

## JORDAN HIGHER DERIVATIONS ON PRIME HILBERT $C^*$ -MODULES

SAYED KHALIL EKRAMI

**ABSTRACT.** Let  $\mathcal{M}$  be a Hilbert  $C^*$ -module. A sequence of linear mappings  $\{\varphi_n : \mathcal{M} \rightarrow \mathcal{M}\}_{n=0}^{+\infty}$  with  $\varphi_0 = I$ , is said to be a Hilbert  $C^*$ -module Jordan higher derivation on  $\mathcal{M}$ , if it satisfies the equation

$$\varphi_n(\langle a, b \rangle a) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b) \rangle \varphi_k(a),$$

for all  $a, b \in \mathcal{M}$  and each non-negative integer  $n$ . In this paper, we show that, if  $\mathcal{M}$  is prime, then every Hilbert  $C^*$ -module Jordan higher derivation  $\{\varphi_n\}_{n=0}^{+\infty}$  on  $\mathcal{M}$ , is a Hilbert  $C^*$ -module higher derivation on  $\mathcal{M}$ . As a consequence, we show that every Hilbert  $C^*$ -module Jordan derivation on  $\mathcal{M}$ , is a Hilbert  $C^*$ -module derivation on  $\mathcal{M}$ .

### 1. INTRODUCTION

The notion of a Hilbert  $C^*$ -module initiated as a generalization of a Hilbert space in which the inner product takes its values in a  $C^*$ -algebra (see [13]). Let  $\mathcal{A}$  be a  $C^*$ -algebra. An inner product  $\mathcal{A}$ -module is a complex linear space  $\mathcal{M}$  which is a left  $\mathcal{A}$ -module with compatible scalar multiplication  $\lambda(ax) = (\lambda a)x = a(\lambda x)$  ( $\lambda \in \mathbb{C}, x \in \mathcal{M}, a \in \mathcal{A}$ ), together with an  $\mathcal{A}$ -valued inner product  $(x, y) \mapsto \langle x, y \rangle : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{A}$  such that for each  $x, y, z \in \mathcal{M}$ ,  $\alpha, \beta \in \mathbb{C}$  and  $a \in \mathcal{A}$ ,

- (i)  $\langle x, x \rangle \geq 0$  and the equality holds if and only if  $x = 0$ ;
- (ii)  $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ ;
- (iii)  $\langle ax, y \rangle = a \langle x, y \rangle$ ;

---

*Key words and phrases.* Derivation, Jordan derivation, higher derivation, Jordan higher derivation, Hilbert  $C^*$ -module.

2020 *Mathematics Subject Classification.* Primary: 46L08. Secondary: 16W25.  
<https://doi.org/10.46793/KgJMat2605.817E>

*Received:* June 08, 2023.

*Accepted:* May 10, 2024.

$$(iv) \langle x, y \rangle^* = \langle y, x \rangle.$$

It follows from the above conditions that  $\langle x, x \rangle$  is a positive element in  $C^*$ -algebra  $\mathcal{A}$ , the inner product is conjugate-linear in its second variable and  $\langle x, ay \rangle = \langle x, y \rangle a^*$  for all  $x, y \in \mathcal{M}$  and  $a \in \mathcal{A}$ . An inner product  $\mathcal{A}$ -module  $\mathcal{M}$  which is complete with respect to the norm  $\|x\|_{\mathcal{M}} = \|\langle x, x \rangle\|_{\mathcal{A}}^{\frac{1}{2}}$  is called a Hilbert  $\mathcal{A}$ -module or a Hilbert  $C^*$ -module over the  $C^*$ -algebra  $\mathcal{A}$ . For example, every  $C^*$ -algebra  $\mathcal{A}$  is a Hilbert  $\mathcal{A}$ -module under the  $\mathcal{A}$ -valued inner product  $\langle a, b \rangle = ab^*$  ( $a, b \in \mathcal{A}$ ). Every complex Hilbert space is a left Hilbert  $\mathbb{C}$ -module. The notion of a right Hilbert  $\mathcal{A}$ -module can be defined similarly.

A Hilbert  $C^*$ -module  $\mathcal{M}$  is said to be *prime*, if for elements  $a, b$  of  $\mathcal{M}$ ,  $\langle a, \mathcal{M} \rangle b = 0$  implies that  $a = 0$  or  $b = 0$ . Equivalently,  $\mathcal{M}$  is called prime, if for elements  $a, b$  of  $\mathcal{M}$ , validity the equation  $\langle a, x \rangle b = 0$  for all  $x \in \mathcal{M}$ , implies that  $a = 0$  or  $b = 0$ .  $\mathcal{M}$  is said to be *semiprime*, if  $\langle a, \mathcal{M} \rangle a = 0$  implies that  $a = 0$ . Trivially any prime Hilbert  $C^*$ -module  $\mathcal{M}$  is semiprime.

Let  $\mathcal{M}$  and  $\mathcal{N}$  be Hilbert  $C^*$ -modules over a  $C^*$ -algebra  $\mathcal{A}$ . A mapping  $T : \mathcal{M} \rightarrow \mathcal{N}$  is said to be adjointable, if there exists a mapping  $S : \mathcal{N} \rightarrow \mathcal{M}$  such that  $\langle T(x), y \rangle = \langle x, S(y) \rangle$  for all  $x \in D_T \subseteq \mathcal{M}$ ,  $y \in D_S \subseteq \mathcal{N}$ . The unique mapping  $S$  is denoted by  $T^*$  and is called the adjoint of  $T$ . It is well known that any adjointable mapping  $T : \mathcal{M} \rightarrow \mathcal{N}$  is  $\mathcal{A}$ -linear (that is  $T(ax + \lambda y) = aT(x) + \lambda T(y)$  for all  $x, y \in \mathcal{M}$ ,  $a \in \mathcal{A}$ ,  $\lambda \in \mathbb{C}$ ) and bounded.

A linear mapping  $\psi : \mathcal{M} \rightarrow \mathcal{M}$  is called a *Hilbert  $C^*$ -module derivation* on  $\mathcal{M}$ , if it satisfies the equation

$$\psi(\langle a, b \rangle c) = \langle \psi(a), b \rangle c + \langle a, \psi(b) \rangle c + \langle a, b \rangle \psi(c),$$

for all  $a, b, c \in \mathcal{M}$ .  $\psi$  is called a *Hilbert  $C^*$ -module Jordan derivation* on  $\mathcal{M}$ , if it satisfies the equation

$$\psi(\langle a, b \rangle a) = \langle \psi(a), b \rangle a + \langle a, \psi(b) \rangle a + \langle a, b \rangle \psi(a),$$

for all  $a, b \in \mathcal{M}$ . Note that every Hilbert  $C^*$ -module derivation is a Hilbert  $C^*$ -module Jordan derivation. But the converse is not true in general.

*Remark 1.1.* Every adjointable mapping  $\psi : \mathcal{M} \rightarrow \mathcal{M}$  satisfying  $\psi^* = -\psi$  is a Hilbert  $C^*$ -module derivation. Infact if  $\psi^* = -\psi$ , then  $\langle \psi(a), b \rangle c + \langle a, \psi(b) \rangle c = 0$  for all  $a, b, c \in \mathcal{M}$ . Moreover

$$\begin{aligned} \langle \psi(\langle a, b \rangle c), x \rangle &= \langle \langle a, b \rangle c, \psi^*(x) \rangle = \langle a, b \rangle \langle c, \psi^*(x) \rangle = \langle a, b \rangle \langle \psi(c), x \rangle \\ &= \langle \langle a, b \rangle \psi(c), x \rangle, \end{aligned}$$

for all  $a, b, c, x \in \mathcal{M}$  which implies that  $\psi(\langle a, b \rangle c) = \langle a, b \rangle \psi(c)$  for all  $a, b, c \in \mathcal{M}$ .

*Example 1.1.* Let  $M_2(\mathbb{C})$  be the  $C^*$ -algebra of  $2 \times 2$  complex matrices. The mapping  $\psi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$  defined by

$$\psi(A) = \begin{bmatrix} a_{21} & a_{22} \\ -a_{11} & -a_{12} \end{bmatrix},$$

for all  $A = [a_{ij}] \in M_2(\mathbb{C})$ , is a Hilbert  $C^*$ -module derivation on  $M_2(\mathbb{C})$ .

A sequence of linear mappings  $\{\varphi_n : \mathcal{M} \rightarrow \mathcal{M}\}_{n=0}^{+\infty}$ , with  $\varphi_0 = I$  (the identity mapping on  $\mathcal{M}$ ) is called a Hilbert  $C^*$ -module higher derivation on  $\mathcal{M}$ , if it satisfies the equation

$$\varphi_n(\langle a, b \rangle c) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b) \rangle \varphi_k(c),$$

for all  $a, b, c \in \mathcal{M}$  and each non-negative integer  $n$ .

*Example 1.2.* Let  $\psi$  be a Hilbert  $C^*$ -module derivation on  $\mathcal{M}$ . Then the sequence  $\{\varphi_n\}_{n=0}^{+\infty}$  of linear mappings on  $\mathcal{M}$  defined by  $\varphi_0 = I$  and

$$\varphi_n(\langle a, b \rangle c) = \sum_{\substack{i+j+k=n \\ 0 \leq i, j, k \leq n}} \frac{1}{i!j!k!} \langle \psi^i(a), \psi^j(b) \rangle \psi^k(c),$$

for all  $a, b, c \in \mathcal{M}$  and each non-negative integer  $n$  (in which  $\psi^0 = I$ ), is a Hilbert  $C^*$ -module higher derivation on  $\mathcal{M}$ . The four terms of this Hilbert  $C^*$ -module higher derivation are

$$\begin{aligned} \varphi_0(\langle a, b \rangle c) &= \langle a, b \rangle c, \\ \varphi_1(\langle a, b \rangle c) &= \langle \psi(a), b \rangle c + \langle a, \psi(b) \rangle c + \langle a, b \rangle \psi(c), \\ \varphi_2(\langle a, b \rangle c) &= \frac{1}{2} \langle \psi^2(a), b \rangle c + \frac{1}{2} \langle a, \psi^2(b) \rangle c + \frac{1}{2} \langle a, b \rangle \psi^2(c) \\ &\quad + \langle \psi(a), \psi(b) \rangle c + \langle \psi(a), b \rangle \psi(c) + \langle a, \psi(b) \rangle \psi(c), \\ \varphi_3(\langle a, b \rangle c) &= \frac{1}{6} \langle \psi^3(a), b \rangle c + \frac{1}{6} \langle a, \psi^3(b) \rangle c + \frac{1}{6} \langle a, b \rangle \psi^3(c) \\ &\quad + \frac{1}{2} \langle \psi^2(a), \psi(b) \rangle c + \frac{1}{2} \langle \psi^2(a), b \rangle \psi(c) + \frac{1}{2} \langle \psi(a), \psi^2(b) \rangle c \\ &\quad + \frac{1}{2} \langle a, \psi^2(b) \rangle \psi(c) + \frac{1}{2} \langle \psi(a), b \rangle \psi^2(c) + \frac{1}{2} \langle a, \psi(b) \rangle \psi^2(c) \\ &\quad + \langle \psi(a), \psi(b) \rangle \psi(c). \end{aligned}$$

A sequence of linear mappings  $\{\varphi_n : \mathcal{M} \rightarrow \mathcal{M}\}_{n=0}^{+\infty}$ , with  $\varphi_0 = I$ , is called a Hilbert  $C^*$ -module Jordan higher derivation on  $\mathcal{M}$ , if

$$\varphi_n(\langle a, b \rangle a) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b) \rangle \varphi_k(a),$$

for all  $a, b \in \mathcal{M}$  and each non-negative integer  $n$ .

When  $\{\varphi_n\}_{n=0}^{+\infty}$  is a Hilbert  $C^*$ -module higher derivation (Jordan higher derivation),  $\varphi_1$  is a Hilbert  $C^*$ -module derivation (Jordan derivation). Trivially every Hilbert  $C^*$ -module higher derivation is a Hilbert  $C^*$ -module Jordan higher derivation. But the converse is not true in general.

The classical result due to Herstein [11] was extended for higher derivations by Haetinger [9], who proved that every Jordan higher derivation on a prime ring of characteristic different from two is a higher derivation. Further, Ferrero and Haetinger

[8] established that on a 2-torsion free semiprime ring every Jordan triple higher derivation, is a higher derivation. In this paper we prove that if  $\mathcal{M}$  is a prime Hilbert  $C^*$ -module, then every Hilbert  $C^*$ -module Jordan higher derivation on  $\mathcal{M}$ , is a Hilbert  $C^*$ -module higher derivation on  $\mathcal{M}$ . As a consequence, we show that every Hilbert  $C^*$ -module Jordan derivation on  $\mathcal{M}$ , is a Hilbert  $C^*$ -module derivation on  $\mathcal{M}$ .

For more information about Hilbert  $C^*$ -module derivations and Hilbert  $C^*$ -module higher derivations the reader can see [6, 16]. Also for information about derivations and higher derivations on algebras, the reader refer to [1–5, 7, 10, 12, 14, 15, 17, 18].

## 2. THE RESULT

Let  $\mathcal{M}$  be a Hilbert  $C^*$ -module and  $I$  be the identity mapping on  $\mathcal{M}$ . A sequence of linear mappings  $\{\varphi_n : \mathcal{M} \rightarrow \mathcal{M}\}_{n=0}^{+\infty}$ , with  $\varphi_0 = I$ , is said to be a

(i) *Hilbert  $C^*$ -module higher derivation* on  $\mathcal{M}$ , if it satisfies the equation

$$(2.1) \quad \varphi_n(\langle a, b \rangle c) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b) \rangle \varphi_k(c),$$

for all  $a, b, c \in \mathcal{M}$  and each non-negative integer  $n$ ;

(ii) *Hilbert  $C^*$ -module Jordan higher derivation* on  $\mathcal{M}$ , if it satisfies the equation

$$(2.2) \quad \varphi_n(\langle a, b \rangle a) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b) \rangle \varphi_k(a),$$

for all  $a, b \in \mathcal{M}$  and each non-negative integer  $n$ .

Trivially every Hilbert  $C^*$ -module higher derivation is a Hilbert  $C^*$ -module Jordan higher derivation. But the converse is not true in general. In this section, we prove that on a prime Hilbert  $C^*$ -module  $\mathcal{M}$ , every Hilbert  $C^*$ -module Jordan higher derivation is a Hilbert  $C^*$ -module higher derivation. Before proving the result, we need some lemmas.

**Lemma 2.1.** *Let  $\mathcal{M}$  be a Hilbert  $C^*$ -module and  $\{\varphi_n : \mathcal{M} \rightarrow \mathcal{M}\}_{n=0}^{+\infty}$  be a Hilbert  $C^*$ -module Jordan higher derivation on  $\mathcal{M}$ . Then,*

$$(2.3) \quad \varphi_n(\langle a, b \rangle c + \langle c, b \rangle a) = \sum_{i+j+k=n} (\langle \varphi_i(a), \varphi_j(b) \rangle \varphi_k(c) + \langle \varphi_i(c), \varphi_j(b) \rangle \varphi_k(a)),$$

for all  $a, b, c \in \mathcal{M}$  and each non-negative integer  $n$ .

*Proof.* Replacing  $a$  by  $a + c$  in (2.2), we get

$$\varphi_n(\langle a + c, b \rangle (a + c)) = \sum_{i+j+k=n} \langle \varphi_i(a + c), \varphi_j(b) \rangle \varphi_k(a + c),$$

which implies that

$$\begin{aligned} & \varphi_n(\langle a, b \rangle a + \langle c, b \rangle a + \langle a, b \rangle c + \langle c, b \rangle c) \\ &= \sum_{i+j+k=n} (\langle \varphi_i(a), \varphi_j(b) \rangle \varphi_k(a) + \langle \varphi_i(a), \varphi_j(b) \rangle \varphi_k(c) \\ & \quad + \langle \varphi_i(c), \varphi_j(b) \rangle \varphi_k(a) + \langle \varphi_i(c), \varphi_j(b) \rangle \varphi_k(c)), \end{aligned}$$

for all  $a, b, c \in \mathcal{M}$ . Since  $\varphi_n(\langle a, b \rangle a) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b) \rangle \varphi_k(a)$  and  $\varphi_n(\langle c, b \rangle c) = \sum_{i+j+k=n} \langle \varphi_i(c), \varphi_j(b) \rangle \varphi_k(c)$ , canceling these terms from both sides of the above equation, we get the equation (2.3).  $\square$

**Lemma 2.2.** *Let  $\mathcal{M}$  be a 2-torsion-free semiprime Hilbert  $C^*$ -module and  $a, b \in \mathcal{M}$ . If  $\langle a, x \rangle b + \langle b, x \rangle a = 0$  for all  $x \in \mathcal{M}$ , then  $\langle a, x \rangle b = \langle b, x \rangle a = 0$  for all  $x \in \mathcal{M}$ . If  $\langle a, x \rangle b = 0$  for all  $x \in \mathcal{M}$ , then  $\langle b, x \rangle a = 0$  for all  $x \in \mathcal{M}$ .*

*Proof.* Let  $a, b \in \mathcal{M}$ . Suppose that  $\langle a, x \rangle b + \langle b, x \rangle a = 0$  for all  $x \in \mathcal{M}$ . Then, we have

$$\begin{aligned} \langle \langle a, x \rangle b, y \rangle \langle a, x \rangle b &= -\langle \langle b, x \rangle a, y \rangle \langle a, x \rangle b = -\langle b, x \rangle \langle a, y \rangle \langle a, x \rangle b = -\langle b, \langle y, a \rangle x \rangle \langle a, x \rangle b \\ &= -\langle \langle b, \langle y, a \rangle x \rangle a, x \rangle b = \langle \langle a, \langle y, a \rangle x \rangle b, x \rangle b = \langle \langle a, x \rangle \langle a, y \rangle b, x \rangle b \\ &= \langle a, x \rangle \langle a, y \rangle \langle b, x \rangle b = \langle a, x \rangle \langle \langle a, y \rangle b, x \rangle b = -\langle a, x \rangle \langle \langle b, y \rangle a, x \rangle b \\ &= -\langle a, x \rangle \langle b, y \rangle \langle a, x \rangle b = -\langle \langle a, x \rangle b, y \rangle \langle a, x \rangle b, \end{aligned}$$

for all  $y \in \mathcal{M}$ , which implies that  $\langle \langle a, x \rangle b, y \rangle \langle a, x \rangle b = 0$  for all  $y \in \mathcal{M}$ . Since  $\mathcal{M}$  is semiprime, we get  $\langle a, x \rangle b = 0$  and so  $\langle b, x \rangle a = 0$  for all  $x \in \mathcal{M}$ .

Now suppose that  $\langle a, x \rangle b = 0$  for all  $x \in \mathcal{M}$ . Then, we have

$$\langle \langle b, x \rangle a, y \rangle \langle b, x \rangle a = \langle b, x \rangle \langle a, y \rangle \langle b, x \rangle a = \langle b, x \rangle \langle \langle a, y \rangle b, x \rangle a = 0,$$

for all  $y \in \mathcal{M}$ . Then semiprimeness of  $\mathcal{M}$  implies that  $\langle b, x \rangle a = 0$  for all  $x \in \mathcal{M}$ .  $\square$

**Lemma 2.3.** *Let  $\mathcal{M}$  be a 2-torsion-free Hilbert  $C^*$ -module. Then the following conditions are equivalent.*

- (i)  $\mathcal{M}$  is a prime Hilbert  $C^*$ -module.
- (ii) For  $a, b \in \mathcal{M}$ , validity of  $\langle a, x \rangle b + \langle b, x \rangle a = 0$  for all  $x \in \mathcal{M}$ , implies that  $a = 0$  or  $b = 0$ .
- (iii) For  $a, b \in \mathcal{M}$ , validity of  $\langle a, x \rangle a = \langle b, x \rangle b$  for all  $x \in \mathcal{M}$ , implies that  $a = b$  or  $a = -b$ .

*Proof.* (i) $\Rightarrow$ (ii) If  $\mathcal{M}$  is a prime Hilbert  $C^*$ -module and  $\langle a, x \rangle b + \langle b, x \rangle a = 0$  for all  $x \in \mathcal{M}$ , then by Lemma 2.2,  $\langle a, x \rangle b = 0$  for all  $x \in \mathcal{M}$  and then by primeness of  $\mathcal{M}$ ,  $a = 0$  or  $b = 0$ .

(ii) $\Rightarrow$ (i) Suppose that  $\langle a, x \rangle b = 0$  for all  $x \in \mathcal{M}$ . Then  $\langle \langle b, x \rangle a, y \rangle \langle b, x \rangle a = \langle b, x \rangle \langle a, y \rangle \langle b, x \rangle a = \langle b, x \rangle \langle \langle a, y \rangle b, x \rangle a = 0$  which implies that  $\langle b, x \rangle a = 0$  for all  $x \in \mathcal{M}$ . Hence  $\langle a, x \rangle b + \langle b, x \rangle a = 0$  for all  $x \in \mathcal{M}$  and therefore  $a = 0$  or  $b = 0$ . Thus,  $\mathcal{M}$  is a prime.

(ii) $\Rightarrow$ (iii) Let  $\langle a, x \rangle a = \langle b, x \rangle b$  for all  $x \in \mathcal{M}$ . Then  $\langle a - b, x \rangle (a + b) + \langle a + b, x \rangle (a - b) = 0$  for all  $x \in \mathcal{M}$ . Thus,  $a - b = 0$  or  $a + b = 0$ .

(iii) $\Rightarrow$ (ii) Let  $\langle a, x \rangle b + \langle b, x \rangle a = 0$  for all  $x \in \mathcal{M}$ . Then,  $\langle a - b, x \rangle (a - b) = \langle a + b, x \rangle (a + b)$  for all  $x \in \mathcal{M}$ . Hence,  $a - b = a + b$  or  $a - b = -(a + b)$ . That is  $a = 0$  or  $b = 0$ .  $\square$

**Lemma 2.4.** *Let  $\mathcal{M}$  be a 2-torsion-free semiprime Hilbert  $C^*$ -module and  $\Delta, \Omega : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$  be mappings which are additive in each variable and  $\Delta(a, b, a) = \Omega(a, b, a) = 0$  for all  $a, b \in \mathcal{M}$ . If*

$$(2.4) \quad \langle \Delta(a, b, c), x \rangle \Omega(a, b, c) = 0,$$

*for all  $a, b, c, x \in \mathcal{M}$ , then  $\langle \Delta(a, b, c), x \rangle \Omega(c, b, a) = 0$  for all  $a, b, c, x \in \mathcal{M}$ .*

*Proof.* Suppose that  $\langle \Delta(a, b, c), x \rangle \Omega(a, b, c) = 0$  for all  $a, b, c, x \in \mathcal{M}$ . Then, by Lemma 2.2, we get  $\langle \Omega(a, b, c), x \rangle \Delta(a, b, c) = 0$  for all  $a, b, c, x \in \mathcal{M}$ .

Replacing  $a$  and  $c$  by  $a + c$  in (2.4), we have

$$\langle \Delta(a + c, b, a + c), x \rangle \Omega(a + c, b, a + c) = 0,$$

which implies that

$$\langle \Delta(a, b, c), x \rangle \Omega(c, b, a) + \langle \Delta(c, b, a), x \rangle \Omega(a, b, c) = 0,$$

for all  $a, b, c, x \in \mathcal{M}$ . It follows from

$$\begin{aligned} & \langle \langle \Delta(a, b, c), x \rangle \Omega(c, b, a), y \rangle \langle \Delta(a, b, c), x \rangle \Omega(c, b, a) \\ &= - \langle \langle \Delta(a, b, c), x \rangle \Omega(c, b, a), y \rangle \langle \Delta(c, b, a), x \rangle \Omega(a, b, c) \\ &= - \langle \Delta(a, b, c), x \rangle \langle \Omega(c, b, a), y \rangle \langle \Delta(c, b, a), x \rangle \Omega(a, b, c) \\ &= - \langle \Delta(a, b, c), x \rangle \langle \langle \Omega(c, b, a), y \rangle \Delta(c, b, a), x \rangle \Omega(a, b, c) = 0, \end{aligned}$$

and semiprimeness of  $\mathcal{M}$  that  $\langle \Delta(a, b, c), x \rangle \Omega(c, b, a) = 0$  for all  $a, b, c, x \in \mathcal{M}$ .  $\square$

**Lemma 2.5.** *Let  $\mathcal{M}$  be a Hilbert  $C^*$ -module. Then for all  $a, b, c, x \in \mathcal{M}$  we have*

$$\left\langle a, \left\langle b, \langle c, x \rangle c \right\rangle b \right\rangle a = \left\langle \langle a, b \rangle c, x \right\rangle \langle c, b \rangle a.$$

*Proof.* Let  $a, b, c, x \in \mathcal{M}$ , then

$$\begin{aligned} \left\langle a, \left\langle b, \langle c, x \rangle c \right\rangle b \right\rangle a &= \left\langle a, \langle b, c \rangle \langle x, c \rangle b \right\rangle a = \left\langle a, \langle x, c \rangle b \right\rangle \langle c, b \rangle a \\ &= \langle a, b \rangle \langle c, x \rangle \langle c, b \rangle a = \left\langle \langle a, b \rangle c, x \right\rangle \langle c, b \rangle a. \end{aligned} \quad \square$$

**Theorem 2.1.** *Let  $\mathcal{M}$  be a 2-torsion-free prime Hilbert  $C^*$ -module. Then, every Hilbert  $C^*$ -module Jordan higher derivation on  $\mathcal{M}$  is a Hilbert  $C^*$ -module higher derivation on  $\mathcal{M}$ .*

*Proof.* Let  $\{\varphi_n\}_{n=0}^{+\infty}$  be a Hilbert  $C^*$ -module Jordan higher derivation on  $\mathcal{M}$  and  $a, b, c \in \mathcal{M}$ . Define

$$(2.5) \quad \Delta_n(a, b, c) := \varphi_n(\langle a, b \rangle c) - \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b) \rangle \varphi_k(c),$$

for each non-negative integer  $n$  and  $\Omega(a, b, c) := \langle a, b \rangle c - \langle c, b \rangle a$ . Trivially  $\Delta_n(a, b, a) = \Omega(a, b, a) = 0$  for all  $n \in \mathbb{N}$ ,  $\Delta_n(a, b, c) + \Delta_n(c, b, a) = 0$  and  $\Omega(a, b, c) + \Omega(c, b, a) = 0$ .

We have

$$\begin{aligned}
S &= \varphi_n \left( \langle a, \langle b, \langle c, x \rangle c \rangle b \rangle a + \langle c, \langle b, \langle a, x \rangle a \rangle b \rangle c \right) \\
&= \sum_{i+j+k=n} \left( \langle \varphi_i(a), \varphi_j(\langle b, \langle c, x \rangle c \rangle b) \rangle \varphi_k(a) + \langle \varphi_i(c), \varphi_j(\langle b, \langle a, x \rangle a \rangle b) \rangle \varphi_k(c) \right) \\
&= \sum_{i+p+q+r+k=n} \left( \langle \varphi_i(a), \langle \varphi_p(b), \varphi_q(\langle c, x \rangle c) \rangle \varphi_r(b) \rangle \varphi_k(a) \right. \\
&\quad \left. + \langle \varphi_i(c), \langle \varphi_p(b), \varphi_q(\langle a, x \rangle a) \rangle \varphi_r(b) \rangle \varphi_k(c) \right) \\
&= \sum_{i+p+s+t+u+r+k=n} \left( \langle \varphi_i(a), \langle \varphi_p(b), \langle \varphi_s(c), \varphi_t(x) \rangle \varphi_u(c) \rangle \varphi_r(b) \rangle \varphi_k(a) \right. \\
&\quad \left. + \langle \varphi_i(c), \langle \varphi_p(b), \langle \varphi_s(a), \varphi_t(x) \rangle \varphi_u(a) \rangle \varphi_r(b) \rangle \varphi_k(c) \right) \\
&= \sum_{i+p+s+t+u+r+k=n} \left( \langle \langle \varphi_i(a), \varphi_p(b) \rangle \varphi_s(c), \varphi_t(x) \rangle \langle \varphi_u(c), \varphi_r(b) \rangle \varphi_k(a) \right. \\
&\quad \left. + \langle \langle \varphi_i(c), \varphi_p(b) \rangle \varphi_s(a), \varphi_t(x) \rangle \langle \varphi_u(a), \varphi_r(b) \rangle \varphi_k(c) \right),
\end{aligned}$$

for all  $x \in \mathcal{M}$ . On the other hand, using Lemmas 2.5 and 2.1, we get

$$\begin{aligned}
S &= \varphi_n \left( \langle \langle a, b \rangle c, x \rangle \langle c, b \rangle a + \langle \langle c, b \rangle a, x \rangle \langle a, b \rangle c \right) \\
&= \sum_{i+j+k=n} \left( \langle \varphi_i(\langle a, b \rangle c), \varphi_j(x) \rangle \varphi_k(\langle c, b \rangle a) + \langle \varphi_i(\langle c, b \rangle a), \varphi_j(x) \rangle \varphi_k(\langle a, b \rangle c) \right),
\end{aligned}$$

for all  $x \in \mathcal{M}$ . It follows from above equations that

$$\begin{aligned}
(2.6) \quad & \sum_{i+j+k=n} \left( \langle \varphi_i(\langle a, b \rangle c), \varphi_j(x) \rangle \varphi_k(\langle c, b \rangle a) + \langle \varphi_i(\langle c, b \rangle a), \varphi_j(x) \rangle \varphi_k(\langle a, b \rangle c) \right) \\
&= \sum_{i+p+s+t+u+r+k=n} \left( \langle \langle \varphi_i(a), \varphi_p(b) \rangle \varphi_s(c), \varphi_t(x) \rangle \langle \varphi_u(c), \varphi_r(b) \rangle \varphi_k(a) \right. \\
&\quad \left. + \langle \langle \varphi_i(c), \varphi_p(b) \rangle \varphi_s(a), \varphi_t(x) \rangle \langle \varphi_u(a), \varphi_r(b) \rangle \varphi_k(c) \right),
\end{aligned}$$

for all  $x \in \mathcal{M}$ .

Now we use induction on  $n$ . Putting  $n = 1$  in the above equation and canceling the like terms from both sides of this equation and then arranging them, we get

$$\begin{aligned}
& \langle \Delta_1(a, b, c), x \rangle \langle c, b \rangle a + \langle \langle c, b \rangle a, x \rangle \Delta_1(a, b, c) \\
& + \langle \Delta_1(c, b, a), x \rangle \langle a, b \rangle c + \langle \langle a, b \rangle c, x \rangle \Delta_1(c, b, a) = 0,
\end{aligned}$$

for all  $x \in \mathcal{M}$ . Since  $\Delta_1(c, b, a) = -\Delta_1(a, b, c)$ , we get

$$\begin{aligned}
& \langle \Delta_1(a, b, c), x \rangle \langle c, b \rangle a + \langle \langle c, b \rangle a, x \rangle \Delta_1(a, b, c) \\
& - \langle \Delta_1(a, b, c), x \rangle \langle a, b \rangle c - \langle \langle a, b \rangle c, x \rangle \Delta_1(a, b, c) = 0,
\end{aligned}$$

which implies that

$$\langle \Delta_1(a, b, c), x \rangle \Omega(c, b, a) + \langle \Omega(c, b, a), x \rangle \Delta_1(a, b, c) = 0,$$

for all  $x \in \mathcal{M}$  and since  $\Omega(c, b, a) = -\Omega(a, b, c)$ , then

$$\langle \Delta_1(a, b, c), x \rangle \Omega(a, b, c) + \langle \Omega(a, b, c), x \rangle \Delta_1(a, b, c) = 0,$$

for all  $x \in \mathcal{M}$ . Since  $\mathcal{M}$  is semiprime, it follows from Lemma 2.2, that

$$\langle \Delta_1(a, b, c), x \rangle \Omega(a, b, c) = \langle \Omega(a, b, c), x \rangle \Delta_1(a, b, c) = 0,$$

for all  $x \in \mathcal{M}$ . Since  $\mathcal{M}$  is prime, it follows from Lemma 2.3 that  $\Delta_1(a, b, c) = 0$  or  $\Omega(a, b, c) = 0$ . If  $\Delta_1(a, b, c) = 0$ , then  $\varphi_1(\langle a, b \rangle c) = \langle \varphi_1(a), b \rangle c + \langle a, \varphi_1(b) \rangle c + \langle a, b \rangle \varphi_1(c)$ , and so  $\varphi_1$  is a Hilbert  $C^*$ -module derivation. If  $\Omega(a, b, c) = 0$ , then  $\langle a, b \rangle c = \langle c, b \rangle a$ . Thus it follows from Lemma 2.1 that  $\varphi_1$  is a Hilbert  $C^*$ -module derivation.

Now suppose that for all  $1 \leq \ell \leq n-1$ ,  $\varphi_\ell$  satisfies the equation

$$(2.7) \quad \varphi_\ell(\langle a, b \rangle c) = \sum_{i+j+k=\ell} \langle \varphi_i(a), \varphi_j(b) \rangle \varphi_k(c).$$

We will show that the equation (2.7) is true for  $\ell = n$ .

Note that equation (2.6) can be written as

$$(2.8) \quad \sum_{j=0}^n \sum_{i+k=n-j} \left( \langle \varphi_i(\langle a, b \rangle c), \varphi_j(x) \rangle \varphi_k(\langle c, b \rangle a) + \langle \varphi_i(\langle c, b \rangle a), \varphi_j(x) \rangle \varphi_k(\langle a, b \rangle c) \right) \\ = \sum_{t=0}^n \sum_{i+p+s+u+r+k=n-t} \left( \langle \langle \varphi_i(a), \varphi_p(b) \rangle \varphi_s(c), \varphi_t(x) \rangle \langle \varphi_u(c), \varphi_r(b) \rangle \varphi_k(a) \right. \\ \left. + \langle \langle \varphi_i(c), \varphi_p(b) \rangle \varphi_s(a), \varphi_t(x) \rangle \langle \varphi_u(a), \varphi_r(b) \rangle \varphi_k(c) \right),$$

for all  $x \in \mathcal{M}$ . In (2.8), for  $1 \leq j \leq n$  we have  $i+k = n-j < n$  and then  $i, k < n$ . This implies that  $\varphi_i, \varphi_k$  satisfy (2.7). Thus we can cancel the like terms of both sides of equation (2.8). In fact the equation (2.8) reduces to the following equation for the case that  $j = 0$ :

$$\sum_{i+k=n} \left( \langle \varphi_i(\langle a, b \rangle c), x \rangle \varphi_k(\langle c, b \rangle a) + \langle \varphi_i(\langle c, b \rangle a), x \rangle \varphi_k(\langle a, b \rangle c) \right) \\ = \sum_{i+p+s+u+r+k=n} \left( \langle \langle \varphi_i(a), \varphi_p(b) \rangle \varphi_s(c), x \rangle \langle \varphi_u(c), \varphi_r(b) \rangle \varphi_k(a) \right. \\ \left. + \langle \langle \varphi_i(c), \varphi_p(b) \rangle \varphi_s(a), x \rangle \langle \varphi_u(a), \varphi_r(b) \rangle \varphi_k(c) \right),$$

which implies that

$$\begin{aligned}
& \langle \varphi_n(\langle a, b \rangle c), x \rangle \langle c, b \rangle a + \langle \varphi_n(\langle c, b \rangle a), x \rangle \langle a, b \rangle c \\
& + \langle \langle a, b \rangle c, x \rangle \varphi_n(\langle c, b \rangle a) + \langle \langle c, b \rangle a, x \rangle \varphi_n(\langle a, b \rangle c) \\
& + \sum_{\substack{i+k=n \\ 1 \leq i, k \leq n-1}} \left( \langle \varphi_i(\langle a, b \rangle c), x \rangle \varphi_k(\langle c, b \rangle a) + \langle \varphi_i(\langle c, b \rangle a), x \rangle \varphi_k(\langle a, b \rangle c) \right) \\
& = \sum_{i+p+s=n} \left( \langle \varphi_i(a), \varphi_p(b) \rangle \varphi_s(c), x \right) \langle c, b \rangle a + \left( \langle \varphi_i(c), \varphi_p(b) \rangle \varphi_s(a), x \right) \langle a, b \rangle c \\
& + \sum_{u+r+k=n} \left( \langle \langle a, b \rangle c, x \rangle \langle \varphi_u(c), \varphi_r(b) \rangle \varphi_k(a) + \langle \langle c, b \rangle a, x \rangle \langle \varphi_u(a), \varphi_r(b) \rangle \varphi_k(c) \right) \\
& + \sum_{\substack{i+p+s+u+r+k=n \\ 1 \leq i+p+s, u+r+k \leq n-1}} \left( \left( \langle \varphi_i(a), \varphi_p(b) \rangle \varphi_s(c), x \right) \langle \varphi_u(c), \varphi_r(b) \rangle \varphi_k(a) \right. \\
& \left. + \left( \langle \varphi_i(c), \varphi_p(b) \rangle \varphi_s(a), x \right) \langle \varphi_u(a), \varphi_r(b) \rangle \varphi_k(c) \right).
\end{aligned}$$

Canceling the like terms from both sides of the above equation and then arranging them, we get

$$\begin{aligned}
& \langle \Delta_n(a, b, c), x \rangle \langle c, b \rangle a + \langle \langle c, b \rangle a, x \rangle \Delta_n(a, b, c) \\
& + \langle \Delta_n(c, b, a), x \rangle \langle a, b \rangle c + \langle \langle a, b \rangle c, x \rangle \Delta_n(c, b, a) = 0,
\end{aligned}$$

for all  $x \in \mathcal{M}$ . A similar argument as used for  $n = 1$ , shows that

$$\langle \Delta_n(a, b, c), x \rangle \Omega(a, b, c) = \langle \Omega(a, b, c), x \rangle \Delta_n(a, b, c) = 0,$$

for all  $x \in \mathcal{M}$ . It follows from primeness of  $\mathcal{M}$  that  $\Delta_n(a, b, c) = 0$  or  $\Omega(a, b, c) = 0$ . In each case, it follows that the equation (2.7) holds for  $\ell = n$ . This proves that  $\{\varphi_n\}_{n=0}^{+\infty}$  is a Hilbert  $C^*$ -module higher derivation on  $\mathcal{M}$ .  $\square$

The next corollary follows from Theorem 2.1.

**Corollary 2.1.** *Let  $\mathcal{M}$  be a 2-torsion-free prime Hilbert  $C^*$ -module. Then every Hilbert  $C^*$ -module Jordan derivation on  $\mathcal{M}$  is a Hilbert  $C^*$ -module derivation on  $\mathcal{M}$ .*

## REFERENCES

- [1] M. J. Atteya, C. Haetinger and D. I. Rasen,  $(\sigma, \tau)$ -derivations of semiprime rings, Kragujevac J. Math. **43**(2) (2019), 239–246.
- [2] M. Brešar, Jordan derivations on semiprime rings, Proc. Amer. Math. Soc. **104** (1988), 1003–1006.
- [3] W. Cortes and C. Haetinger, On Jordan generalized higher derivations in rings, Turk. J. Math. **29** (2005), 1–10.
- [4] S. Kh. Ekrami, A note on characterization of higher derivations and their product, J. Mahani Math. Res. **13**(1) (2024), 403–415. <https://doi.org/10.22103/jmmr.2023.21376.1432>

- [5] S. Kh. Ekrami, *Approximate orthogonally higher ring derivations*, Control Optim. App. Math. **7**(1) (2022), 93–106. <https://doi.org/10.30473/coam.2021.59727.1166>
- [6] S. Kh. Ekrami, *Characterization of Hilbert  $C^*$ -module higher derivations*, Georgian Math. J. (2023). <https://doi.org/10.1515/gmj-2023-2085>
- [7] S. Kh. Ekrami, *Jordan higher derivations, a new approach*, J. Algebr. Syst. **10**(1) (2022), 167–177. <https://doi.org/10.22044/JAS.2021.10636.1527>
- [8] M. Ferrero and C. Haetinger, *Higher derivations and a theorem by Herstein*, Quaest. Math. **25**(2) (2002), 249–257.
- [9] C. Haetinger, *Derivações de ordem superior em anéis primos e semiprimos*, Ph.D. Thesis, UFRGS, Porto Alegre, Brazil, (2000).
- [10] C. Haetinger, *Higher derivations on Lie ideals*, Trends Comp. App. Math. **3** (2002), 141–145.
- [11] I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1104–1110.
- [12] N. P. Jewell, *Continuity of module and higher derivations*, Pacific J. Math. **68** (1977), 91–98.
- [13] I. Kaplansky, *Modules over operator algebras*, Amer. J. Math. **75** (1953), 839–858.
- [14] M. Mirzavaziri, *Characterization of higher derivations on algebras*, Comm. Algebra **38** (2010), 981–987.
- [15] A. Roy and R. Sridharan, *Higher derivations and central simple algebras*, Nagoya Math. J. **32** (1968), 21–30.
- [16] H. Saidi, A. R. Janfada and M. Mirzavaziri, *Kinds of derivations on Hilbert  $C^*$ -modules and their operator algebras*, Miskolc Math. Notes. **16**(1) (2015), 453–461.
- [17] E. Tafazzoli and M. Mirzavaziri, *Inner higher derivations on algebras*, Kragujevac J. Math. **44**(2) (2019), 267–273.
- [18] S. Xu and Z. Xiao, *Jordan higher derivation revisited*, Gulf J. Math. **2**(1) (2014), 11–21.

DEPARTMENT OF MATHEMATICS,  
 PAYAME NOOR UNIVERSITY,  
 P.O. BOX 19395-3697, TEHRAN, IRAN.  
*Email address:* ekrami@pnu.ac.ir, khalil.ekrami@gmail.com  
 ORCID iD: <https://orcid.org/0000-0002-6233-5741>