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NUMERICAL TREATMENT OF VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER AND ITS CONVERGENCE ANALYSIS

ABHILIPSA PANDA[1] AND JUGAL MOHAPATRA[2]

Abstract. This work deals with semi-analytical and numerical methods to solve a class of fractional order Volterra-Fredholm integro-differential equations. First, a semi-analytical method is proposed using the Chebyshev and Bernstein polynomials in the Adomian decomposition method. The uniqueness of the solution and convergence of the method are proved. Further, we solve the model using a numerical scheme comparing the L1 scheme for the fractional order derivative in combination with appropriate quadrature rules for the integral parts. Numerical experiments are done by the proposed methods to show their efficiency through a few tabular data and plots. Some comparisons with the existing results show that the proposed methods are highly productive and reliable.

1. Introduction

The considerable interest in integro-differential equations (IDEs) has mainly arisen due to its major applications in the theory of mechanical engineering, elasticity [29] and several others. The well-known mathematician Niels Henrik Abel obtained the famous integral equation of the first kind with kernel function $\mathcal{K}(x,t) = (x-t)^{-\mu},$ for $\mu = 1/2$ by solving the mechanical problem of *Tautochrone* as described in [23], which he then generalized it for $0 < \mu < 1$. The theory given by Abel in [23] further paved the way for researchers to look deep into the idea of fractional order integro-differential equations (FracIDEs). The wide application of FracIDEs for electromagnetic waves in dielectric media and unsteady aerodynamics have generated great interest in exploring more in

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this field. Several analytical and numerical techniques have been introduced to obtain the solutions of FracIDEs with singular and nonsingular kernels [12, 17]. Aghajani et al., established the existence of solutions for FracIDEs in [2]. The operational tau approximation method based on orthogonal polynomials was implemented on a class of FracIDEs by Vanani and Aminataei in [30]. Heydari and Hooshmandasl [14] used the Chebyshev wavelet method to solve the nonlinear FracIDEs on a large interval by converting the fractional differential and integral parts of the FracIDE to some operational matrices. Then, they obtained the solutions by solving a set of algebraic equations. Also, based on the Haar wavelet collocation method, Marasi and Derakhshan in [20] focused on finding a numerical method for solving the variable-order Caputo-Prabhakar FracIDEs. Higher order FracIDEs, such as the fourth-order FracIDEs, were solved by Amer et al. [5] using the Adomian decomposition method (ADM) and variational iteration method (VIM), where the solution was given by an infinite convergent series. Also, quite a few approximated techniques described in [9, 24] have been discussed in the past to solve the linear and nonlinear FracIDEs.

But all the model problems solved have considered the source term as a polynomial function which is comparatively easier to approximate. Thus, we propose a new modification of ADM for obtaining the solution of a class of FracIDE where the source function is not a polynomial one. The general way of ADM was first introduced by G. Adomian [1] to solve linear and nonlinear problems. Gradually, ADM was improvised using the Chebyshev [15], Legendre [19] and Bernstein polynomials [25]. These modified techniques are used to solve a class of ordinary and partial differential equations where the source function is exponential, trigonometric, or hyperbolic functions rather than the polynomial one. The approximation of functions by polynomials is extremely important as different scientific experiments rely on them, such as the study of statistics in population dynamics [28], temperatures, and also in the approximation theory [7]. Moreover, polynomials are the best mathematical techniques to approximate as they can be characterized, figured, separated, and incorporated effortlessly. Orthogonal polynomials such as the Chebyshev polynomials have been widely used in approximating functions in a wide variety of problems. These are the eigen functions of singular Sturm-Liouville problems. It is well known that these eigen functions allow the approximation of functions in $C^\infty[a, b]$, where the truncation error approaches zero faster than any function used in the approximation as described in [8]. Gottlieb in [10] described this effect as the “spectral accuracy”. For more information, one may refer [6]. In this article, we are using the first kind orthogonal Chebyshev polynomials $\{\mathcal{T}_k\}_{k=0}^\infty$ given as

$$\sqrt{1-x^2}\mathcal{T}'(x) + \frac{k^2}{\sqrt{1-x^2}}\mathcal{T}_k(x) = 0.$$

Also, we have used the Bernstein polynomials for the modification of ADM. These polynomials approximate the function with a few terms in comparison to the approximations done using the Taylor series. They are utilized in the fields of connected arithmetic and material science as well as computer-assisted geometric outlines. They are also used in conjunction with other techniques like the Galerkin and collocation methods to solve some differential and integral problems.

Though researchers have widely studied the semi-analytical approaches for solving the mixed FracIDE, a few numerical solutions to such model problems have been studied in the past. Certain works are done, such as Ali et al. [3] used the hybrid orthonormal Bernstein and block-pulse functions wavelet method, Alkan and Hatipoglu [4] introduced the sinc-collocation method for solving the mixed FracIDE. One may also refer to the work done in [16, 21, 26]. Keeping this literature gap in mind, this article also proposes an efficient numerical scheme for finding the numerical solution of a class of Volterra-Fredholm FracIDE. The novel L1 scheme is applied for the fractional derivatives and the quadrature rule for the integral parts. The Composite trapezoidal scheme approximates the Volterra integral whereas the Fredholm integral is solved using the rectangular rule. The error analysis is briefly carried out. Computational data in the numerical section prove the robustness of the proposed numerical technique.

The paper is structured as follows. Section 2 outlines some of the definitions and properties, while the model problem is defined in Section 3. Section 4 describes the semi-analytical approximations along with the convergence analysis. The numerical approximation of the solution and the error analysis are described in Section 5. Some test examples are considered in Section 6, satisfying the theoretical findings and finally, Section 7 draws the concluding remarks.

2. Some Definitions and Properties

Definition 2.1. The Bernstein basis polynomials of degree m over the interval $[0,1]$ are defined as:

$$\mathcal{B}_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad i = 0, 1, \ldots, m.$$

Definition 2.2. The Riemann-Liouville fractional integral of order $\mu > 0$ for a function f is defined as:

$$\mathbb{I}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-\tau)^{\mu-1} f(\tau) d\tau.$$

Definition 2.3. The Caputo derivative of order $\mu \in \mathbb{R}^+$ for a function f is defined as:

$$\mathbb{D}^\mu f(x) = \begin{cases} \frac{1}{\Gamma(n-\mu)} \int_0^x (x-\tau)^{n-\mu-1} f(\tau) d\tau, & n-1 < \mu < n, \\ \frac{d^n f(x)}{dx^n}, & \mu = n, n \in \mathbb{N}. \end{cases}$$

Some of the important properties of fractional derivatives and integrals are discussed as follows.

- $\mathbb{D}^\mu \mathbb{I}^\mu f(x) = f(x)$ and $\mathbb{I}^\mu \mathbb{D}^\mu f(x) = f(x) - f(0+)$, $0 < \mu \leq 1$, where $f(0+) = \lim_{h \to 0} f(0+h)$.

- Linearity property is sustained while defining the derivative in the Caputo sense, given as:

$$\mathbb{D}^{\mu}(\psi_1 m(x) + \psi_2 n(x)) = \psi_1 \mathbb{D}^{\mu} m(x) + \psi_2 \mathbb{D}^{\mu} n(x).$$

- For $0 < \mu \leq 1$ and $\vartheta \in \mathbb{R}$, $\mathbb{I}^{\mu} x^{\vartheta} = \frac{\Gamma(\vartheta+1)}{\Gamma(\mu+\vartheta+1)}$, $\mu > 0$, $\vartheta > -1$, $x > 0$.

3. Model Problem

Consider the Volterra-Fredholm FracIDE of order μ described as:

(3.1)
$$\begin{cases} \mathbb{D}^{\mu} z(x) + a(x)z(x) = f(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{N}_1(z(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{N}_2(z(s))ds, \\ z(0) = z_0, \end{cases}$$

where $0 < \mu \leq 1$ and $x \in [0,1]$. The fractional order derivative $\mathbb{D}^{\mu}$ is defined in the Caputo sense which is assumed to be invertible. The source function $f(x)$, $a(x)$, and the nonlinear operators denoted by $\mathcal{N}_1$ and $\mathcal{N}_2$ are continuous functions on $[0,1]$. $\mathcal{K}_1(x,s)$ and $\mathcal{K}_2(x,s)$ are smooth kernel functions defined on $[0,1] \times [0,1]$. The given initial condition is symbolized as z_0. Throughout this article, for a function $f(x)$, defined on $\Omega = [0,1]$, we define $||f(x)||_{\infty} = \max_{x \in \Omega} |f(x)|$ and C is defined as a generic constant, independent of μ.

4. Semi-Analytical Approximations

4.1. **Adomian decomposition method (ADM).** A brief description of the modified ADM is discussed in this section. Consider the FracIDE (3.1). The nonlinear operator is approximated using the Adomian polynomials $\mathcal{A}_n$. One may refer to [1,24] for the formula of $\mathcal{A}_n$. The solution z is represented as a series solution given by $z = \sum_{n=0}^{\infty} z_n$. Operating the inverse operator $\mathbb{I}^{\mu}$ on both sides of (3.1), we get

$$\begin{aligned} z(x) = & z(0^+) + \mathbb{I}^{\mu} \left[f(x) - a(x)z(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{N}_1(z(s))ds \right. \\ & \left. + \int_0^1 \mathcal{K}_2(x,s)\mathcal{N}_2(z(s))ds \right]. \end{aligned} \tag{4.1}$$

Following the classical ADM, the recurrence relation for the solution of (4.1) is obtained as:

$$\begin{cases} z_0 = \mathbb{I}^{\mu}(f(x)) + z(0^+), \\ z_1 = \mathbb{I}^{\mu} \left(-a(x)z_0(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_0(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_0(s))ds \right), \\ z_2 = \mathbb{I}^{\mu} \left(-a(x)z_1(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_1(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_1(s))ds \right), \\ \vdots \end{cases}$$

Finally, we calculate the solution as $z = \sum_{n=0}^{\infty} z_n$, if $\sum_{n=0}^{\infty} z_n$ converges.

4.2. **ADM based on Chebyshev polynomials (ADM-CP).** In the usual algorithm of ADM, the approximation of f is made using Taylor's series expansion as $f(x) = \sum_{i=0}^{n} \frac{f^n(0)}{n!} x^n$ for an arbitrary $\mathbb{N}$. Hosseini [15] modified the ADM by expanding f using the Chebyshev polynomial approximation

$$f_C(x) \approx \sum_{i=0}^{n} C_n \mathcal{T}_n(x), \tag{4.2}$$

where $\mathcal{T}_n(x)$ is the first kind of orthogonal Chebyshev polynomial. Some of the Chebyshev polynomials are noted below:

$$\begin{cases} \mathcal{T}_0(x) = 1, \\ \mathcal{T}_1(x) = x, \\ \mathcal{T}_2(x) = 2x^2 - 1, \\ \mathcal{T}_3(x) = 4x^3 - 3x, \\ \vdots \\ \mathcal{T}_{n+1}(x) = 2x\mathcal{T}_n - \mathcal{T}_{n-1}, \quad n \geq 1. \end{cases} \tag{4.3}$$

Using (4.2) and (4.3), the following approximations for the solution of (3.1) are obtained as

(4.4)

$$\begin{cases} z_0 = \mathbb{I}^{\mu}(C_0\mathcal{T}_0(x) + C_1\mathcal{T}_1(x) + C_2\mathcal{T}_2(x) + \cdots + C_n\mathcal{T}_n(x)) + z(0^+), \\ z_1 = \mathbb{I}^{\mu}\left(-a(x)z_0(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_0(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_0(s))ds\right), \\ z_2 = \mathbb{I}^{\mu}\left(-a(x)z_1(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_1(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_1(s))ds\right), \\ \vdots \end{cases}$$

This work will prove that the approximated solution obtained by (4.4) is more reliable than any other existing methods. In addition, one may also approximate using the following algorithm as described in [15]

$$\begin{cases} z_n = \mathbb{I}^{\mu}(C_n\mathcal{T}_n(x)) + z(0^+), \quad n = 0, \\ z_{n+1} = \mathbb{I}^{\mu}\left(C_{n+1}\mathcal{T}_{n+1}(x) - a(x)z_n(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_n(s))ds\right) \\ \qquad + \mathbb{I}^{\mu}\left(\int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_n(s))ds\right), \quad n \geq 1. \end{cases} \tag{4.5}$$

Now, (4.2) can also be written in the standard form as $f(x) \approx p_0 + p_1x + p_2x^2 + \cdots + p_rx^r$,

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & -3 & 0 & 5 & \cdots \\ 0 & 0 & 2 & 0 & -8 & 0 & \cdots \\ 0 & 0 & 0 & 4 & 0 & -20 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_r \end{bmatrix}$$

and

$$\begin{cases} z_0 = \mathbb{I}^{\mu}(p_0) + z(0^+), \\ z_1 = \mathbb{I}^{\mu}\left(p_1 - a(x)z_0(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_0(s))ds\right) + \mathbb{I}^{\mu}\left(\int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_0(s))ds\right), \\ z_2 = \mathbb{I}^{\mu}\left(p_2 - a(x)z_1(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_1(s))ds\right) + \mathbb{I}^{\mu}\left(\int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_1(s))ds\right). \end{cases}$$

Finally, using (4.5), the series solution $z(x)$ is obtained as follows:

$$z(x) = z_0(x) + z_1(x) + z_2(x) + \cdots . \tag{4.6}$$

4.3. **ADM based on Bernstein polynomials (ADM-BP).** In this segment, in order to improve the accuracy and reliability of ADM, the source function is expressed in the form of Bernstein polynomial approximation

$$f_B(x) = \sum_{i=0}^{n} D_i \mathcal{B}_i(x), \tag{4.7}$$

where $\mathcal{B}_i(x)$ are the Bernstein polynomials. Using (4.1) and (4.7), the approximated solution for FracIDE (3.1) is obtained as follows:

(4.8)
$$\begin{cases} z_0 = \mathbb{I}^{\mu}(D_0\mathcal{B}_0(x) + D_1\mathcal{B}_1(x) + D_2\mathcal{B}_2(x) + \cdots + D_n\mathcal{B}_n(x)) + z(0^+), \\ z_1 = \mathbb{I}^{\mu}\left(-a(x)z_0(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_0(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_0(s))ds\right), \\ z_2 = \mathbb{I}^{\mu}\left(-a(x)z_1(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_1(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_1(s))ds\right), \\ \vdots \end{cases}$$

The Bernstein polynomials of degree m are obtained as $\mathcal{B}_m f(x) = \sum_{i=0}^{m} \binom{m}{i} x^i$ $(1-x)^{(m-i)} f\left(\frac{i}{m}\right)$. For each function $f : [0,1] \to \mathrm{R}$, we have $\lim_{m \to +\infty} \mathcal{B}_m f(x) = f(x)$. Finally using (4.8), the solution is obtained as

$$z(x) = z_0(x) + z_1(x) + z_2(x) + \cdots . \tag{4.9}$$

4.4. **Convergence analysis.**

4.4.1. *Existence and uniqueness of the solution.* In this segment, some of the hypothesies are stated, which will be further used in the analysis.

- (**H1**) Consider two Lipschitz constants $C_1, C_2 > 0$ such that $\mathcal{N}_1(z(x))$ and $\mathcal{N}_2(z(x))$ satisfy the Lipschitz conditions given as
$$\begin{cases} ||\mathcal{N}_1(z_1(x)) - \mathcal{N}_1(z_2(x))|| \le C_1||z_1 - z_2||, \\ ||\mathcal{N}_2(z_1(x)) - \mathcal{N}_2(z_2(x))|| \le C_2||z_1 - z_2||. \end{cases}$$
- (**H2**) Consider $\mathcal{Q} = \{(x,t) \in \mathbb{R} \times \mathbb{R} : 0 \le t \le x \le 1\}$ and $\mathcal{K}_1^*,\ \mathcal{K}_2^* \in C(\mathcal{Q}, \mathbb{R}^+)$, such that
$$\mathcal{K}_1^* = \sup_{x \in [0,1]} \int_0^x |\mathcal{K}_1(x,s)|dt < +\infty, \quad \mathcal{K}_2^* = \sup_{x \in [0,1]} \int_0^x |\mathcal{K}_2(x,s)|dt < +\infty.$$

Theorem 4.1. *Assuming that* (**H1**) *and* (**H2**) *hold, if* $\frac{\|a\|_\infty+(\mathcal{K}_1^*C_1+\mathcal{K}_2^*C_2)}{\Gamma(\mu+1)} < 1$, *then there exists a unique solution* $z(x) \in C[0,1]$ *for* (3.1).

Proof. The proof of the above theorem is well explained in Theorem 7 of [13]. Here we provide the outline of the proof in very few lines. Applying $\mathbb{I}^\mu$ on both sides of (3.1) we get, $z(x) = Tz(x)$, where

$$\begin{aligned}(Tz)(x) =& z_0 + \mathbb{I}^\mu \Big[-a(x)z(x) + f(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{N}_1(z(s))ds \\ &- \int_0^1 \mathcal{K}_2(x,s)\mathcal{N}_2(z(s))ds)\Big].\end{aligned}$$

Since, we know $z_1(x), z_2(x) \in C[0,1]$, so

$$\begin{aligned}|(Tz_1)(x) - (Tz_2)(x)| \leq& \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}|a(s)||z_1(s)-z_2(s)|ds \\ &+ \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}\Big[\int_0^t |\mathcal{K}_1(t,s)|\cdot|\mathcal{N}_1(z_1(s)) - \mathcal{N}_1(z_2(s))|ds \\ &+ \int_0^1 |\mathcal{K}_2(t,s)|\cdot|\mathcal{N}_2(z_1(s)) - \mathcal{N}_2(z_2(s))|ds\Big]dt \\ \leq& \frac{\|a\|_\infty}{\Gamma(\mu+1)}|z_1 - z_2| + \frac{\mathcal{K}_1^*}{\Gamma(\mu+1)}\int_0^x (x-s)^{\mu-1} \\ &\times \Big[\int_0^t |z_1(s) - z_2(s)|ds\Big]dt + \frac{\mathcal{K}_2^*}{\Gamma(\mu+1)}\int_0^x (x-s)^{\mu-1} \\ &\times \Big[\int_0^1 |z_1(s) - z_2(s)|ds\Big]dt \\ \leq& \frac{\|a\|_\infty}{\Gamma(\mu+1)}|z_1 - z_2| + \frac{\mathcal{K}_1^*C_1 + \mathcal{K}_2^*C_2}{\Gamma(\mu+1)}|z_1 - z_2|.\end{aligned}$$

As $\frac{\|a\|_\infty+(\mathcal{K}_1^*C_1+\mathcal{K}_2^*C_2)}{\Gamma(\mu+1)} < 1$, we have, $||T(z_1(x)) - T(z_2(x))|| \leq ||z_1 - z_2||$. This proves that T is a contraction mapping in Banach space $C([0,1], ||\cdot||)$. So, we can conclude that (3.1) has a unique solution in $C[0,1]$ using the Banach contraction principle. ☐

Theorem 4.2. *Suppose* $C([0,1], ||\cdot||)$ *is the Banach space of all continuous functions on* Ω. *Then* $z = \sum_{i=0}^\infty z_i(x)$ *uniformly converges to the exact solution on* $[0,1]$.

Proof. As proved in [12], consider $|z_1(x)| < +\infty$ for all $x \in [0,1]$. The sequence of the partial sum of the series is denoted as s_p. Let s_p and s_q be arbitrary partial sums with $p \geq q$. We need to prove that $s_p = \sum_{i=0}^p z_i(x)$ is a Cauchy sequence in $C([0,1], ||\cdot||)$. We have

$$\begin{aligned}\|s_p - s_q\|_\infty &= \max_{x\in[0,1]} |s_p - s_q| \\ &= \max_{x\in[0,1]} \left|\sum_{i=0}^p z_i(x) - \sum_{i=0}^q z_i(x)\right|\end{aligned}$$

$$
\begin{aligned}
&= \max_{x\in[0,1]} \left| \sum_{i=q+1}^{p} z_i(x) \right| \\
&= \max_{x\in[0,1]} \left| \sum_{i=q+1}^{p} \left(\frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} \left[a(t)z_i(t) + \int_0^t \mathcal{K}_1(t,s)\mathcal{A}_{1i}(s)ds \right.\right.\right. \\
&\quad \left.\left.\left. + \int_0^1 \mathcal{K}_2(t,s)\mathcal{A}_{2i}(s)ds \right] \right) dt \right| \\
&= \max_{x\in[0,1]} \left| \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} \left[a(t) \sum_{i=q}^{p-1} z_i(t) + \int_0^t \mathcal{K}_1(t,s) \sum_{i=q}^{p-1} \mathcal{A}_{1i}(s)ds \right.\right. \\
&\quad \left.\left. + \int_0^1 \mathcal{K}_2(t,s) \sum_{i=q}^{p-1} \mathcal{A}_{2i}(s)ds \right] dt \right|.
\end{aligned}
$$

Since, we know $\sum_{i=q}^{p-1} \mathcal{A}_{1i} = \mathcal{N}_1(s_{p-1}) - \mathcal{N}_1(s_{q-1})$, $\sum_{i=q}^{p-1} \mathcal{A}_{2i} = \mathcal{N}_2(s_{p-1}) - \mathcal{N}_2(s_{q-1})$ and $\sum\limits_{i=q}^{p-1} z_i = z(s_{p-1}) - z(s_{q-1})$. So, we reach at

$$
\begin{aligned}
||s_p - s_q||_\infty &= \max_{x\in[0,1]} \left(\left| \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} \left[a(t)(z(s_{p-1}) - z(s_{q-1})) \right.\right.\right. \\
&\quad + \int_0^t \mathcal{K}_1(t,s) \Big(\mathcal{N}_1(s_{p-1}) - \mathcal{N}_1(s_{q-1}) \Big) ds \\
&\quad \left.\left.\left. + \int_0^1 \mathcal{K}_2(t,s) \Big(\mathcal{N}_2(s_{p-1}) - \mathcal{N}_2(s_{q-1}) \Big) ds \right] dt \right| \right) \\
&\le \max_{x\in[0,1]} \left(\frac{1}{\Gamma(\mu)} \int_0^x |x-t|^{\mu-1} \left[|a(t)| \, |z(s_{p-1}) - z(s_{q-1})| \right.\right. \\
&\quad + \int_0^t |\mathcal{K}_1(t,s)| \cdot |\mathcal{N}_1(s_{p-1}) - \mathcal{N}_1(s_{q-1})| \, ds \\
&\quad \left.\left. + \int_0^1 |\mathcal{K}_2(t,s)| \cdot |\mathcal{N}_2(s_{p-1}) - \mathcal{N}_2(s_{q-1})| \, ds \right] dt \right) \\
&\le \frac{1}{\Gamma(\mu+1)} \Big[\|a(t)\|_\infty \|s_{p-1} - s_{q-1}\|_\infty + \mathcal{K}_1^* C_1 ||s_{p-1} - s_{q-1}||_\infty \\
&\quad + \mathcal{K}_2^* C_2 ||s_{p-1} - s_{q-1}||_\infty \Big] \\
&= \left(\frac{\|a\|_\infty + \mathcal{K}_1^* C_1 + \mathcal{K}_2^* C_2}{\Gamma(\mu+1)} \right) ||s_{p-1} - s_{q-1}||_\infty = \gamma_1 ||s_{p-1} - s_{q-1}||_\infty,
\end{aligned}
$$

where

$$
\gamma_1 = \left(\frac{\|a\|_\infty + \mathcal{K}_1^* C_1 + \mathcal{K}_2^* C_2}{\Gamma(\mu+1)} \right). \tag{4.10}
$$

Also, for $p = q + 1$,

$$
\|s_p - s_q\|_\infty \le \gamma_1 \|s_q - s_{q-1}\|_\infty \le \gamma_1^2 \|s_{q-1} - s_{q-2}\|_\infty \le \gamma_1^3 \|s_{q-2} - s_{q-3}\|_\infty
$$

$$\leq \cdots \leq \gamma_1^q \|s_1 - s_0\|_\infty.$$

So, we can write

$$\begin{aligned}\|s_p - s_q\|_\infty &\leq \|s_{q+1} - s_q\|_\infty + |s_{q+2} - s_{q+1}\|_\infty + \cdots + |s_p - s_{p-1}\|_\infty \\ &\leq \left[\gamma_1^q + \gamma_1^{q+1} + \cdots + \gamma_1^{p-1}\right] ||s_1 - s_0||_\infty \\ &\leq \gamma_1^q \left(\frac{1-\gamma_1^{p-q}}{1-\gamma_1}\right) ||z_1||_\infty.\end{aligned}$$

Since $0 < \gamma_1 < 1$, we have $(1 - \gamma_1^{p-q}) < 1$, then $||s_p - s_q||_\infty \leq \frac{\gamma_1^q}{1-\gamma_1}||z_1||_\infty$. As $||z_1(x)|| < \infty$ and $m \to \infty$, we get $||s_p - s_q||_\infty \to 0$. Hence, it can be concluded that s_p is a Cauchy sequence in $C[0,1]$ and $z = \lim_{n\to\infty} z_n$. Thus, the series is proved to be convergent by Weierstrass M-test. □

4.4.2. *Error bound.* The exact solution for (3.1) is given by $z(x) = \lim_{N\to\infty} z_N$ and the numerical solution can be obtained by truncating the series (4.6) and (4.9) up to a finite number of terms. If z_N gives the N terms approximated solution then, the absolute pointwise error bound depends on the partial sum $\sum_{n=0}^{N-1} z_n(x)$ which is bounded by $\frac{\mathcal{M}\gamma_1^N}{1-\gamma_1}$. γ_1 is defined in (4.10) which satisfies $0 < \gamma_1 < 1$ and $z_0 \leq \mathcal{M}$.

5. Numerical Approximation

In this section, we propose the numerical solution for (3.1). The approximation of fractional derivative $\mathbb{D}^\mu$ is made using the L1 scheme in [22]. The composite trapezoidal rule is used for approximating the Volterra integral and the rectangular rule for the Fredholm integral.

Now, to construct the mesh points, consider $\mathcal{N}$ to be any positive integer and $h = 1/\mathcal{N}$. Then, the mesh can be obtained as $\{x_n = nh : n = 0, 1, \ldots, \mathcal{N}\}$. The Caputo fractional order derivative is defined as

$$\mathbb{D}^\mu z(x_n) = \frac{1}{\Gamma(1-\mu)} \sum_{i=0}^{n-1} \int_{p=x_i}^{x_{i+1}} \frac{z'(p)}{(x_n - p)^\mu} dp. \tag{5.1}$$

Approximating $\mathbb{D}^\mu$ in (5.1) using the L1 approach at each x_n for $1 \leq n \leq \mathcal{N}$, we reach at

$$\begin{aligned}\mathbb{D}^\mu z(x_n) \approx \mathbb{D}^\mu_{\mathcal{N}} z_n &:= \frac{1}{\Gamma(1-\mu)} \sum_{i=0}^{n-1} \frac{z(x_{i+1}) - z(x_i)}{h} \int_{p=x_i}^{x_{i+1}} \frac{dp}{(x_n - p)^\mu} \\ &= \frac{1}{h^\mu \Gamma(2-\mu)} \sum_{i=0}^{n-1} (z(x_{i+1}) - z(x_i)) c_{n-i} + \mathcal{R}_n^{(1)},\end{aligned} \tag{5.2}$$

where $c_k = k^{1-\mu} - (c-1)^{1-\mu}$, $k \geq 1$. Approximating an integral part using the composite trapezoidal rule for the Volterra integral and rectangular rule for the

Fredholm integral for $1 \leq n \leq N$, we have

$$\int_0^{x_n} \mathcal{K}_1(x_n, s)z(s)ds = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \mathcal{K}_1(x_n, s)z(s)ds,$$

$$= \frac{h}{2} \sum_{i=0}^{n-1} [\mathcal{K}_1(x_n, x_{i+1})z(x_{i+1}) + \mathcal{K}_1(x_n, x_i)z(x_i)] + \mathcal{R}_n^{(2)}, \tag{5.3}$$

$$\int_0^{l} \mathcal{K}_2(x_n, s)z(s)ds = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \mathcal{K}_2(x_n, s)z(s)ds$$

$$= h \sum_{i=1}^{n} [\mathcal{K}_2(x_n, x_i)z(x_i)] + \mathcal{R}_n^{(3)}, \tag{5.4}$$

where the remainder terms $\mathcal{R}_n^{(i)}$ for $i = 1, 2, 3$ are given by

$$\mathcal{R}_n^{(1)} = (\mathbb{D}^\mu - \mathbb{D}_{\mathcal{N}}^\mu)z(x_n) = \left[\frac{1}{\Gamma(1-\mu)} \sum_{i=0}^{n-1} \frac{x_{i+1} + x_i - 2p}{(x_n - p)^\mu} + O(h^2)\right], \tag{5.5}$$

$$\mathcal{R}_n^{(2)} = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (x_{i+1/2} - p) \frac{\partial}{\partial p} [\mathcal{K}_1(x_n, p)z(p)]\, dp + O(h^2), \tag{5.6}$$

$$\mathcal{R}_n^{(3)} = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (x_{i-1} - p) \frac{\partial}{\partial p} [\mathcal{K}_2(x_n, p)z(p)]\, dp + O(h). \tag{5.7}$$

Finally, using (5.2), (5.3) and (5.4), we construct the difference scheme as

$$\mathbb{D}_{\mathcal{N}}^\mu z(x_n) + a(x_n)z(x_n) + \frac{h}{2} \sum_{i=0}^{n-1} [\mathcal{K}_1(x_n, x_{i+1})z(x_{i+1}) + \mathcal{K}_1(x_n, x_i)z(x_i)]$$
$$+ h \sum_{i=1}^{n} [\mathcal{K}_2(x_n, x_i)z(x_i)] = f(x_n) + \mathcal{R}_n^{(i)}, \quad \text{for } n = 1, 2, \dots \mathcal{N},$$
$$z(0) = z_0,$$

where $\mathcal{R}_n^{(i)} = \mathcal{R}_n^{(1)} + \mathcal{R}_n^{(2)} + \mathcal{R}_n^{(3)}$ described as in (5.5), (5.6) and (5.7). Neglecting the remainder terms for $n = 1, 2, \dots, \mathcal{N}$, we get the fully discrete scheme as

$$\mathbb{D}_{\mathcal{N}}^\mu z_n + a_n z_n + \frac{h}{2} \sum_{i=0}^{n-1} [\mathcal{K}_1(x_n, x_{i+1})z_{i+1} + \mathcal{K}_1(x_n, x_i)z_i] + h \sum_{i=1}^{n} [\mathcal{K}_2(x_n, x_i)z_i] = f_n, \tag{5.8}$$
$$z(0) = z_0^N.$$

5.1. **Convergence analysis.** In this section, we find the error estmates for approximating (3.1) using the numerical scheme (5.8).

Lemma 5.1. *For all $\mu \in [0, 1]$ and $n \geq 1$. If*

$$B(n) = n^{1-\mu} + 2\Big((n-1)^{1-\mu} + (n-2)^{1-\mu} + (n-3)^{1-\mu} + \cdots + 1^{1-\mu}\Big) - \frac{2}{2-\mu} n^{2-\mu},$$

then $|B(n)| \leq C$, where C is independent of n and μ.

Proof. The detailed proof of this lemma is discussed in [18]. □

Theorem 5.1. *For a constant C and fractional order derivative $\mu \in (0,1)$, the following inequality follows:*

$$|\mathcal{R}_n^{(1)}| \leq Ch^{2-\mu}. \tag{5.9}$$

Proof. Solving the L.H.S of (5.9)

$$\begin{aligned}
&\frac{1}{\Gamma(1-\mu)}\sum_{i=0}^{n-1}\frac{x_{i+1}+x_i-2p}{(x_n-p)^{\mu}}dp\\
=&\frac{-1}{\Gamma(1-\mu)}\sum_{i=0}^{n-1}\frac{1}{1-\mu}(2i+1)h^{2-\mu}\left[(n-i-1)^{1-\mu}-(n-i)^{1-\mu}\right]\\
&+\frac{1}{\Gamma(1-\mu)}\sum_{i=0}^{n-1}\frac{2}{1-\mu}h^{2-\mu}\Big[(i+1)(n-i-1)^{1-\mu}-i(n-i)^{1-\mu}\\
&+\frac{1}{\Gamma(1-\mu)}\sum_{i=0}^{n-1}\frac{2}{(2-\mu)(1-\mu)}h^{2-\mu}[(n-i-1)^{2-\mu}-(n-i)^{2-\mu}]\Big]\\
=&\frac{h^{2-\mu}}{\Gamma(2-\mu)}\left[n^{1-\mu}+2((n-1)^{1-\mu}+(n-2)^{1-\mu})+\cdots+1^{1-\mu}\right]-\frac{2h^{2-\mu}}{\Gamma(3-\mu)}n^{2-\mu}\\
=&\frac{h^{2-\mu}}{\Gamma(2-\mu)}\left[n^{1-\mu}+2((n-1)^{1-\mu}+(n-2)^{1-\mu}+\cdots+1^{1-\mu})-\frac{2}{2-\mu}n^{2-\mu}\right].
\end{aligned}$$

Let $B(n)=n^{1-\mu}+2((n-1)^{1-\mu}+(n-2)^{1-\mu}+(n-3)^{1-\mu}+\cdots+1^{1-\mu})-\frac{2}{2-\mu}n^{2-\mu}$. From Lemma 5.1, $|B(n)|$ is bounded for all $\mu\in[0,1]$ and all $n\geq 1$. So, taking into fact that $\frac{1}{\Gamma(2-\mu)}\leq 2$ for all $\mu\in[0,1]$, we get

$$\left|\frac{1}{\Gamma(1-\mu)}\sum_{i=0}^{n-1}\frac{x_{i+1}+x_i-2p}{(x_n-p)^{\mu}}dp\right|\leq 2h^{2-\mu}. \tag{5.10}$$

As a result, from (5.10), we obtain $\mathcal{R}_{nr}^{(1)}\leq C\mathcal{N}^{-(2-\mu)}$. □

The above theorem proves that the solution obtained using the L1 scheme on a uniform mesh is $O(\mathcal{N}^{-(2-\mu)})$ accurate. But, when the solutions have a mild singularity at the initial mesh point $x=0$, then the order of accuracy will be $O(\mathcal{N}^{-\mu})$ and $O(\mathcal{N}^{-1})$ on any sub-domain that is bounded away from $x=0$. For the analysis of such cases, one may refer to [11, 22]. We have considered $\mathcal{R}_n^{(1)}=\mathcal{R}_{ns}^{(1)}+\mathcal{R}_{nr}^{(1)}$, where $\mathcal{R}_{ns}^{(1)}$ is the remainder term for the case, where there is a mild singularity at $x=0$ and $\mathcal{R}_{nr}^{(1)}$ is the remainder term for the case where the solution is regular. The following lemma gives the truncation error for the Caputo order derivative due to the presence of weak singularity at the initial mesh point.

Lemma 5.2. *For each mesh point x_n, $n=1,2,\ldots,\mathcal{N}$, we have the following estimate while there is a mild singularity at the initial mesh point $x=0$*

$$|\mathcal{R}_{ns}^{(1)}|\leq Cn^{-(\mu+1)},\quad \textit{for all}\, n=1,2\ldots,\mathcal{N}.$$

Proof. One may refer to [22, 27] for the detailed proof of the lemma. □

Lemma 5.3. *The remainder term* $\mathcal{R}_n^{(2)}$, $n = 1, 2, \dots, \mathcal{N}$, *satisfies the following estimate:*

$$|\mathcal{R}_n^{(2)}| \leq C\mathcal{N}^{-1}.$$

Proof. From (5.6), we get

$$\begin{aligned}
|\mathcal{R}_n^{(2)}| &= \left|\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (x_{i+1/2} - p) \frac{d}{dp}[\mathcal{K}(x_n, p)z(p)]dp\right| \\
&\leq \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (x_{i+1/2} - p) \left|\frac{d}{dp}[\mathcal{K}(x_n, p)z(p)]\right| dp \\
&\leq \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (x_{i+1/2} - p) \left|\frac{\partial}{\partial p}[\mathcal{K}(x_n, p)z(p)] + \frac{\partial}{\partial z}[\mathcal{K}(x_n, p)z(p)]z'(p)\right| dp \\
&\leq Ch \int_0^{x_n} (1 + z'(p))dp \leq Ch \leq C\mathcal{N}^{-1}, \quad \text{for all } n = 1, 2, \dots, \mathcal{N},
\end{aligned}$$

which is the desired bound. □

Lemma 5.4. *Assuming that* $\mathcal{K}_2$ *is a continuous bounded function on* $[0, 1]$. *The remainder term* $\mathcal{R}_n^{(3)}$, $n = 1, 2, \dots, \mathcal{N}$, *satisfies the following estimate:*

$$|\mathcal{R}_n^{(3)}| \leq C\mathcal{N}^{-1}.$$

Proof. From (5.7), we get

$$\begin{aligned}
|\mathcal{R}_n^{(3)}| &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (x_i - p) \left|\frac{\partial}{\partial p}[\mathcal{K}_2(x_n, p)z(p)]dp\right| \\
&\leq h \int_0^1 \left|\frac{\partial}{\partial p}[\mathcal{K}_2(x_n, p)z(p)]dp\right| \\
&\leq h \int_0^1 \left\{\left|\frac{\partial \mathcal{K}_2(x_n, p)}{\partial p}\right| |z(p)| + |\mathcal{K}_2(x_n, p)||z'(p)|\right\} dp \\
&\leq Ch \leq C\mathcal{N}^{-1}, \quad \text{for all } n = 1, 2, \dots, \mathcal{N}.
\end{aligned}$$

This proves the required estimate. □

Consider e_n to be the error function. $\{z(x_n)\}_{n=1}^{\mathcal{N}}$ be the exact solution of the continuous problem (3.1) and $\{z_n\}_{n=1}^{\mathcal{N}}$ be the numerical solution of (5.8), then the error function is defined as:

$$\mathrm{e}_n = |z(x_n) - z_n|, \quad \mathrm{e}_0 = 0, \quad \text{for } n = 1, 2, \dots, \mathcal{N}. \tag{5.11}$$

Theorem 5.2. *If* $\{z(x_n)\}_{n=1}^{\mathcal{N}}$ *is the exact solution to the continuous problem* (3.1) *and* $\{z_n\}_{n=1}^{\mathcal{N}}$ *is the numerical solution of* (5.8), *then the error bound when there exists a weak singularity at the initial mesh point is given by*

$$|\mathrm{e}_n| \leq |z(x_n) - z_n| \leq h^\mu \Gamma(2 - \mu) \sum_{i=1}^{n} |\mathcal{R}_n^{(i)}| \leq C \left[hx_n^{\mu-1} + h^\mu \mathcal{N}^{-(1-\mu)}\right]. \tag{5.12}$$

Proof. From (5.11) and using Lemma 3 of [27], we have

$$\begin{aligned} |\mathrm{e}_n| \le |z(x_n) - z_n| &\le h^{\mu}\Gamma(2-\mu)\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(i)}| \\ &\le Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_{ns}^{(1)}| + Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(2)}| + Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(3)}|. \end{aligned} \tag{5.13}$$

Applying Lemma 5.2, Lemma 5.3 and Lemma 5.4, (5.13) reduces to

$$|\mathrm{e}_n| \le Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}i^{-(1+\mu)} + Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}\mathcal{N}^{-1}.$$

Finally employing Lemma 3 of [11] and Lemma 4.3 of [22] to the above inequality, the desired result (5.12) is obtained. □

Theorem 5.3. *If* $\{z(x_n)\}_{n=1}^{\mathcal{N}}$ *is the exact solution to the continuous problem* (3.1) *and* $\{z_n\}_{n=1}^{\mathcal{N}}$ *is the numerical solution of* (5.8), *then the error bound is given by*

$$|\mathrm{e}_n| \le |z(x_n) - z_n| \le CX^{\mu}h, \quad n = 1, 2, \dots, \mathcal{N}.$$

Proof. We have

$$\begin{aligned} |z(x_n) - z_n| &\le h^{\mu}\Gamma(2-\mu)\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(i)}| \\ &\le Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(1)}| + Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(2)}| + Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(3)}|. \end{aligned}$$

Combining Theorem 5.1, Lemma 5.3 and Lemma 5.4, we get

$$|z(x_n) - z_n| \le C\mathcal{N}^{-1} + C\gamma_{n-1}^{-1}h^2.$$

By the definition of γ_n, we have $n^{-\mu}\gamma_{n-1}^{-1} \le \frac{1}{1-\mu}$, $n = 1, 2, \dots, \mathcal{N}$. Consequently, for all n such that $nh \le X$, we have

$$\begin{aligned} |z(x_n) - z_n| &\le C\mathcal{N}^{-1} + C\gamma_{n-1}^{-1}h^2 \\ &= C\mathcal{N}^{-1} + Cn^{-\mu}\gamma_{n-1}^{-1}h^2 = C\mathcal{N}^{-1} + Cn^{-\mu}n^{-\mu}\gamma_{n-1}^{-1}n^{\mu}h^2 \\ &= C\mathcal{N}^{-1} + C\left(\frac{1}{1-\mu}\right)(nh)^{\mu}h^{2-\mu} \le C\mathcal{N}^{-1} + CX^{\mu}h^{2-\mu}, \end{aligned}$$

which gives the desired result. □

$\mathcal{E}_{\mu}^{\mathcal{N}} = \max_{0\le n\le \mathcal{N}}|z(x_n) - z_n|$ denotes the pointwise error while using the numerical scheme, while $\mathcal{P}_{\mu}^{\mathcal{N}} = \frac{\mathcal{E}_{\mu}^{\mathcal{N}}/\mathcal{E}_{\mu}^{2\mathcal{N}}}{\ln 2}$ denotes the order of convergence.

6. Numerical Experiments

This section consists of two numerical examples which clearly depict the efficiency of the proposed techniques.

Example 6.1. Consider the following model

$$\mathbb{D}^{\mu} z(x) + a(x)z(x) = f(x) + \int_0^x \mathcal{K}_1(x,s)z(s)ds + \int_0^1 \mathcal{K}_2(x,s)z(s)ds,$$

with the initial condition $z(0) = 1$. Here $f(x) = \exp(x) - 1 + x^{1-\mu} E_{1,2-\mu}(x)$, $E_{1,2-\mu}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+2-\mu)}$, $a(x) = 0$, $\mathcal{K}_1(x,s) = 1$ and $\mathcal{K}_2(x,s) = 2s - 1$. The exact solution is $z(x) = \exp(x)$.

First approximating $f(x)$ using the Chebyshev polynomials, $f_C(x) = \sum_{i=0}^{5} C_i \mathcal{T}_i(2x - 1)$, $x \in [0,1]$. Here,

$$C_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(0.5x + 0.5)\mathcal{T}_0(x)}{\sqrt{1-x^2}} dx,$$
$$C_i = \frac{2}{\pi} \int_{-1}^{1} \frac{f(0.5x + 0.5)\mathcal{T}_i(x)}{\sqrt{1-x^2}} dx, \quad i = 0, 1, \dots, 5.$$

So, we get

$$\begin{aligned} f_C(x) \approx & \frac{x^{1-\mu}}{\Gamma(2-\mu)} + \frac{x^{2-\mu}}{\Gamma(3-\mu)} + \frac{x^{3-\mu}}{\Gamma(4-\mu)} + \frac{x^{4-\mu}}{\Gamma(5-\mu)} + \frac{x^{5-\mu}}{\Gamma(6-\mu)} + \frac{x^{6-\mu}}{\Gamma(7-\mu)} \\ & - 1.0002x - 0.499197x^2 - 0.166489x^3 - 0.0437939x^4 - 0.00868682x^5 \\ & - 0.00004. \end{aligned} \tag{6.1}$$

Substituting (6.1) and applying (4.4), we obtain the two term approximated solution as follows:

$$\begin{aligned} z(x) =& z_0(x) + z_1(x) \\ =& 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} - 1.0002\frac{x^{1+\mu}}{\Gamma(2+\mu)} - 0.499197\frac{(x^{2+\mu})}{\Gamma(3+\mu)} \\ & - 0.166489\frac{x^{3+\mu}}{\Gamma(4+\mu)} - 0.0437939\frac{x^{4+\mu}}{\Gamma(5+\mu)} - 0.0086862\frac{x^{5+\mu}}{\Gamma(6+\mu)} \\ & - 4.00e - 05\frac{x^{\mu}}{\Gamma(1+\mu)} + 0.000198413\frac{x^{7+\mu}\Gamma(8)}{\Gamma(8+\mu)} + 0.00138889\frac{x^{6+\mu}\Gamma(7)}{\Gamma(7+\mu)} \\ & + 0.00833333\frac{x^{5+\mu}\Gamma(6)}{\Gamma(6+\mu)} + 0.0416667\frac{x^{4+\mu}\Gamma(5)}{\Gamma(5+\mu)} + 0.166667\frac{x^{3+\mu}\Gamma(4)}{\Gamma(4+\mu)} \\ & + 0.5\frac{x^{2+\mu}\Gamma(3)}{\Gamma(3+\mu)} + \frac{x^{1+\mu}\Gamma(2)}{\Gamma(2+\mu)} - 1.0002\frac{x^{2+2\mu}\Gamma(3+\mu)}{\Gamma(3+2\mu)(2+\mu)} \\ & - 0.499197\frac{x^{3+2\mu}\Gamma(4+\mu)}{\Gamma(4+2\mu)(3+\mu)} - 0.166489\frac{x^{4+2\mu}\Gamma(5+\mu)}{\Gamma(5+2\mu)(4+\mu)} \end{aligned}$$

$$- 0.0437939\frac{x^{5+2\mu}\Gamma(6+\mu)}{\Gamma(6+2\mu)(5+\mu)} - 0.0086862\frac{x^{6+2\mu}\Gamma(7+\mu)}{\Gamma(7+2\mu)(6+\mu)}$$
$$- 0.00004\frac{x^{1+2\mu}\Gamma(2+\mu)}{\Gamma(2+2\mu)(1+\mu)}.$$

Simplify, the problem using the Bernstein polynomials $\sum_{i=0}^{n}\mathcal{D}_i\mathcal{B}_i(x)$ with $i=5$ gives the approximation for $f(x)$ as

$$\begin{aligned} f_B(x) \approx& 1 + \frac{x^{1-\mu}}{\Gamma(2-\mu)} + \frac{x^{2-\mu}}{\Gamma(3-\mu)} + \frac{x^{3-\mu}}{\Gamma(4-\mu)} + \frac{x^{4-\mu}}{\Gamma(5-\mu)} + \frac{x^{5-\mu}}{\Gamma(6-\mu)} + \frac{x^{6-\mu}}{\Gamma(7-\mu)} \\ &- 1.10701379x - 0.490191813x^2 - 0.108529819x^3 - 0.0120144007x^4 \\ &- 0.00053200429x^5. \end{aligned}$$

Using (4.8), the series solution using ADM-BP is obtained. We get the two term approximated solution as follows:

$$\begin{aligned} z(x) =& z_0(x) + z_1(x) \\ =& 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} - 1.10701\frac{x^{1+\mu}}{\Gamma(2+\mu)} - 0.49019\frac{x^{2+\mu}}{\Gamma(3+\mu)} \\ &- 0.10853\frac{x^{3+\mu}}{\Gamma(4+\mu)} - 0.012014\frac{x^{4+\mu}}{\Gamma(5+\mu)} - 0.00053\frac{x^{5+\mu}}{\Gamma(6+\mu)} \\ &+ 0.00019\frac{x^{7+\mu}\Gamma(8)}{\Gamma(8+\mu)} + 0.001389\frac{x^{6+\mu}\Gamma(7)}{\Gamma(7+\mu)} + 0.00833\frac{x^{5+\mu}\Gamma(6)}{\Gamma(6+\mu)} \\ &+ 0.04167\frac{x^{4+\mu}\Gamma(5)}{\Gamma(5+\mu)} + 0.16667\frac{x^{3+\mu}\Gamma(4)}{\Gamma(4+\mu)} + 0.5\frac{x^{2+\mu}\Gamma(3)}{\Gamma(3+\mu)} + \frac{x^{1+\mu}\Gamma(2)}{\Gamma(2+\mu)} \\ &- 1.10701\frac{x^{2+2\mu}\Gamma(3+\mu)}{\Gamma(3+2\mu)(2+\mu)} - 0.49019\frac{x^{3+2\mu}\Gamma(4+\mu)}{\Gamma(4+2\mu)(3+\mu)} \\ &- 0.10853\frac{x^{4+2\mu}\Gamma(5+\mu)}{\Gamma(5+2\mu)(4+\mu)} - 0.01201\frac{x^{5+2\mu}\Gamma(6+\mu)}{\Gamma(6+2\mu)(5+\mu)} \\ &- 0.00053\frac{x^{6+2\mu}\Gamma(7+\mu)}{\Gamma(7+2\mu)(6+\mu)}. \end{aligned}$$

Finally, using the classical ADM and approximating $f(x)$ using Taylor's polynomial denoted as $f_T(x)$, we get

$$\begin{aligned} f_T(x) \approx& 1 + \frac{x^{1-\mu}}{\Gamma(2-\mu)} + \frac{x^{2-\mu}}{\Gamma(3-\mu)} + \frac{x^{3-\mu}}{\Gamma(4-\mu)} + \frac{x^{4-\mu}}{\Gamma(5-\mu)} + \frac{x^{5-\mu}}{\Gamma(6-\mu)} \\ &+ \frac{x^{6-\mu}}{\Gamma(7-\mu)} - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} - \frac{x^5}{5!}. \end{aligned}$$

Recursively, using the scheme for ADM, the solution is obtained as

$$z(x) = z_0(x) + z_1(x)$$

$$
\begin{aligned}
=&1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}-\frac{x^{1+\mu}}{\Gamma(2+\mu)}-\frac{x^{2+\mu}}{\Gamma(3+\mu)}-\frac{x^{3+\mu}}{\Gamma(4+\mu)}\\
&-\frac{x^{4+\mu}}{\Gamma(5+\mu)}-\frac{x^{5+\mu}}{\Gamma(6+\mu)}+0.00019\frac{x^{7+\mu}\Gamma(8)}{\Gamma(8+\mu)}+0.00139\frac{x^{6+\mu}\Gamma(7)}{\Gamma(7+\mu)}\\
&+0.008333\frac{x^{5+\mu}\Gamma(6)}{\Gamma(6+\mu)}+0.041667\frac{x^{4+\mu}\Gamma(5)}{\Gamma(5+\mu)}+0.16667\frac{x^{3+\mu}\Gamma(4)}{\Gamma(4+\mu)}\\
&+0.5\frac{x^{2+\mu}\Gamma(3)}{\Gamma(3+\mu)}+\frac{x^{1+\mu}\Gamma(2)}{\Gamma(2+\mu)}-\frac{x^{2+2\mu}\Gamma(3+\mu)}{\Gamma(3+2\mu)(2+\mu)}-\frac{x^{3+2\mu}\Gamma(4+\mu)}{\Gamma(4+2\mu)(3+\mu)}\\
&-\frac{x^{4+2\mu}\Gamma(5+\mu)}{\Gamma(5+2\mu)(4+\mu)}-\frac{x^{5+2\mu}\Gamma(6+\mu)}{\Gamma(6+2\mu)(5+\mu)}-\frac{x^{6+2\mu}\Gamma(7+\mu)}{\Gamma(7+2\mu)(6+\mu)}.
\end{aligned}
$$

For the semi-analytical methods, the error is calculated using $\mathbb{E}_n^\infty = |z(x) - \sum_{i=0}^n z_i(x)|$. Figure 1(a) shows the error plot at $\mu = 0.25$ using the two term expansion of the modified ADM and the classical ADM. One can observe the robustness of ADM-BP and ADM-CP over the classical ADM, as the decrement in error is more in the case of our proposed techniques as compared to the classical technique. Similarly, Figure 1(b) depicts the comparison of $\mathbb{E}_2^\infty$ between all the three techniques. The error in the case of ADM-CP and ADM-BP is minimal compared to the classical ADM which makes it efficient for use when the source term in the model problem is any function rather than a polynomial function. The solution plots are graphically shown in Figure 2 for the proposed techniques and the classical ADM. The accuracy of the semi-analytical methods can be seen. Table 1 shows the error computed with one term and two term solutions. The data depicts that the error decreases gradually with the increase in number of iterations. Tables 2 and 3 give the pointwise error for $x \in [0,1]$ at $\mu = 0.01$ and $\mu = 0.95$, respectively. At some points close to zero, the error in classical ADM seems less than our proposed methods. But at rest all of the node points, the proposed methods prove to be more accurate and efficient which clearly shows their reliability.

Example 6.2. Consider the following Volterra-Fredholm FracIDE

$$\mathbb{D}^{0.75}z(x)+\frac{x^2e^x}{5}z(x)=\frac{6}{\Gamma(2.25)}x^{2.25}+\int_0^x e^x s z(s)ds+\int_0^1(4-s^{-3})z(s)ds,$$

with the initial condition $z(0) = 0$. The exact solution is $z(x) = x^3$.

Here, the source function is in the form of a polynomial function. We first approximate $f(x)$ using the Chebyshev polynomials, and then apply the recursive algorithm to obtain the series solution.

$$f_C(x)=\sum_{i=0}^{4}C_i\mathcal{T}_i(2x-1),\quad 0\le x\le 1,$$

where $C_0 = \frac{1}{\pi}\int_{-1}^{1}\frac{f(0.5x+0.5)\mathcal{T}_0(x)}{\sqrt{1-x^2}}dx$ and $C_i = \frac{2}{\pi}\int_{-1}^{1}\frac{f(0.5x+0.5)\mathcal{T}_i(x)}{\sqrt{1-x^2}}dx$, $i = 1, 2, \ldots, 6$. It implies that $f_C(x) \approx -0.2293888x^4 + 0.9765696x^3 + 1.666256x^2 - 0.0610748x +$

0.0010169. Applying (4.4), we obtain the approximated solution as

$$
\begin{aligned}
z(x) = &-0.2293888x^{4.75}\frac{\Gamma(5)}{\Gamma(5.75)} + 0.9765696x^{3.75}\frac{\Gamma(4)}{\Gamma(4.75)} + 1.666256x^{2.75}\frac{\Gamma(3)}{\Gamma(3.75)} \\
&-0.0610748x^{1.75}\frac{\Gamma(2)}{\Gamma(2.75)} + 0.0010169x^{0.75}\frac{\Gamma(1)}{\Gamma(1.75)}.
\end{aligned}
$$

$f(x)$ is approximated using the Bernstein polynomials $f_B(x) = \sum_{i=0}^{n} \mathcal{D}_i \mathcal{B}_i(x)$ with $i = 10$. Then applying the recursive algorithm for ADM to obtain the series solution, we get

$$
\begin{aligned}
f_B(x) \approx &-0.00056944194 93x^{10} + 0.006482451697x^9 - 0.03408x^8 \\
&+ 0.11017x^7 - 0.247127x^6 + 0.416952685x^5 - 0.584450992x^4 \\
&+ 0.9119425917x^3 + 1.641949867x^2 + 0.132354172x.
\end{aligned}
$$

Substituting $f_B(x)$ in (4.8), we obtain the approximated solution, which converges to the exact solution as shown in Figure 3(a). Also, the pointwise errors of the proposed techniques are shown using Figure 3(b). Hamoud and Ghadle in [12] solved this example using the classical ADM and obtained the exact solution in the first iteration. Since, the source term is already a polynomial function (in Taylor's series expansion), the proposed techniques (ADM-BP and ADM-CP) do not contribute much to decreasing the error in comparison to the solutions obtained in [12]. Table 4 shows the pointwise error obtained after the first term series solution using ADM-BP and ADM-CP. Though the error is less, the proposed methods are still ineffective for such model problems. Hence, one can conclude that the proposed techniques are suitable for the model problems where the source term is any other function except the polynomials.

We have also solved this example using the proposed numerical scheme (5.8). The solution is regular in its considered domain. The computed results are recorded in Table 5. One can clearly observe that the order of accuracy is almost first order accurate over the entire domain which satisfies the theoretical estimates. Figure 4(a) shows the solution plot for both the approximated and the exact solution at $\mu = 0.75$.

Example 6.3. Consider the following numerical experiment:

$$
\mathbb{D}^{\mu} z(x) + a(x)z(x) = f(x) + \int_0^x s z(s)ds + \int_0^1 (x-s)z(s)ds,
$$

where $a(x) = 0$ and the exact solution is $z(x) = x^{\mu} + x$.

The problem is solved using the proposed numerical scheme (5.8). Table 6 shows the error and rate of convergence for Example 6.3. Due to the presence of weak singularity, the order of accuracy is $O(\mathcal{N}^{-\mu})$ over the entire domain. A sharp singularity is present at the initial mesh point $x = 0$ which is evident from Figure 4(b) at $\mu = 0.1$.

Example 6.4. Consider a nonlinear model of Volterra-Fredholm FracIDE:

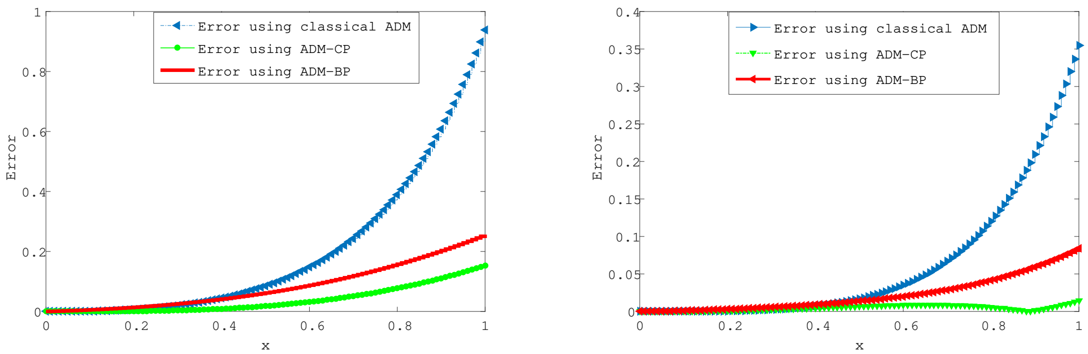
$$\mathbb{D}^{\mu} z(x) + a(x)z(x) = f(x) + \int_0^x z^4(s)ds - \int_0^1 f(x+s)z(s)ds,$$

where $a(x) = 0$, $f(x) = \frac{t^4\Gamma(5+\mu)}{24} + t^{5+\mu} - \frac{t^{17+4\mu}}{17+4\mu} + \frac{t}{5+\mu} + \frac{1}{6+\mu}$ and the exact solution is $z(x) = x^{\mu+4}$.

Table 7 shows the computed values of maximum pointwise error and order of convergence for arbitrary order fractional derivatives. The tabular data proves that the proposed numerical scheme also works well for a class of nonlinear Volterra-Fredholm FracIDEs.

7. Conclusion

This article intends to solve the fractional order Volterra-Fredholm integro-differential equations using semi-analytical and numerical methods. At first, we used the modified Adomian decomposition technique for the model problem where the source term is generalized as any kind of function (other than the polynomial function). The uniqueness and existence of the solutions are properly established and convergence of the method is carried out. Secondly, we have developed a fully discrete scheme for obtaining the numerical solution. Error analysis is done and it is validated with the help of a few numerical experiments. Finally, a comparison with some existing methods shows that the proposed methods are more efficient and robust.



(a) $\mu = 0.25$ *(b)* $\mu = 0.75$

FIGURE 1. Error plots using semi-analytical methods for Example 6.1.

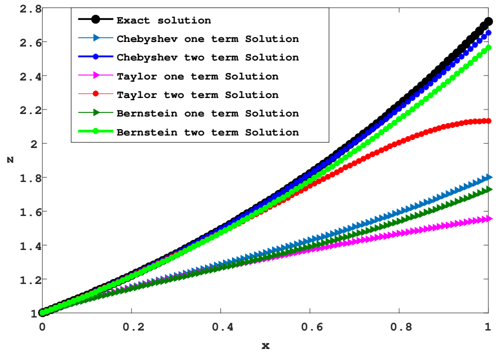
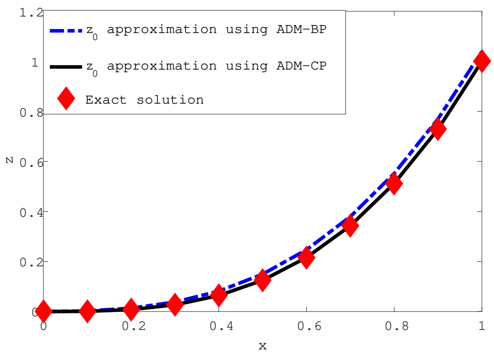
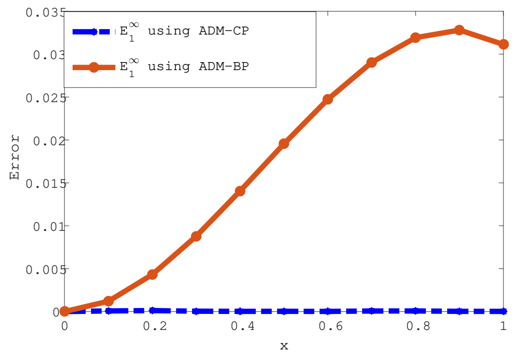


FIGURE 2. Solution plots using semi-analytical methods at $\mu = 0.5$ for Example 6.1.

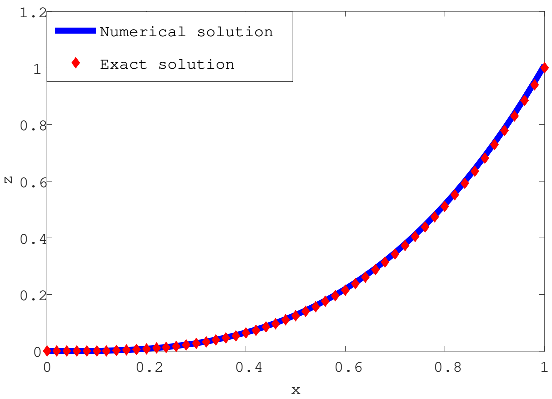


(a) Solution plots

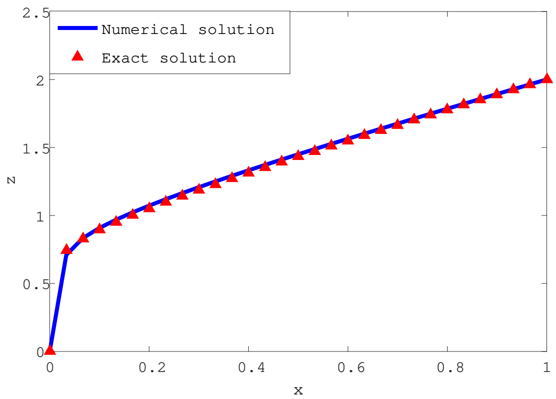


(b) Error plots

FIGURE 3. Plots using semi-analytical methods for Example 6.2.



(a) Example 6.2.



(b) Example 6.3.

FIGURE 4. Solution plots using the numerical method.

TABLE 1. Absolute pointwise errors using semi-analytical methods with $\mu = 0.5$ for Example 6.1.

x	ADM-BP		ADM-CP	
	$\mathbb{E}_1^\infty$	$\mathbb{E}_2^\infty$	$\mathbb{E}_1^\infty$	$\mathbb{E}_2^\infty$
0.2	7.7156e-2	6.2421e-3	7.0056e-2	1.0425e-3
0.4	2.2598e-1	1.8957e-2	2.0617e-1	2.2681e-3
0.6	4.2975e-1	4.2329e-2	3.9409e-1	2.2862e-3
0.8	6.8471e-1	8.3729e-2	6.3132e-1	2.1288e-2
1.0	9.9004e-1	1.5230e-1	9.1806e-1	6.6316e-2

TABLE 2. Absolute pointwise errors using semi-analytical methods with $\mu = 0.01$ for Example 6.1.

x	$\mathbb{E}_2^\infty$ using ADM-BP	$\mathbb{E}_2^\infty$ using ADM-CP	$\mathbb{E}_2^\infty$ using classical ADM
0.2	3.2232e-2	9.6072e-3	2.1724e-2
0.4	8.8292e-2	3.9878e-2	1.0156e-1
0.6	1.6813e-1	9.2759e-2	2.7822e-1
0.8	2.7053e-1	1.7013e-1	6.4637e-1
1.0	3.9279e-1	2.7399e-1	1.4248

TABLE 3. Absolute pointwise errors using semi-analytical methods with $\mu = 0.95$ for Example 6.1.

x	$\mathbb{E}_2^\infty$ using ADM-BP	$\mathbb{E}_2^\infty$ using ADM-CP	$\mathbb{E}_2^\infty$ using classical ADM
0.2	1.7740e-3	6.4018e-4	1.9533e-4
0.4	5.3420e-3	4.0840e-3	3.3104e-3
0.6	1.0931e-2	1.0136e-2	1.8948e-2
0.8	2.2620e-2	1.4696e-2	7.2710e-2
1.0	4.9022e-2	8.5299e-3	2.3222e-1

TABLE 5. Absolute pointwise errors using numerical approximation with $\mu = 0.75$ for Example 6.2.

$\mathcal{N}$	100	200	400	800	1600	3200
$\mathcal{E}_\mu^\mathcal{N}$	3.8450e-3	1.6347e-3	6.9200e-4	2.9214e-4	1.2312e-4	5.1839e-5
$\mathcal{P}_\mu^\mathcal{N}$	1.234	1.240	1.244	1.246	1.248	

TABLE 4. Absolute pointwise errors using semi-analytical methods with $\mu = 0.75$ for Example 6.2.

x	$\mathbb{E}_1^\infty$ using ADM-BP	$\mathbb{E}_1^\infty$ using ADM-CP
0.1	1.5167e-3	7.7086e-5
0.2	5.4081e-3	1.1525e-4
0.3	1.0989e-2	4.4583e-5
0.4	1.7565e-2	2.3456e-5
0.5	2.4440e-2	2.9868e-5
0.6	3.0919e-2	1.6814e-5
0.7	3.6305e-2	6.7225e-5
0.8	3.9902e-2	7.1810e-5
0.9	4.1014e-2	2.6809e-5
1	3.8945e-2	2.2190e-5

TABLE 6. Absolute pointwise errors using numerical approximation for Example 6.3.

$\mathcal{N}$	100	200	400	800	1600	3200
$\mu = 0.2$	5.3434e-2 0.144	4.8349e-2 0.173	4.2892e-2 0.186	3.7690e-2 0.193	3.2964e-2 0.197	2.8763e-2
$\mu = 0.4$	3.1270e-2 0.364	2.4291e-2 0.382	1.8635e-2 0.391	1.4209e-2 0.396	1.0801e-2 0.398	8.1980e-3
$\mu = 0.6$	1.2472e-2 0.566	8.4267e-3 0.583	5.6251e-3 0.592	3.7329e-3 0.596	2.4700e-3 0.598	1.6319e-3
$\mu = 0.8$	3.5118e-3 0.709	2.1476e-3 0.744	1.2819e-3 0.770	7.5155e-4 0.785	4.3619e-4 0.791	2.5213e-4

TABLE 7. Absolute pointwise errors using numerical approximation for Example 6.4.

$\mathcal{N}$	100	200	400	800	1600	3200
$\mu = 0.5$	1.9248e-2 0.975	9.7900e-3 0.518	6.8373e-3 0.502	4.8276e-3 0.501	3.4111e-3 0.501	2.4111e-3
$\mu = 0.7$	2.3163e-2 1.015	1.1464e-2 1.023	5.6412e-3 1.024	2.7745e-3 1.022	1.3666e-3 1.019	6.7455e-4
$\mu = 0.9$	3.2869e-2 1.009	1.6331e-2 1.020	8.0508e-3 1.026	3.9539e-3 1.028	1.9390e-3 1.028	9.5055e-4

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[1]Dept. of Mathematics,
National Institute of Technology Rourkela, India
Email address: `519ma2004@nitrkl.ac.in`

[1]Dept. of Mathematics,
National Institute of Technology Rourkela, India
Email address: `jugal@nitrkl.ac.in`