neighborhood of the limit point $x = \alpha$ such that

$$h(\alpha) = 0 \tag{4}$$

and

$$h'(\alpha) = 1, \tag{5}$$

formula

$$x_{n+1} = f(x_n) - \frac{1}{k} f'(x_n) h(x_n), \quad n = 0, 1, 2, \cdots \tag{6}$$

is an iterative method of order $\geq k + 1$.

Proof. In the method (1) the iteration function is $f(x)$, and in the method (6) the iteration function is

$$g(x) = f(x) - \frac{1}{k} f'(x) h(x). \tag{7}$$

For the function $g(x)$ we shall prove that

$$g(\alpha) = \alpha, \quad g'(\alpha) = g''(\alpha) = \cdots = g^{(k)}(\alpha) = 0. \tag{8}$$

By hypothesis, (1) is an iterative method of order $k \geq 1$ and therefore the relations (3) hold.

From (7) we have

$$g^{(r)}(x) = f^{(r)}(x) - \frac{1}{k}\Big(f^{(r+1)}(x) h(x) + r f^{(r)}(x) h'(x) + \\ + \binom{r}{2} f^{(r-1)}(x) h''(x) + \cdots + f'(x) h^{(r)}(x) \Big). \tag{9}$$

For $k \geq 1$, in view of (3) and (4), we obtain from (7)

$$g(\alpha) = \alpha. \tag{10}$$

Because of (3) and (4), we obtain from (9)

$$g^{(r)}(\alpha) = 0, \qquad textfor \quad 1 \leq r \leq k - 1,$$

that is

$$g'(\alpha) = 0, \; g''(\alpha) = 0, \cdots, g^{(k-1)}(\alpha) = 0. \tag{11}$$

On account of (3), (4) and (5), for $r = k$, we obtain from (9)

$$g^{(k)}(\alpha) = f^{(k)}(\alpha) - \frac{1}{k} \cdot k f^{(k)}(\alpha) = 0. \tag{12}$$

In view of (10), (11) and (12), we conclude that the relations (8) are satisfied for $k \geq 1$, which means that the iterative method (6) is of order $\geq k + 1$. □

3. Some Forms of the Function $h(x)$

Taking for the function $h(x)$ different forms, we can obtain from (6) several particular results. Here we give some forms for the function $h(x)$.

3.1. For

$$h(x) = \frac{u(x)}{u'(x)} v(x), \tag{13}$$

where the functions $u(x)$ and $v(x)$ are $k+1$ times differentiable in a neighborhood of the limit point $x = \alpha$ such that $u(\alpha) = 0$, $u'(\alpha) \neq 0$ and $v(\alpha) = 1$, we have $h(\alpha) = 0$ and $h'(\alpha) = 1$. In this case formula (6) reduces to

$$x_{n+1} = f(x_n) - \frac{1}{k} f'(x_n) \frac{u(x_n)}{u'(x_n)} v(x_n), \quad n = 0, 1, 2, \cdots . \tag{14}$$

For different forms of the function $u(x)$ and $v(x)$, from (14) we can obtain the particular results.

3.1.1. For $u(x) = x - f(x)$ and $v(x) = 1$, where $u(\alpha) = 0$, $u'(\alpha) \neq 0$, from (14) we obtain the iterative method

$$\begin{aligned} x_{n+1} &= f(x_n) - \frac{1}{k} f'(x_n) \frac{x_n - f(x_n)}{1 - f'(x_n)} = \\ &= x_n - \left(1 + \frac{1}{k} \frac{f(x_n)}{1 - f'(x_n)}\right) \big(x_n - f(x_n)\big), \quad n = 0, 1, 2, \cdots , \end{aligned} \tag{15}$$

which is the result obtained in [7].

3.1.2. For $u(x) = x - f(x)$ and

$$v(x) = \frac{1 - f'(x)}{1 - \frac{1}{k} f'(x)},$$

where $u(\alpha) = 0$, $u'(\alpha) \neq 0$ and $v(\alpha) = 1$, from (14) we obtain the iterative method

$$\begin{aligned} x_{n+1} &= f(x_n) - f'(x_n) \frac{x_n - f(x_n)}{k - f'(x_n)} = \\ &= x_n - \frac{x_n - f(x_n)}{1 - \frac{1}{k} f'(x_n)}, \quad n = 0, 1, 2, \cdots , \end{aligned} \tag{16}$$

which is the result obtained by B. Jovanović [4].

3.2. Let $x = \alpha$ is single root of the equation $F(x) = 0$ and let the function $F(x)$ is $k+1$ times differentiable in a neighbourhood of the limit point $x = \alpha$. Then we have $F(\alpha) = 0$ and $F'(\alpha) \neq 0$.

For $u(x) = F(x)$, from (13) we obtain

$$h(x) = \frac{F(x)}{F'(x)} v(x), \tag{17}$$

where $h(\alpha) = 0$ and $h'(\alpha) = 1$. In this case formula (6) reduces to

$$x_{n+1} = f(x_n) - \frac{1}{k} f'(x_n) \frac{F(x_n)}{F'(x_n)} v(x_n), \qquad n = 0, 1, 2, \cdots \tag{18}$$

3.2.1. For $v(x) = 1$, from (18) we obtain the iterative method

$$x_{n+1} = f(x_n) - \frac{1}{k} f'(x_n) \frac{F(x_n)}{F'(x_n)}, \quad n = 0, 1, 2, \cdots \tag{19}$$

3.3. For $h(x) = x - f(x)$ and for $k \geq 2$ we have $h(\alpha) = 0$ and $h'(\alpha) = 1$. In this case formula (6) reduces to

$$\begin{aligned} x_{n+1} &= f(x_n) - \frac{1}{k} f'(x_n)\big(x_n - f(x_n)\big) = \\ &= x_n - \Big(1 + \frac{1}{k} f'(x_n)\Big)\big(x_n - f(x_n)\big), \qquad n = 0, 1, 2, \cdots \end{aligned} \tag{20}$$

which is the result obtained by G. Milovanović [5].

4. EXAMPLES

1) Let (1) be regula falsi, which means

$$x_{n+1} = \frac{aF(x_n) - x_n F(a)}{F(x_n) - F(a)}, \qquad n = 0, 1, 2, \cdots \tag{21}$$

where

$$f(x) = \frac{aF(x) - xF(a)}{F(x) - F(a)}.$$

The method (21) is of order $k = 1$.

For $v(x) = \frac{F(x) - F(a)}{-F(a)}$, where $v(\alpha) = 1$, from (18) we obtain Newton's iterative method of order $k = 2$ for finding of the single root $x = \alpha$ of the equation $F(x) = 0$, namely

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}, \qquad n = 0, 1, 2, \cdots \tag{22}$$

2) If (1) represents Newton's method (22) for finding a single root $x = \alpha$ of the equation $F(x) = 0$, which means that

$$f(x) = x - \frac{F(x)}{F'(x)}$$

and $k = 2$, then we obtain from (18) the iterative method

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \left(1 + \frac{F(x_n)F''(x_n)}{2\big(F'(x_n)\big)^2} v(x_n)\right), \qquad n = 0, 1, 2, \cdots \tag{23}$$

According to Theorem 1, the iterative method (23) is of order $k \geq 3$, since as we know Newton's method (22) is of order 2.

For different forms of the function $v(x)$, from (23) we can obtain particular results.

a) For $v(x)=1$, we obtain from (23)

$$x_{n+1}=x_n-\frac{F(x_n)}{F'(x_n)}\frac{2\big(F'(x_n)\big)^2+F(x_n)F''(x_n)}{2\big(F'(x_n)\big)^2},\qquad n=0,1,2,\cdots \tag{24}$$

which is Chebyshev's method (see [1]).

b) For

$$v(x)=\frac{2(F'(x))^2}{2(F'(x))^2-F(x)F''(x)},$$

we obtain from (23)

$$x_{n+1}=x_n-\frac{F(x_n)}{F'(x_n)}\frac{2(F'(x_n))^2}{2(F'(x_n))^2-F(x_n)F''(x_n)},\qquad n=0,1,2,\cdots \tag{25}$$

which represents Halley's method (see [2, 3]).

c) For

$$v(x)=\frac{(F'(x))^2}{(F'(x))^2-F(x)F''(x)},$$

we obtain from (23)

$$x_{n+1}=x_n-\frac{F(x_n)}{F'(x_n)}\frac{2(F'(x_n))^2-F(x_n)F''(x_n)}{2(F'(x_n))^2-2F(x_n)F''(x_n)},\qquad n=0,1,2,\cdots \tag{26}$$

which is the method obtained in [7].

d) For

$$v(x)=\frac{2}{\left(1-\frac{F(x)F''(x)}{(F'(x))^2}\right)^{\frac12}\left(1+\left(1-\frac{F(x)F''(x)}{(F'(x))^2}\right)^{\frac12}\right)},$$

we obtain from (23)

$$x_{n+1}=x_n-\frac{F(x_n)}{F'(x_n)}\left(1-\frac{F(x_n)F''(x_n)}{(F'(x_n))^2}\right)^{-\frac12},\qquad n=0,1,2,\cdots, \tag{27}$$

which represents Ostrowski's square root method (see [6]).

e) For

$$v(x)=\frac{2m}{\left(1+(m-1)\left(1-\frac{m}{m-1}\frac{F(x)F''(x)}{(F'(x))^2}\right)^{\frac12}\right)\left(1+\left(1-\frac{m}{m-1}\frac{F(x)F''(x)}{(F'(x))^2}\right)^{\frac12}\right)},$$

when $F(x)$ is a polynomial of degree $m\geq 2$, we obtain from (23)

$$x_{n+1}=x_n-\frac{F(x_n)}{F'(x_n)}\frac{m}{1+(m-1)\left(1-\frac{m}{m-1}\frac{F(x_n)F''(x_n)}{(F'(x_n))^2}\right)^{\frac12}},\qquad n=0,1,\cdots \tag{28}$$

which is the Laguerre's method (see [3]).

f) For

$$v(x) = \frac{2(\pm 1)}{\left(\pm\left(1-(\pm 1)\frac{F(x)F''(x)}{(F'(x))^2}\right)^{\frac{1}{2}}\right)\left(1+\left(1-(\pm 1)\frac{F(x)F''(x)}{(F'(x))^2}\right)^{\frac{1}{2}}\right)},$$

where β is fixed finite parameter, we obtain from (23)

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \frac{\pm 1}{\pm\left(1-(\pm 1)\frac{F(x_n)F''(x_n)}{(F'(x_n))^2}\right)^{\frac{1}{2}}}, \qquad n = 0, 1, 2, \cdots \tag{29}$$

which represents a one parameter family of iterative formulas obtained by E. Hansen and M. Patrick [3].

In all previous cases we have $v(\alpha) = 1$.

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