

**CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS  
DEFINED BY LINEAR MULTIPLIER FRACTIONAL  
 $q$ -DIFFERENTIAL OPERATOR**

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**ABSTRACT.** This paper introduces a novel subclass of analytic and bi-univalent functions that are linked to a linear multiplier fractional  $q$ -differential operator, defined in the open unit disk  $\mathbb{D}$ . The authors establish the upper bounds for the coefficients  $|a_2|$  and  $|a_3|$  for the functions that belong to this new subclass and its subclasses.

1. INTRODUCTION AND PRELIMINARIES

Let the class of functions  $\mathcal{A}$  be of the form:

$$(1.1) \quad \eta(z) = z + \sum_{k=2}^{+\infty} a_k z^k,$$

which are analytic on the open unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . Also let  $S$  indicates the functions of all subclasses in  $\mathcal{A}$ , which are univalent in  $\mathbb{D}$ . Since univalent functions are one-to-one, they are invertible. Although the inverse functions of single-valued functions are inverse functions, they do not need to be defined for the entire unit disk  $\mathbb{D}$ . Certainly, according to Koebe's quarter theorem [1], the disk with radius  $\frac{1}{4}$  is in the image  $\mathbb{D}$ . Thus, every univalent function  $\eta$  has an inverse  $\eta^{-1}$  that satisfies  $\eta^{-1}(\eta(z)) = z$ ,  $z \in \mathbb{D}$ , and  $\zeta(w) = \eta^{-1}(\eta(w)) = w$ ,  $|w| < r_0(\eta)$ ,  $r_0(\eta) \geq \frac{1}{4}$ , where

$$(1.2) \quad \eta^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

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A function  $\eta \in \mathcal{A}$  is bi-univalent in  $\mathbb{D}$  if both  $\eta(z)$  and  $\eta^{-1}(z)$  are univalent in  $\mathbb{D}$ . Let  $\Sigma$  be the class of bi-univalent functions on  $\mathbb{D}$  given by (1.1). Example of functions in the class  $\Sigma$  are

$$\frac{z}{1-z}, \quad \log \frac{1}{1-z}, \quad \log \sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is not a member of  $\Sigma$ . Other common examples of functions in  $\mathbb{D}$  such as

$$\frac{2z - z^2}{2} \quad \text{and} \quad \frac{z}{1 - z^2}$$

are also not members of  $\Sigma$ .

The widely-cited by Srivastava et al. [2] actually revived the study of analytic and bi-univalent functions in recent years, and it has also led to a flood of papers on the subject by (see, for example, [3–23]).

If  $|q| < 1$ , the  $q$ -shifted factorial, also known as the  $q$ -Pochhammer symbol, is defined for all  $n \in \mathbb{N}$  by

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k),$$

where  $a$  and  $q$  are complex numbers. When  $n = +\infty$ , the product becomes

$$(a; q)_{+\infty} = \prod_{k=0}^{+\infty} (1 - aq^k).$$

If  $|q| < 1$ , then the product converges absolutely, and we can define the  $q$ -shifted factorial for  $n = +\infty$  as the limit of the sequence of partial products

$$(a; q)_{+\infty} = \lim_{n \rightarrow +\infty} (a; q)_n = \lim_{n \rightarrow +\infty} \prod_{k=0}^{n-1} (1 - aq^k).$$

Therefore, when  $|q| < 1$ , the  $q$ -shifted factorial remains meaningful for  $n = +\infty$  as a convergent infinite product.

The  $q$ -gamma function is a  $q$ -analogue of the gamma function, defined by the recurrence relation  $\Gamma_q(y + 1) = [y]_q \Gamma_q(y)$ , where  $[y]_q = \frac{(1 - q^y)}{(1 - q)}$  is the  $q$ -analogue of  $y$ .

Jackson’s [24]  $q$ -derivative and  $q$ -integral of a function  $\eta$  defined on a subset of  $\mathbb{C}$  are given by

$$D_q^a \eta(x) = \frac{\eta(q^a x) - \eta(x)}{(1 - q^a)x - x}, \quad I_q^a \eta(x) = (1 - q^a)x \sum_{n=0}^{+\infty} q^{an} \eta(q^n x),$$

where  $a \in \mathbb{C}$  is a fixed parameter. These operators are also known as the  $q$ -difference and  $q$ -integral operators, respectively. The theory of  $q$ -calculus operators are used in describing and solving various problems in applied science such as ordinary fractional calculus, optimal control,  $q$ -difference and  $q$ -integral equations, as well as geometric function theory of complex analysis. The application of  $q$ -calculus was initiated by Jackson [24]. Recently, many researchers studied  $q$ -calculus such as Srivastava et al.

[25], Muhammad and Darus [26], Kanas and Răducanu [27], (see also, [28–33]) and also the reference cited therein.

**Definition 1.1** ([34]). The fractional integral operator  $I_{q,z}^\delta$  of order  $\delta > 0$ , for the function  $\eta(z)$  is defined by

$$I_{q,z}^\delta = D_{q,z}^{-\delta}\eta(z) = \frac{1}{\Gamma_q(\delta)} \int_0^z (z - rq)_{1-\delta}\eta(r)d_qr,$$

where  $\eta(z)$  is the analytic of the simply connected regions of the  $z$  plane containing the origin. Here, the term  $(z - rq)_{\delta-1}$  is a  $q$ -binomial function defined by

$$(z - rq)_{\delta-1} = z^{\delta-1} \prod_{k=0}^{+\infty} \left[ \frac{1 - (\frac{rq}{z})q^k}{1 - (\frac{rq}{z})q^{\delta+k-1}} \right] = z^\delta {}_1\phi_0 \left[ q^{-\delta+1}; -; q, \frac{rq^\delta}{z} \right].$$

**Definition 1.2.** The fractional  $q$ -derivative operator  $D_{q,z}^\delta$  of a  $\eta(z)$  of order  $0 \leq \delta < 1$ , is defined by

$$D_{q,z}^\delta\eta(z) = D_{q,z}I_{q,z}^{1-\delta}\eta(z) = \frac{1}{\Gamma_q(1-\delta)}D_q \int_0^z (z - rq)_{-\delta}\eta(r)d_qr,$$

where  $\eta(z)$  is suitably constrained and the multiplicity of  $(z - rq)_{-\delta}$  is removed as in Definition 1.1 above.

**Definition 1.3.** Under the hypotheses of Definition 1.2, the fractional  $q$ -derivative for the function  $\eta(z)$  of order  $\delta$  is defined by

$$D_{q,z}^\delta\eta(z) = D_{q,z}^n I_{q,z}^{n-\delta}\eta(z),$$

where  $n - 1 \leq \delta < n, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

**Definition 1.4** ([35]). The definition of the fractional  $q$ -differintegral operator  $\Omega_{q,z}^\delta$  is as follows. For a function  $\eta(z)$  of the form (1.1), we define

$$\Omega_{q,z}^\delta\eta(z) = \Gamma_q(2 - \delta)z^\delta D_{q,z}^\delta\eta(z),$$

where  $D_{q,z}^\delta$  denotes the fractional  $\delta$  order of the  $q$ -integral  $\eta(z)$  when  $-\infty < \delta < 0$  and the fractional  $\delta$  order  $q$ -derivative of  $\eta(z)$  if  $0 < \delta < 2$ .

The expression for  $\Omega_{q,z}^\delta\eta(z)$  in terms of the coefficients  $a_k$  of the power series expansion of  $\eta(z)$  is given by

$$\Omega_{q,z}^\delta\eta(z) = z + \sum_{k=2}^{+\infty} \frac{\Gamma_q(k+1)\Gamma_q(2-\delta)}{\Gamma_q(k+1-\delta)} a_k z^k.$$

**Definition 1.5** ([34]). A linear multiplier fractional  $q$ -differintegral operator is defined as

$$\begin{aligned}
 \mathcal{L}_{q,\lambda}^{\delta,0}\eta(z) &= \eta(z), \\
 \mathcal{L}_{q,\lambda}^{\delta,1}\eta(z) &= (1 - \lambda)\Omega_q^\delta\eta(z) + \lambda z\mathcal{L}_q\left(\Omega_q^\delta\eta(z)\right), \\
 \mathcal{L}_{q,\lambda}^{\delta,2}\eta(z) &= \mathcal{L}_{q,\lambda}^{\delta,1}\left(\mathcal{L}_{q,\lambda}^{\delta,1}\eta(z)\right), \\
 &\vdots \\
 \mathcal{L}_{q,\lambda}^{\delta,n}\eta(z) &= \mathcal{L}_{q,\lambda}^{\delta,1}\left(\mathcal{L}_{q,\lambda}^{\delta,n-1}\eta(z)\right).
 \end{aligned}
 \tag{1.3}$$

We note that if  $f \in \mathcal{A}$  is given by (1.1), then by (1.3), we have

$$\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z) = z + \sum_{k=2}^{+\infty} C(k, \delta, \lambda, n, q) a_k z^k,$$

where

$$C(k, \delta, \lambda, n, q) = \left( \frac{\Gamma_q(k+1)\Gamma_q(2-\delta)}{\Gamma_q(k+1-\delta)} \left[ ([k]_q - 1)\lambda + 1 \right] \right)^n.$$

We define two new subclasses of the function class  $\Sigma$  by utilizing the linear multiplier fractional  $q$ -differential operator of a function  $\eta \in \mathcal{A}$ . Then, we provide coefficient estimates for  $|a_2|$  and  $|a_3|$  for functions belonging to these new subclasses of the function class  $\Sigma$ .

First, we have to follow the lemma to get the main results.

**Lemma 1.1** ([36]). *Let  $\mathcal{H}$  be the family of all functions  $\mathfrak{h}$  that are analytic in the open unit disk  $\mathbb{D}$  and satisfy  $\mathfrak{h}(0) = 1$  and  $\Re(\mathfrak{h}(z)) > 0$  for all  $z \in \mathbb{D}$ . If a function  $\mathfrak{h} \in \mathcal{H}$  is given by  $\mathfrak{h}(z) = 1 + d_1z + d_2z^2 + \dots$  for  $z \in \mathbb{D}$ , then  $|d_k| \leq 2$  for all  $k \in \mathbb{N}$ .*

## 2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $M_\Sigma(q, \alpha, \tau, \delta, \lambda, n)$

**Definition 2.1.** A function  $\eta(z)$  given by (1.1) is said to be in the class  $M_\Sigma(q, \alpha, \tau, \delta, \lambda, n)$  if the following conditions are satisfied:  $\eta \in \Sigma$  and

$$\left| \frac{zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)}{\tau zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)} \right| < \frac{\alpha\pi}{2},$$

where  $0 < \alpha \leq 1$ ,  $0 \leq \tau < 1$ ,  $\delta \leq 2$ ,  $\lambda > 0$ ,  $n \in \mathbb{N}_0$ ,  $z \in \mathbb{D}$ , and

$$\left| \frac{wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)}{\tau wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)} \right| < \frac{\alpha\pi}{2},$$

where  $0 < \alpha \leq 1$ ,  $0 \leq \tau < 1$ ,  $\delta \leq 2$ ,  $\lambda > 0$ ,  $n \in \mathbb{N}_0$ ,  $w \in \mathbb{D}$  and function  $\zeta$  is given by

$$\zeta(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots.
 \tag{2.1}$$

We note that the following hold.

- (a) When we set  $\delta = 0$ ,  $\lambda = 1$ , and  $q \rightarrow 1^-$ , the class  $M_\Sigma(q, \alpha, \tau, \delta, \lambda, n)$  reduces to the class  $S_\Sigma^{m,\tau}(\alpha)$ , where  $0 < \alpha \leq 1$ ,  $0 \leq \tau < 1$ , and  $n \in \mathbb{N}_0$ . This class was previously introduced and studied by Jothibasu [37].
- (b) If we set  $\delta = 0$ ,  $\lambda = 1$ ,  $q \rightarrow 1^-$ ,  $n = 0$ , and  $\tau = 0$  in the class  $M_\Sigma(q, \alpha, \tau, \delta, \lambda, n)$ , it reduces to the class of strongly bi-starlike functions  $S_\Sigma^*(\alpha)$  of order  $\alpha$  introduced and studied by Brannan and Taha [38], where  $0 < \alpha \leq 1$ .

**Theorem 2.1.** *Let  $\eta(z)$  given by (1.1) be in the class  $M_\Sigma(q, \alpha, \tau, \delta, \lambda, n)$ ,  $0 < \alpha \leq 1$ ,  $0 \leq \tau < 1$ ,  $\delta \leq 2$ ,  $\lambda > 0$ . Then*

$$(2.2) \quad |a_2| \leq \frac{2\alpha}{\sqrt{2\alpha Yq(q+1)(1-\tau) - 2X^2\alpha q(1-\tau)[\tau q + 1] + X^2(1-\alpha)^2(1-\tau)^2}}$$

and

$$(2.3) \quad |a_3| \leq \frac{4\alpha^2}{X^2q^2(1-\tau)^2} + \frac{2\alpha}{Yq(q+1)(1-\tau)},$$

where  $X = C(2, \delta, \lambda, n, q)$  and  $Y = C(3, \delta, \lambda, n, q)$ .

*Proof.* It follows from the Definition 2.1

$$(2.4) \quad \frac{zD_q(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z))}{\tau zD_q(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)} = [s(z)]^\alpha$$

and

$$(2.5) \quad \frac{wD_q(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w))}{\tau wD_q(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)} = [t(w)]^\alpha,$$

respectively, where  $s(z)$  and  $t(w)$  satisfy the following inequalities:  $\Re(s(z)) > 0$ ,  $z \in \mathbb{D}$ , and  $\Re(t(w)) > 0$ ,  $w \in \mathbb{D}$ .

Furthermore, the functions  $s(z)$  and  $t(w)$  have the forms

$$(2.6) \quad s(z) = 1 + s_1z + s_2z^2 + s_3z^3 + \dots,$$

$$(2.7) \quad t(w) = 1 + t_1w + t_2w^2 + t_3w^3 + \dots.$$

Now, equating the coefficients in (2.4) and (2.5), we get

$$(2.8) \quad a_2Xq(1-\tau) = \alpha s_1,$$

$$(2.9) \quad a_3Yq(q+1)(1-\tau) - a_2^2X^2q(1-\tau)[\tau q + 1] = \alpha s_2 + \frac{\alpha(\alpha-1)}{2}s_1^2,$$

$$(2.10) \quad -a_2Xq(1-\tau) = \alpha t_1$$

and

$$(2.11) \quad -a_3Yq(q+1)(1-\tau) + 2a_2^2Yq(q+1)(1-\tau) - a_2^2X^2q(1-\tau)[\tau q + 1] = \alpha t_2 + \frac{\alpha(\alpha-1)}{2}t_1^2.$$

From (2.8) and (2.10), we get

$$(2.12) \quad s_1 = -t_1$$

and

$$(2.13) \quad 2a_2^2 X^2 q^2 (1 - \tau)^2 = \alpha^2 (s_1^2 + t_1^2).$$

From (2.9), (2.11) and (2.13), we obtain

$$a_2^2 = \frac{\alpha^2 (s_2 + t_2)}{2\alpha Y q(q+1) (1 - \tau) - 2X^2 \alpha q (1 - \tau) [\tau q + 1] + X^2 (1 - \alpha) q^2 (1 - \tau)^2}.$$

Applying Lemma 1.1 to the coefficients  $s_2$  and  $t_2$ , we immediately get

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha Y q(q+1) (1 - \tau) - 2X^2 \alpha q (1 - \tau) [\tau q + 1] + X^2 (1 - \alpha) q^2 (1 - \tau)^2}}.$$

This gives the value of  $|a_2|$  as shown in (2.2)

Next, in order to find the bound on  $|a_3|$ , by subtracting (2.11) from (2.9), we get

$$(2.14) \quad \begin{aligned} & 2a_3 Y q(q+1) (1 - \tau) - 2a_2^2 Y q(q+1) (1 - \tau) \\ &= \alpha (s_2 - t_2) + \frac{\alpha(\alpha - 1)}{2} (s_1^2 - t_1^2). \end{aligned}$$

It follows from (2.12), (2.13) and (2.14) that

$$|a_3| = \frac{\alpha^2 (s_1^2 + t_1^2)}{2X^2 q^2 (1 - \tau)^2} + \frac{\alpha (s_2 - t_2)}{2Y q(q+1) (1 - \tau)}.$$

Applying Lemma 1.1 again to the coefficients  $s_1$ ,  $s_2$ ,  $t_1$  and  $t_2$ , we easily get

$$|a_3| \leq \frac{4\alpha^2}{X^2 q^2 (1 - \tau)^2} + \frac{2\alpha}{Y q(q+1) (1 - \tau)}.$$

This ends the proof of Theorem 2.1.  $\square$

Utilizing the parameters setting of Definition 2.1 in the Theorem 2.1, we get the following corollaries.

**Corollary 2.1.** *If  $\eta(z)$  given by (1.1) be in the class  $S_{\Sigma}^{m,\tau}(\alpha)$ ,  $0 < \alpha \leq 1$ ,  $0 \leq \tau < 1$  and  $n \in \mathbb{N}_0$ . Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{4\alpha(1 - \tau) 3^n + [2\alpha(\tau^2 - 1) - (\alpha - 1)(1 - \tau)^2] 2^{2n}}}$$

and

$$|a_3| \leq \frac{\alpha}{3^n (1 - \tau)} + \frac{4\alpha^2}{2^{2n} (1 - \tau)^2}.$$

**Corollary 2.2.** *If  $\eta(z)$  given by (1.1) and in the class  $S_{\Sigma}^*(\alpha)$ ,  $0 < \alpha \leq 1$ . Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha + 1}} \quad \text{and} \quad |a_3| \leq 4\alpha^2 + \alpha.$$

3. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS  $B_\Sigma(q, \gamma, \tau, \delta, \lambda, n)$

**Definition 3.1.** A function  $\eta(z)$  given by (1.1) is said to be in the class  $B_\Sigma(q, \gamma, \tau, \delta, \lambda, n)$  if the following conditions are satisfied:  $\eta \in \Sigma$  and

$$\Re\left(\frac{zD_q(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z))}{\tau zD_q(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)}\right) > \gamma,$$

where  $0 \leq \gamma < 1, 0 \leq \tau < 1, \delta \leq 2, \lambda > 0, n \in \mathbb{N}_0, z \in \mathbb{D}$ , and

$$\Re\left(\frac{wD_q(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w))}{\tau wD_q(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)}\right) > \gamma,$$

where  $0 \leq \gamma < 1, 0 \leq \tau < 1, \delta \leq 2, \lambda > 0, n \in \mathbb{N}_0, w \in \mathbb{D}$ .

The function  $\zeta$  is defined as given in equation (2.1).

- (a) If we set  $\delta = 0, \lambda = 1$ , and  $q \rightarrow 1^-$  in the class  $B_\Sigma(q, \gamma, \tau, \delta, \lambda, n)$ , it reduces to the class  $S_\Sigma^{n,\tau}(\gamma)$  introduced and studied by Jothibasu [37], where  $0 \leq \gamma < 1, 0 \leq \tau < 1$  and  $n \in \mathbb{N}_0$ .
- (b) When  $\delta = 0, \lambda = 1, q \rightarrow 1^-, n = 0$  and  $\tau = 0$ , the class  $B_\Sigma(q, \gamma, \tau, \delta, \lambda, n)$  simplifies to the class of strongly bi-starlike functions  $S_\Sigma^*(\gamma)$  of order  $\gamma$  introduced and studied by Brannan and Taha [38].

**Theorem 3.1.** Let  $\eta(z)$  given by (1.1) be in the class  $B_\Sigma(q, \gamma, \tau, \delta, \lambda, n), 0 \leq \gamma < 1, 0 \leq \tau < 1, \delta \leq 2, \lambda > 0$ . Then

$$(3.1) \quad |a_2| \leq \sqrt{\frac{2(1-\gamma)}{Yq(q+1)(1-\tau) - X^2q(1-\tau)[\tau q + 1]}}$$

and

$$(3.2) \quad |a_3| \leq \frac{4(1-\gamma)^2}{X^2q^2(1-\tau)^2} + \frac{2(1-\gamma)}{Yq(q+1)(1-\tau)},$$

where  $X = C(2, \delta, \lambda, n, q)$  and  $Y = C(3, \delta, \lambda, n, q)$ .

*Proof.* It follows from the Definition 3.1 that there exist  $s(z)$  and  $t(w) \in \mathcal{H}$  such that

$$(3.3) \quad \frac{zD_q(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z))}{\tau zD_q(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)} = \gamma + (1-\gamma)s(z),$$

$$(3.4) \quad \frac{wD_q(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w))}{\tau wD_q(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)} = \gamma + (1-\gamma)t(w),$$

where  $s(z)$  and  $t(w)$  in  $\mathcal{H}$  and have the forms (2.6) and (2.7), respectively.

Equating the coefficients in (3.3) and (3.4) yields

$$(3.5) \quad a_2 X q (1 - \tau) = (1 - \gamma) s_1,$$

$$(3.6) \quad a_3 Y q (q + 1) (1 - \tau) - a_2^2 X^2 q (1 - \tau) [\tau q + 1] = (1 - \gamma) s_2,$$

$$(3.7) \quad -a_2 X q (1 - \tau) = (1 - \gamma) t_1$$

and

$$(3.8) \quad -a_3 Y q (q + 1) (1 - \tau) + 2a_2^2 Y q (q + 1) (1 - \tau) - a_2^2 X^2 q (1 - \tau) [\tau q + 1] \\ = (1 - \gamma) t_2.$$

From (3.5) and (3.7), we get  $s_1 = -t_1$  and

$$(3.9) \quad 2a_2^2 X^2 q^2 (1 - \tau)^2 = (1 - \gamma)^2 (s_1^2 + t_1^2).$$

Also, from (3.6) and (3.8), we find that

$$2a_2^2 Y q (q + 1) (1 - \tau) - 2a_2^2 X^2 q (1 - \tau) [\tau q + 1] = (1 - \gamma) (s_2 + t_2).$$

Applying Lemma 1.1 to the coefficients  $s_2$  and  $t_2$ , we immediately get

$$|a_2| \leq \sqrt{\frac{2(1 - \gamma)}{Y q (q + 1) (1 - \tau) - X^2 q (1 - \tau) [\tau q + 1]}}$$

which is the bound on  $|a_2|$  as given in (3.1). Then, to get the limit of  $|a_3|$  by subtracting (3.8) from (3.6),

$$2a_3 Y q (q + 1) (1 - \tau) - 2a_2^2 Y q (q + 1) (1 - \tau) = (1 - \gamma) (s_2 - t_2),$$

or, equivalently

$$a_3 = a_2^2 + \frac{(1 - \gamma) (s_2 - t_2)}{2Y q (q + 1) (1 - \tau)}.$$

Substituting the values of  $a_2^2$  into (3.9), we get

$$a_3 = \frac{(1 - \gamma)^2 (s_1^2 + t_1^2)}{2X^2 q^2 (1 - \tau)^2} + \frac{(1 - \gamma) (s_2 - t_2)}{2Y q (q + 1) (1 - \tau)}.$$

After applying Lemma 1.1 to the coefficients  $s_1$ ,  $s_2$ ,  $t_1$  and  $t_2$ , we get

$$|a_3| \leq \frac{4(1 - \gamma)^2}{X^2 q^2 (1 - \tau)^2} + \frac{2(1 - \gamma)}{Y q (q + 1) (1 - \tau)}.$$

This completes the proof of Theorem 3.1.  $\square$

Utilizing the parameters setting of Definition 3.1 in the Theorem 3.1, we get the following corollaries.

**Corollary 3.1.** *If  $\eta(z)$  given by (1.1) is in the class  $S_{\Sigma}^{m,\tau}(\gamma)$ ,  $0 \leq \gamma < 1$ ,  $0 \leq \tau < 1$  and  $n \in \mathbb{N}_0$ , then*

$$|a_2| \leq \sqrt{\frac{2(1 - \gamma)}{2^{2n} (\tau^2 - 1) + 2(1 - \tau) 3^n}}$$

and

$$|a_3| \leq \frac{4(1-\gamma)^2}{2^{2n}(1-\tau)^2} + \frac{(1-\gamma)}{3^n(1-\tau)}.$$

**Corollary 3.2.** *If  $\eta(z)$  given by (1.1) and in the class  $S_{\Sigma}^*(\gamma)$ ,  $0 \leq \gamma < 1$ , then*

$$|a_2| \leq \sqrt{2(1-\gamma)} \quad \text{and} \quad |a_3| \leq 4(1-\gamma)^2 + (1-\gamma).$$

#### 4. CONCLUSIONS

The main contribution of this paper is the introduction of new subclasses of bi-univalent functions defined by the linear multiplier fractional  $q$ -differential operator. Additionally, we provide upper bounds for the coefficients  $|a_2|$  and  $|a_3|$  for functions belonging to this new subclass and its subclasses.

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