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FIXED POINT THEOREMS ON F_Λ-ORBITALLY COMPLETE NORMED SPACES

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ABSTRACT. Let X be a normed space and $x_0 \in X$. In this paper we proves the convergence of a convex sequence $x_n = \lambda x_{n-1} + (1-\lambda) f(x_{n-1}), \lambda \in (0,1)$, to the fixed point of the f, where $f : X \to X$ is the nonexpansive completely continuous operator, which satisfies some nonexpansive conditions.

Let X be a Banach's space with uniformly convex sphere, E be a closed, bounded and convex subset of X and $f : E \to E$ nonexpansive completely continuous operator. M. A. Krasnoselskij [1] proved that the sequence $x_n = 2^{-1}(x_{n-1} + f(x_{n-1}))$ converges to a fixed point of mapping f, for each $x_0 \in E$. In [2] we considered a fixed point result's for certain mapping, by used convergence of a convex sequence's defined by

$$x_n = \lambda x_{n-1} + (1-\lambda) f(x_n), \lambda \in (0,1). \tag{1}$$

Let X be a vector space, $f : X \to X$ and $x \in X$. Let $\lambda \in (0,1)$ and $O_\lambda(x, f) \subseteq X$ be a set defined by

$$O_\lambda(x,f) = \{g_0(x, f(x)),\ g_1(x, f(x)),\ g_2(x, f(x)), \dots\},$$

where $g_0(x, f(x)) = x$, $g_1(x, f(x)) = \lambda x + (1-\lambda) f(x)$, $g_n(x, f(x)) = g(g_{n-1}(x, f(x)), f(g_{n-1}(x, f(x))))$. Then $O_\lambda(x, f)$ is called convex orbit or **λ-orbit** of the point x defined by f.

Let (X, d) be a metric linear space, $f : X \to X$ and $\lambda \in (0,1)$. X is f_λ-orbitally complete if each Cauchy's sequence from $O_\lambda(x, f)$ is convergent.

Each complete space is λ-orbitally complete, but the inverse statement is not true [3].

Theorem 1. *Let X be a normed space, E be a closed, bounded and convex subset of X, $\lambda \in (0, \frac{1}{2})$, and $f : E \to E$ nonexpansive completely continuous operator. If for each $\lambda \in (0, \frac{1}{2})$ such that X is f_λ-orbitally complete, there exists β, $\frac{2}{1-\lambda} \le \beta \le \frac{2+\lambda}{1-\lambda}$ such that $\beta(\|f(x) - f(y)\| + \|x - y\|)$*

$$\le \|x - f(x)\| + \|y - f(y)\| + \|x - f(y)\| + \|y - f(x)\|, \tag{2}$$

for all $x, y \in E$,then the mapping f has a unique fixed point, which is limit of all sequences defined by (1).

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Proof. If in the equation (2) we put $x = x_{n-1}$ and $y = x_n$, from (1) follows

$$(\beta-1)\|(x_{n-1}-x_n)-\lambda(x_{n-1}-x_n)\|+\beta(1-\lambda)\|x_{n-1}-x_n\| \leq$$
$$\leq (2+\lambda)\|x_{n-1}-x_n\|+\|x_n-x_{n+1}\|,$$

which implies

$$(\beta-1)|\|x_{n+1}-x_n\|-\lambda\|x_{n-1}-x_n\|| \leq$$
$$\leq (2+\lambda-\beta(1-\lambda))\|x_{n-1}-x_n\|+\|x_n-x_{n+1}\|.$$

So we obtain

$$\|x_{n+1}-x_n\| \leq \frac{2-\beta+2\beta\lambda}{\beta-2}\|x_{n-1}-x_n\|.$$

It follows that the sequence (1) is Cauchy's, and since space X for certain $\lambda \in (0, \frac{1}{2})$ is f_λ-orbitally complete and E is a closed and convex subset of X, the sequence (1) converges into E for arbitrary $x_0 \in E$.

Let $\lim_{n\to\infty} x_n = \xi$. If we apply the inequality (2) for $x = \xi$ and $y = x_n$, when $n \to \infty$, we simply get that $\xi = f(\xi)$.

Let ξ and y be two fixed points, and if in the inequality (2) x is replaced by ξ and y is replaced by η,through arranging we get that $(\beta-1)\|\xi-\eta\| \leq 0$, from which it follows that there must be $\xi = \eta$. Theorem 1 is thus proved.

It can be easily checked that the sequence $x_n = f(x_{n-1}), n \in N$ does not converge to a fixed point in Banach's space, for the conditions given in Theorem 1. □

Theorem 2. *Let X be a normed f_λ-orbitally complete space for some $\lambda \in (0,1)$, $E \subseteq X$ its a closed and convex subset and $f : E \to E$. If there exists real numbers α and β such that $\alpha > 2$, $\frac{-1-\lambda}{1-\lambda} \leq \beta < \frac{\alpha-3-(\alpha-1)\lambda}{1-\lambda}$, and for all $x, y \in E$ the following inequality is valid:*

$$\alpha\|f(x)-f(y)\| \leq \beta\|x-y\|+\min\{\|x-f(y)\|,\|x-f(x)\|\} + \min\{\|y-f(x)\|,\|y-f(y)\|\}, \tag{3}$$

then the mapping f has a unique fixed point to which all sequences shaped (1) *converge, for arbitrary $x_0 \in E$.*

Proof. Let

$$\min\{\|x-f(y)\|,\|y-f(x)\|\} = \|x-f(y)\|$$

and

$$\min\{\|y-f(x)\|,\|y-f(y)\|\} = \|y-f(x)\|.$$

For $x = x_{n-1}$, and $y = x_n$ from (3) and (1) we obtain the following inequality

$$|\|x_n-x_{n+1}\|-\lambda\|x_{n-1}-x_n\|| \leq \frac{\beta(1-\lambda)+\lambda+1}{\alpha-1}\|x_n-x_{n-1}\|,$$

which implies

$$\|x_n-x_{n+1}\| \leq \frac{\beta(1-\lambda)+\lambda(\alpha-1)+\lambda+1}{\alpha-1}\|x_n-x_{n-1}\|. \tag{4}$$

We also have

$$(4') \qquad 0 \le \frac{\beta(1-\lambda)+\lambda(\alpha-1)+\lambda+1}{\alpha-1} < 1, \quad \lambda \in (0,1).$$

Let $\min\{\|x-f(y)\|, \|x-f(x)\|\} = \|x-f(y)\|$ and $\min\{\|y-f(x)\|, \|y-f(y)\|\} = \|y-f(y)\|$. For $x = x_{n-1}$, and $y = x_n$, we obtain the inequality

$$(\alpha-1)\,|\|x_n - x_{n+1}\| - \lambda\|x_{n-1}-x_n\|| - \|x_n - x_{n+1}\| \le$$
$$\le (\beta(1-\lambda)+1)\|x_{n-1}-x_n\|.$$

It follows

$$(5) \qquad \|x_n - x_{n+1}\| \le \frac{(\alpha-1)\lambda+\beta(1-\lambda)+1}{\alpha-2}\|x_n - x_{n-1}\|.$$

We also have:

$$(5') \qquad 0 \le \frac{(\alpha-1)\lambda+\beta(1-\lambda)+1}{\alpha-2} < 1, \lambda \in (0,1)$$

Let $\min\{\|x-f(y)\|, \|x-f(x)\|\} = \|x-f(x)\|$ and $\min\{\|y-f(x)\|, \|y-f(y)\|\} = \|y-f(x)\|$. For $x = x_{n-1}$, and $y = x_n$, we get the inequality

$$|\|x_n - x_{n+1}\| - \lambda\|x_{n-1}-x_n\|| \le \frac{\beta(1-\lambda)+\lambda+1}{\alpha}\|x_{n-1}-x_n\|.$$

So

$$(6) \qquad \|x_{n+1} - x_n\| \le \frac{\beta(1-\lambda)+\lambda\alpha+\lambda+1}{\alpha}\|x_{n-1}-x_n\|.$$

We also have

$$(6') \qquad 0 \le \frac{\beta(1-\lambda)+\lambda\alpha+\lambda+1}{\alpha} < 1, \quad \lambda \in (0,1).$$

Let $\min\{\|x-f(y)\|, \|x-f(x)\|\} = \|x-f(x)\|$ and $\min\{\|y-f(x)\|, \|y-f(y)\|\} = \|y-f(y)\|$. For $x = x_{n-1}$, and $y = x_n$, from (1) and (3), we obtain the following inequality

$$\alpha\,|\|x_n - x_{n+1}\| - \lambda\|x_{n-1}-x_n\| - \|x_n - x_{n+1}\|| \le (\beta(1-\lambda)+1)\|x_{n-1}-x_n\|.$$

It follows

$$(7) \qquad \|x_n - x_{n+1}\| \le \frac{\beta(1-\lambda)+\lambda\alpha+1}{\alpha-1}\|x_{n-1}-x_n\|.$$

We also have:

$$(7') \qquad 0 \le \frac{\beta(1-\lambda)+\lambda\alpha+1}{\alpha-1} < 1, \quad \lambda \in (0,1).$$

From the relations (4), (4′), (5), (5′), (6), (6′), (7) and (7′) it follows that the sequence $\{x_n\}_{n\in N}$ defined by (1) is Cauchy's sequences and since space X is f_λ-orbitally complete and E, it converges to a certain point $\xi \in E$, i.e. $\lim_{n\to\infty} x_n = \xi$.

For $x = \xi$ and $y = x_n$, when $n \to \infty$ we get that $\alpha\|\xi - f(\xi)\| \le 0$. It follows that $\xi = f(\xi)$ because $\alpha > 2$.

Let ξ and y be two fixed points and if in relation (3) we replace x by ξ and y by η, we get $(\alpha - \beta)\|\xi - \eta\| < 0$. It follows that $\xi = \eta$ because $\alpha - \beta > 0$. This proves Theorem 2. □

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