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BI-PERIODIC HYPER-FIBONACCI NUMBERS

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Abstract. In the present paper, we introduce and study a new generalization of hyper-Fibonacci numbers, called the bi-periodic hyper-Fibonacci numbers. Furthermore, we give a combinatorial interpretation using the weighted tilings approach and prove several identities relating these numbers. Moreover, we derive their generating function and new identities for the classical hyper-Fibonacci numbers.

1. Introduction

The Fibonacci numbers F_n are defined, as usual, by the recurrence relation

$$F_0 = 0,\ F_1 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 2.$$

The hyper-Fibonacci numbers denoted $F_n^{(r)}$, are introduced by Dil and Mezö [10], for $n, r \in \mathbb{N} \cup \{0\}$, as entries of an infinite matrix arranged such that $F_n^{(r)}$ is the entry of the rth row and nth column, satisfying

$$F_n^{(0)} = F_n,\ F_0^{(r)} = 0 \quad \text{and} \quad F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)}, \quad \text{for } n, r \geq 1. \tag{1.1}$$

The sum of the first $n+1$ elements of row $r-1$ is expressed by $F_n^{(r)}$, i.e.,

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}. \tag{1.2}$$

They satisfy many interesting number theoretical and combinatorial properties, see [9]. Belbachir and Belkhir [3] provided a combinatorial interpretation of the hyper-Fibonacci numbers in terms of linear tilings and gave some combinatorial identities.

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They also defined bivariate hyper-Fibonacci polynomials in [4], as

$$F_n^{(r)}(x,y) = xF_{n-1}^{(r)}(x,y) + yF_n^{(r-1)}(x,y), \quad \text{for } n, r \geq 1, \tag{1.3}$$

with initial conditions $F_n^{(0)}(x,y) = F_n(x,y)$, $F_0^{(r)}(x,y) = 0$, where x, y are real parameters and $F_n(x,y)$ is the nth bivariate Fibonacci polynomial, defined by (see [1,5])

$$F_0(x,y) = 0,\ F_1(x,y) = 1 \quad \text{and} \quad F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y).$$

The bivariate hyper-Fibonacci polynomials are given by the following explicit formula

$$F_{n+1}^{(r)}(x,y) = \sum_{k=r}^{\lfloor n/2\rfloor + r} \binom{n+2r-k}{k} x^{n+2r-2k} y^k. \tag{1.4}$$

The associated generating function is given as follows

$$\sum_{n\geq 0} F_n^{(r)}(x,y) z^n = \frac{y^r z}{(1-xz-yz^2)(1-xz)^r}. \tag{1.5}$$

For $y = 1$, we denote $F_n(x,y)$ by $F_n(x)$.

Edson and Yayenie [12] introduced a new generalization for the Fibonacci sequence, called as bi-periodic Fibonacci sequence, that depends on two real parameters a and b, defined for $n \geqslant 2$, as follows

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even}, \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd}, \end{cases} \tag{1.6}$$

with initial values $q_0 = 0$ and $q_1 = 1$. These sequences are found in the study of continued fraction expansion of the quadratic irrational numbers and combinatorics on words or dynamical system theory [18]. Some well-known sequences, such as the Fibonacci sequence, the Pell sequence and the k-Fibonacci sequence for some positive integer k, are special cases of this sequence. For more results related to this sequence, see [8,11–18]

The generating function of q_n is given by

$$\sum_{n\geq 0} q_n z^n = \frac{z\,(1+az-z^2)}{1-(ab+2)z^2+z^4}. \tag{1.7}$$

Yayenie [18] gave an explicit formula of bi-periodic Fibonacci numbers, as

$$q_{n+1} = a^{\xi(n)} \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{k} (ab)^{\lfloor n/2\rfloor - k}, \tag{1.8}$$

where $\xi(n) = n - 2\lfloor n/2\rfloor$, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd.

In this paper, we define a new generalization of hyper-Fibonacci numbers, which we will also call bi-periodic hyper-Fibonacci numbers. We give a combinatorial interpretation of these numbers using a weighted tilings approach and provide several combinatorial proofs of some identities. We also obtain new identities for the classical hyper-Fibonacci numbers. Moreover, by using the generating function of the bivariate hyper-Fibonacci polynomials, we establish the generating function of the bi-periodic hyper-Fibonacci sequence.

Definition 1.1. For any integers $n, r \geq 1$ and nonzero real numbers a and b, the bi-periodic hyper-Fibonacci numbers, denoted by $q_n^{(r)}$, are defined by

$$q_n^{(r)} = \sum_{k=0}^{n} a^{\xi(k)\xi(n+1)} b^{\xi(k+1)\xi(n)} (ab)^{\lfloor (n-k)/2 \rfloor} q_k^{(r-1)}, \tag{1.9}$$

with initial values $q_0^{(r)} = 0$ and $q_n^{(0)} = q_n$, where q_n is the nth bi-periodic Fibonacci number.

The first few generations are as follows in Table 1.

Table 1. Sequence of bi-periodic hyper-Fibonacci numbers in the first few generations

n	0	1	2	3	4	5	6
$q_n^{(0)}$	0	1	a	$ab+1$	a^2b+2a	$a^2b^2+3ab+1$	$a^3b^2+4a^2b+3a$
$q_n^{(1)}$	0	1	$2a$	$3ab+1$	$4a^2b+3a$	$5a^2b^2+6ab+1$	$6a^3b^2+10a^2b+4a$
$q_n^{(2)}$	0	1	$3a$	$6ab+1$	$10a^2b+4a$	$15a^2b^2+10ab+1$	$21a^3b^2+20a^2b+5a$
$q_n^{(3)}$	0	1	$4a$	$10ab+1$	$20a^2b+5a$	$35a^2b^2+15ab+1$	$56a^3b^2+35a^2b+6a$
$q_n^{(4)}$	0	1	$5a$	$15ab+1$	$35a^2b+6a$	$70a^2b^2+21ab+1$	$126a^3b^2+56a^2b+7a$

From the definition, we have the following recurrence relation:

$$q_n^{(r)} = \begin{cases} aq_{n-1}^{(r)} + q_n^{(r-1)}, & \text{if } n \text{ is even,} \\ bq_{n-1}^{(r)} + q_n^{(r-1)}, & \text{if } n \text{ is odd.} \end{cases} \tag{1.10}$$

Note that, for $a = b = 1$, we obtain the classical hyper-Fibonacci sequence (1.1).

2. Combinatorial Identities

The Fibonacci numbers can be interpreted as the number of ways to tile a board of length n (i.e., an n-board) with cells numbered 1 to n from left to right using only squares and dominoes; see [6, 7]. We expand the results to bi-periodic Fibonacci numbers using weighted tilings. We assign a weight to each square in a tiling based on its position. It is assigned a weight a if it is in an odd position and a weight b if it is in an even position. The weight of a tiling of an n-board is defined as the product of the weights of its individual tiles. The sum of all possible weighted tilings is given by q_{n+1}. Furthermore, the total of all possible weighted tilings of an $(n + 2r)$-board with at least r dominoes is given by the bi-periodic hyper-Fibonacci numbers $q_{n+1}^{(r)}$, as shown in Theorem 2.1.

For example, Figure 1 shows the tilings and the sum of their weights of a 5-board. We have $q_6^{(0)} = q_6 = a^3b^2 + 4a^2b + 3a$.

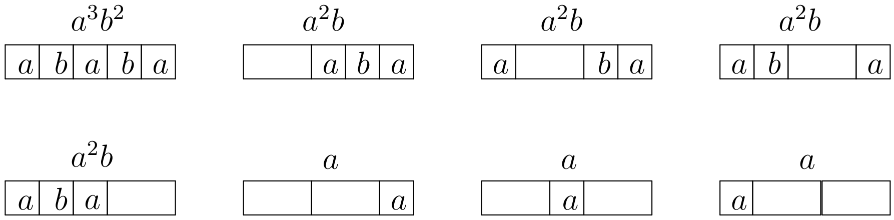

FIGURE 1. Tilings of a 5-board

Figure 2 shows the tilings and the sum of their weights of a 6-board with at least 2 dominoes, there are $q_3^{(2)} = 6ab + 1$ dispositions.

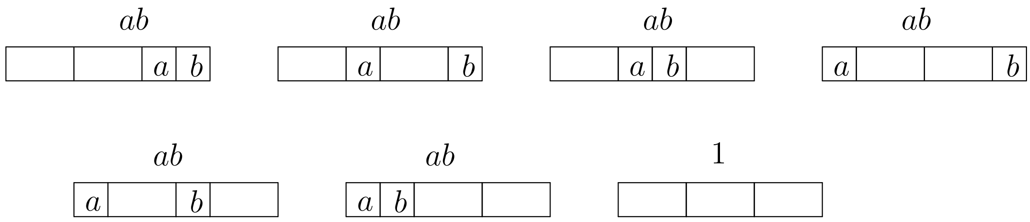

FIGURE 2. Tilings of a 6-board with at least 2 dominos

Therefore, we have the following results.

Theorem 2.1. *For $n, r \geq 0$, $q_{n+1}^{(r)}$ gives the weight of all tilings of an $(n+2r)$-board having at least r dominoes.*

Proof. Given $(n+2r)$-board. If it ends with a square, then there are $bq_n^{(r)}$ ways to tile the $(n+2r-1)$-board for n even and $aq_n^{(r)}$ for n odd. If it ends with a domino, then there are $q_{n+1}^{(r-1)}$ ways to tile the $(n+2(r-1))$-board. When $n = 0$, there is one way to tile a $2r$-board with at least r dominoes and there are q_{n+1} ways to tile a n-board with at least 0 dominoes. There is no way to tile an $(n+2r)$-board with at least r dominoes for $n < 0$. □

Let $f(n, k)$ be the number of weighted tilings having n tiles and exactly k dominoes. Then

$$f(n, k) = a^{\xi(n+k)} b^{\xi(n+k+1)} f(n-1, k) + f(n-1, k-1).$$

In fact, if the $(n+k)$-board ends in a square there are $a^{\xi(n+k)} b^{\xi(n+k+1)} f(n-1, k)$ ways to tile the board. If it ends with a domino, then there are $f(n-1, k-1)$ ways.

Lemma 2.1. *The number of weighted tilings having n tiles and exactly k dominoes is*

$$a^{\xi(n+k)} \binom{n}{k} (ab)^{\lfloor (n-k)/2 \rfloor}.$$

Proof. Let $g(n,k) = a^{\xi(n+k)}\binom{n}{k}(ab)^{\lfloor (n-k)/2\rfloor}$. Then

$$a^{\xi(n+k)}\binom{n}{k}(ab)^{\lfloor (n-k)/2\rfloor} = a^{\xi(n+k)}\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)(ab)^{\lfloor (n-k)/2\rfloor}.$$

Using $\lfloor (n-k)/2\rfloor = \lfloor (n-k-1)/2\rfloor + \xi(n+k+1)$, we get

$$\begin{aligned} a^{\xi(n+k)}\binom{n}{k}(ab)^{\lfloor (n-k)/2\rfloor} &= a^{\xi(n+k)}(ab)^{\xi(n+k+1)}\binom{n-1}{k}(ab)^{\lfloor (n-k-1)/2\rfloor} \\ &\quad + a^{\xi(n+k)}\binom{n-1}{k-1}(ab)^{\lfloor (n-k)/2\rfloor} \\ &= a^{\xi(n+k)}b^{\xi(n+k+1)}g(n-1,k) + g(n-1,k-1). \end{aligned}$$

Since $g(n,k)$ satisfies the same recurrence of $f(n,k)$ and the same initial conditions, we get result. □

In the following theorems, we establish an explicit formula for the bi-periodic hyper-Fibonacci sequence.

Theorem 2.2. *For $n, r \geq 0$, we have*

$$q_{n+1}^{(r)} = a^{\xi(n)} \sum_{k=r}^{\lfloor n/2\rfloor + r} \binom{n+2r-k}{k}(ab)^{\lfloor n/2\rfloor + r - k}. \tag{2.1}$$

Proof. From Theorem 2.1, $q_{n+1}^{(r)}$ counts the number of ways to tile an $(n+2r)$-board with at least r dominoes. On the other hand, using Lemma 2.1, the possible tilings with exactly k dominoes contains $n+2r-2k$ squares and $n+2r-k$ tiles, have cardinality $a^{\xi(n)}\binom{n+2r-k}{k}(ab)^{\lfloor n/2\rfloor+r-k}$. Since it contains at least r dominoes, the sum over $k \geq r$ gives the identity. □

Now, we establish a double-summation formula for even-numbered bi-periodic hyper-Fibonacci numbers $q_{2n+2}^{(r)}$.

Theorem 2.3. *For $n, r \geq 0$, we have*

$$q_{2n+2}^{(r)} = a \sum_{k=r}^{n+r}\sum_{j=0}^{k}(ab)^{\xi(n+r-k)}\binom{n+r-j}{k-j}\binom{n+r-k+j}{j}(ab)^{2\lfloor (n+r-k)/2\rfloor}. \tag{2.2}$$

Proof. Consider an $(n+2r+1)$-board. Since the length of the board is odd, there are an odd number of squares such that we have at least one in each tiling. Suppose there are i dominoes to the left of its median square and j dominoes to its right, whose total is at least r dominoes, i.e., $i+j \geq r$. The median square contributes an $a^{\xi(n+r-i-j+1)}b^{\xi(n+r-i-j)}$ to the weight (according to the position of the median square). Such tiling contains $2n+2r-2i-2j+1$ squares, so there are $n+r-i-j$ squares on each side of the median square. The left side gives $n+r-j$ tiles with i dominos. Hence, there are $a^{\xi(n+r-i-j)}\binom{n+r-j}{i}(ab)^{\lfloor (n+r-i-j)/2\rfloor}$ different ways. Similarly, we have $a^{\xi(n+r-i-j)}\binom{n+r-i}{j}(ab)^{\lfloor (n+r-i-j)/2\rfloor}$ different ways to tile the right side. Thus, the possible tilings have cardinality $a(ab)^{\xi(n+r-i-j)}\binom{n+r-i}{j}(ab)^{2\lfloor (n+r-i-j)/2\rfloor}$. Summing over $i+j\geq r$, we get

$$\begin{aligned}
&a\sum_{r\leq i+j\leq n+r}(ab)^{\xi(n+r-i-j)}\binom{n+r-j}{i}\binom{n+r-i}{j}(ab)^{2\lfloor (n+r-i-j)/2\rfloor}\\
=&a\sum_{k=r}^{n+r}\sum_{i+j=k}(ab)^{\xi(n+r-k)}\binom{n+r-j}{i}\binom{n+r-i}{j}(ab)^{2\lfloor (n+r-k)/2\rfloor}\\
=&a\sum_{k=r}^{n+r}\sum_{j=0}^{k}(ab)^{\xi(n+r-k)}\binom{n+r-j}{k-j}\binom{n+r-k+j}{j}(ab)^{2\lfloor (n+r-k)/2\rfloor}. \qquad \square
\end{aligned}$$

For $a=b=1$, we get the following identity.

Corollary 2.1. *For $n,r\geq 0$, the following identity holds*

$$F_{2n+2}^{(r)}=\sum_{k=r}^{n+r}\sum_{j=0}^{k}\binom{n+r-j}{k-j}\binom{n+r-k+j}{j}. \tag{2.3}$$

From the explicit formulas (1.8) and (2.1), we state the bi-periodic hyper-Fibonacci sequence in terms of the bi-periodic Fibonacci sequence and binomial sum.

Theorem 2.4. *Let $n\geq 0$ and $r\geq 1$ be integers, then we have*

$$q_{n+1}^{(r)}=q_{n+1+2r}-a^{\xi(n)}\sum_{k=0}^{r-1}\binom{n+2r-k}{k}(ab)^{\lfloor n/2\rfloor+r-k}. \tag{2.4}$$

Note that, if we take $a=b=1$, we get the following identity, see [3],

$$F_{n+1}^{(r)}=F_{n+1+2r}-\sum_{k=0}^{r-1}\binom{n+2r-k}{k}.$$

Theorem 2.5. *For $n,r\geq 1$, we have*

$$q_{n+1}^{(r)}=q_{n-1}+\sum_{k=0}^{r}a^{\xi(n)}b^{\xi(n+1)}q_n^{(k)}. \tag{2.5}$$

Proof. There exists $q_{n+1}^{(r)}$ ways to tile a board of length $n+2r$ containing at least r dominoes. Consider the number of dominoes at the end of each tiling. If tiling ends in at least r dominoes, then the final r dominoes cover cells $n+1$ through $n+2r$, while the remaining tilings can be done in q_{n+1} ways. On the other hand, if tilings ends in exactly $r-k$ dominoes for some $1\leq k\leq r$, preceded by a square at position $n+2k$ and contribute $a^{\xi(n)}b^{\xi(n+1)}$ to the weight, then the remaining $(n-1+2k)$-board can be tiled with at least k dominoes in $q_n^{(k)}$ ways. The result follows from the sum of over k, i.e.,

$$q_{n+1}^{(r)}=q_{n+1}+\sum_{k=1}^{r}a^{\xi(n)}b^{\xi(n+1)}q_n^{(k)}=q_{n-1}+\sum_{k=0}^{r}a^{\xi(n)}b^{\xi(n+1)}q_n^{(k)}.$$

□

Note that, if we take $a = b = x$, we get the following hyper-Fibonacci identity.

Corollary 2.2. *For $n, r \geq 1$, we have*

$$F_{n+1}^{(r)}(x) = F_{n-1}(x) + \sum_{k=0}^{r} xF_n^{(k)}(x). \tag{2.6}$$

For $a = b = 1$, we obtain the following identity, see [2],

$$F_{n+1}^{(r)} = F_{n-1} + \sum_{k=0}^{r} F_n^{(k)}.$$

In the following theorem, we give the recurrence relation of the bi-periodic hyper-Fibonacci sequence.

Theorem 2.6. *For $n \geq 0$ and $r \geq 2$, we have*

$$q_{n+2}^{(r)} = abq_n^{(r)} + 2q_{n+2}^{(r-1)} - q_{n+2}^{(r-2)}. \tag{2.7}$$

Proof. We will construct a 3-to-1 correspondence between the following two sets.

- The set of all tiled $(n + 2r - 1)$-boards with at least r dominoes. There are $q_n^{(r)}$ ways.
- The set of all tiled $(n+2r+1)$-boards with at least r dominoes and $(n+2r-3)$-boards with at least $r - 1$ dominoes. There are $q_{n+2}^{(r)} + q_n^{(r-1)}$ ways.

Consider an arbitrary tiling T of length $n + 2r - 1$, we can do the following.

1. Add two squares at the end of T to get an $(n + 2r + 1)$-board ending in a square. Then there are $abq_n^{(r)}$ ways.
2. Add a domino at the end of T to get an $(n+2r+1)$-board ending in a domino. Then there are $q_{n+2}^{(r-1)}$ ways.
3. Condition on whether T ends in a square or a domino.
   i. Suppose T ends in a square, then insert a domino immediately to the left of the square to creates $(n + 2r + 1)$-board ending in a square. Then there are $a^{\xi(n+1)}b^{\xi(n)}q_{n+1}^{(r-1)}$ ways to do it.
   ii. Suppose T ends in a domino, we remove the domino to get an $(n+2r-2)$-board. Then there are $q_n^{(r-1)}$ ways.

So, we conclude that

$$\begin{aligned} q_{n+2}^{(r)} + q_n^{(r-1)} &= abq_n^{(r)} + q_{n+2}^{(r-1)} + a^{\xi(n+1)}b^{\xi(n)}q_{n+1}^{(r-1)} + q_n^{(r-1)} \\ &= abq_n^{(r)} + 2q_{n+2}^{(r-1)} + q_n^{(r-1)} - q_{n+2}^{(r-2)}. \end{aligned}$$

Therefore

$$q_{n+2}^{(r)} = abq_n^{(r)} + 2q_{n+2}^{(r-1)} - q_{n+2}^{(r-2)}.$$ □

Note that, if we take $a = b = 1$, we get the following hyper-Fibonacci identity.

Corollary 2.3. *For $n \geq 0$ and $r \geq 2$, we have*

$$F_{n+2}^{(r)} = F_n^{(r)} + 2F_{n+2}^{(r-1)} - F_{n+2}^{(r-2)}. \tag{2.8}$$

The following theorem gives the nonhomogeneous recurrence relation for the bi-periodic hyper-Fibonacci sequence.

Theorem 2.7. *For $n, r \geq 1$, we have*

$$q_{n+1}^{(r)} = a^{\xi(n)} b^{\xi(n+1)} q_n^{(r)} + q_{n-1}^{(r)} + a^{\xi(n)} (ab)^{\lfloor n/2 \rfloor} \binom{n+r-1}{r-1}. \tag{2.9}$$

Proof. There are $q_{n+1}^{(r)}$ ways to tile a $(n+2r)$-board with at least r dominoes. We consider the last tile in a tiling, which can be either a square or a domino. If the board ends in a square, then there are $bq_n^{(r)}$ ways to tile $(n+2r-1)$-boards with at least r dominoes for n even and $aq_n^{(r)}$ ways to do it for n odd. If the board ends in a domino, we separate the tilings into two disjoint sets A and B. The set A with exactly r dominoes and the set B whose contain tilings with at least $r+1$ dominoes. Having in mind that one domino is fixed, the tilings in the set A has $n+r-1$ tiles with exactly $r-1$ dominoes, then by Lemma 2.1, we have $|A| = a^{\xi(n)} (ab)^{\lfloor n/2 \rfloor} \binom{n+r-1}{r-1}$. The tilings in the set B are equivalent to the tilings of an $(n+2r-2)$-boards with at least r dominoes, i.e., $|B| = q_{n-1}^{(r)}$. Therefore,

$$q_{n+1}^{(r)} = a^{\xi(n)} b^{\xi(n+1)} q_n^{(r)} + |A| + |B|.$$

□

Note that, if we take $a = b = x$, we get the following hyper-Fibonacci identity, see [4],

$$F_{n+1}^{(r)}(x) = xF_n^{(r)}(x) + F_{n-1}^{(r)}(x) + x^n \binom{n+r-1}{r-1}.$$

Theorem 2.8. *For $m, n \in \mathbb{N} \cup \{0\}$ with $m \leq r$, we have*

$$q_{n+m}^{(r)} = \sum_{k=0}^{m} a^{\xi(n+m+1)\xi(n+k)} b^{\xi(n+m)\xi(n+k+1)} \binom{m}{k} (ab)^{\lfloor (m-k)/2 \rfloor} q_{n+k}^{(r-k)}. \tag{2.10}$$

Proof. There exists $q_{n+m}^{(r)}$ ways to tile a board of length $(n+m+2r-1)$ containing at least r dominoes. Consider the number of dominoes among the first m tiles. The k dominoes can be placed among the first m tiles in $\binom{m}{k}$ ways and the remaining tiles which consisting of squares, contribute $a^{\xi(n+m+1)\xi(n+k)} b^{\xi(n+m)\xi(n+k+1)} (ab)^{\lfloor (m-k)/2 \rfloor}$ to the weight. The remaining right board has a length of $n-1+2r-k$, with at least $r-k$ dominos that can be tiled in $q_{n+k}^{(r-k)}$ ways. Summing over all possible k completes the proof. □

Note that, if we take $a = b = x$ and $m = r$, we get the following hyper-Fibonacci identity, see [4],

$$F_{n+r}^{(r)} = \sum_{k=0}^{r} \binom{r}{k} x^{r-k} F_{n+k}^{(r-k)}.$$

The bi-periodic hyper-Fibonacci sequence can be expressed in terms of the combinatorial sum of bi-periodic Fibonacci sequence.

Theorem 2.9. *For $n, r \geq 1$, we have*

$$q_n^{(r)} = \sum_{k=1}^{n} a^{\xi(n+1)\xi(k)} b^{\xi(n)\xi(k+1)} \binom{n+r-k-1}{r-1} (ab)^{\lfloor (n-k)/2 \rfloor} q_k. \tag{2.11}$$

Proof. The left-hand side of this equality counts the number of ways to tile a board of length $n+2r-1$ containing at least r dominoes.

The right-hand side is obtained by conditioning on the location of the rth domino. Suppose that the rth domino occupies cell k and $k+1$ $(1 \leq k \leq n)$ (from the right). The left part is a tiling of some section of length $k-1$ which can be done in q_k ways. The rigth part is a tiling of the remaining portion of length $n+2r-2-k$ (i.e., cells $k+2$ through $n+2r-1$) with exactly $r-1$ dominos, which can be done in a $a^{\xi(n+1)\xi(k)} b^{\xi(n)\xi(k+1)} \binom{n+r-k-1}{r-1} (ab)^{\lfloor (n-k)/2 \rfloor}$ ways (according to the parity of the numbers n and k). The result follows from considering the sum of all possible locations of the r^{th} domino. □

Note that, if we take $a = b = x$, we get the following hyper-Fibonacci identity, see [4],

$$F_n^{(r)}(x) = \sum_{k=1}^{n} x^{n-k} \binom{n+r-k-1}{r-1} F_k(x).$$

In the following theorem, we give the alternating binomial sum of the bi-periodic hyper-Fibonacci numbers.

Theorem 2.10. *For $r, m, n \in \mathbb{N} \cup \{0\}$ with $m \leq r$,we have*

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} q_{n+m}^{(r-j)} = a^{\xi(n)\xi(m)} b^{\xi(n+1)\xi(m)} (ab)^{\lfloor m/2 \rfloor} q_n^{(r)}. \tag{2.12}$$

Proof. We proceed by induction on $m \leq r$. For $m = 1$ and $m = 2$, we get (1.10) and Theorem 2.6, respectively. Suppose that the result holds for all $i \leq m$. Then we can prove it for $m+1$

$$\begin{aligned}\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} q_{n+m+1}^{(r-j)} &= \sum_{j=0}^{m+1} (-1)^j \left(\binom{m}{j} + \binom{m}{j-1} \right) q_{n+m+1}^{(r-j)} \\ &= \sum_{j \geq 0} (-1)^j \binom{m}{j} q_{n+m+1}^{(r-j)} - \sum_{j \geq 0} (-1)^j \binom{m}{j} q_{n+m+1}^{(r-j-1)}.\end{aligned}$$

From (1.10), we obtain

$$\begin{aligned}\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} q_{n+m+1}^{(r-j)} &= \sum_{j \geq 0} (-1)^j \binom{m}{j} a^{\xi(n+m)} b^{\xi(n+m+1)} q_{n+m}^{(r-j)} \\ &= a^{\xi(n+m)} b^{\xi(n+m+1)} a^{\xi(n)\xi(m)} b^{\xi(n+1)\xi(m)} (ab)^{\lfloor m/2 \rfloor} q_n^{(r)}.\end{aligned}$$

Using $\xi(n+m)=\xi(n)+\xi(m)-2\xi(n)\xi(m)$ and $\lfloor m/2\rfloor=\lfloor (m+1)/2\rfloor-\xi(m)$, we get

$$\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}q_{n+m+1}^{(r-j)}=a^{\xi(n)\xi(m+1)}b^{\xi(n+1)\xi(m+1)}(ab)^{\lfloor (m+1)/2\rfloor}q_n^{(r)}.$$

Therefore, the identity is valid for all $m\leq r$. □

Note that, for $a=b=x$, we get the following result.

Corollary 2.4. *The following equality holds for any nonnegative integers $r\geq m$*

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}F_{n+m}^{(r-j)}=x^mF_n^{(r)}. \tag{2.13}$$

The bi-periodic Fibonacci sequence can be expressed in terms of the bi-periodic hyper-Fibonacci sequence.

Theorem 2.11. *For $r,m\in\mathbb{N}\cup\{0\}$, we have*

$$q_{m+1}=\sum_{k=0}^{m}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m)}b^{\xi(k)\xi(m+1)}(ab)^{\lfloor k/2\rfloor}q_{m+1-k}^{(r)}. \tag{2.14}$$

Proof. We proceed by induction on m. This is true for $m=0$. Suppose that the result holds for all $i\leq m$. Then we can prove it for $m+1$. From (1.10), we get

$$\begin{aligned}q_{m+2}&=a^{\xi(m+1)}b^{\xi(m)}q_{m+1}+q_m\\&=a^{\xi(m+1)}b^{\xi(m)}\sum_{k=0}^{m}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m)}b^{\xi(k)\xi(m+1)}(ab)^{\lfloor k/2\rfloor}q_{m+1-k}^{(r)}\\&\quad+\sum_{k=0}^{m-1}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m+1)}b^{\xi(k)\xi(m)}(ab)^{\lfloor k/2\rfloor}q_{m-k}^{(r)}.\end{aligned}$$

Using $\xi(m+1)=\xi(m-k+1)+\xi(k)\xi(m+1)-\xi(k)\xi(m)$ and $\xi(m)=\xi(m-k)+\xi(k)\xi(m)-\xi(k)\xi(m+1)$ we get $\xi(k)\xi(m)+\xi(m+1)=\xi(k)\xi(m+1)+\xi(m-k+1)$ and $\xi(k)\xi(m+1)+\xi(m)=\xi(k)\xi(m)+\xi(m-k)$. Therefore, we have

$$\begin{aligned}q_{m+2}&=\sum_{k=0}^{m}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m+1)+\xi(m-k+1)}b^{\xi(k)\xi(m)+\xi(m-k)}(ab)^{\lfloor k/2\rfloor}q_{m+1-k}^{(r)}\\&\quad+\sum_{k=0}^{m-1}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m+1)}b^{\xi(k)\xi(m)}(ab)^{\lfloor k/2\rfloor}q_{m-k}^{(r)}\\&=\sum_{k\geq 0}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m+1)}b^{\xi(k)\xi(m)}(ab)^{\lfloor k/2\rfloor}\left(a^{\xi(m-k+1)}b^{\xi(m-k)}q_{m+1-k}^{(r)}+q_{m-k}^{(r)}\right)\\&=\sum_{k=0}^{m+1}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m+1)}b^{\xi(k)\xi(m)}(ab)^{\lfloor k/2\rfloor}q_{m+2-k}^{(r)}.\end{aligned}$$

□

Note that, for $a=b=x$, we get the following result.

Corollary 2.5. *The following equality holds for any integers $r, m \geq 0$*

$$F_{m+1}(x) = \sum_{k=0}^{m} \binom{r}{k} (-1)^k x^k F^{(r)}_{m+1-k}(x). \tag{2.15}$$

3. Generating Function

We start by establishing the relationship between the bi-periodic hyper-Fibonacci sequence and the hyper-Fibonacci polynomials.

Lemma 3.1. *For $n, r \geq 0$, we have*

$$q_n^{(r)} = \frac{1}{2}\left(\left(1+\sqrt{\frac{a}{b}}\right) - (-1)^n\left(1-\sqrt{\frac{a}{b}}\right)\right) F_n^{(r)}\left(\sqrt{ab}\right). \tag{3.1}$$

Proof. Using (1.4), (2.1) and $\lfloor n/2 \rfloor = (n - \xi(n))/2$, we have

$$\begin{aligned}
q_n^{(r)} &= a^{\xi(n-1)} \sum_{k=r}^{\lfloor (n-1)/2\rfloor + r} \binom{n-1+2r-k}{k} (ab)^{(n-1-\xi(n-1))/2+r-k} \\
&= \left(\frac{a}{\sqrt{ab}}\right)^{\xi(n-1)} \sum_{k=r}^{\lfloor (n-1)/2\rfloor + r} \binom{n-1+2r-k}{k} \left(\sqrt{ab}\right)^{n-1+2r-2k} \\
&= \left(\sqrt{\frac{a}{b}}\right)^{\xi(n-1)} \sum_{k=r}^{\lfloor (n-1)/2\rfloor + r} \binom{n-1+2r-k}{k} \left(\sqrt{ab}\right)^{n-1+2r-2k} \\
&= \frac{\left(1+\sqrt{\frac{a}{b}}\right) - (-1)^n\left(1-\sqrt{\frac{a}{b}}\right)}{2} \sum_{k=r}^{\lfloor (n-1)/2\rfloor + r} \binom{n-1+2r-k}{k} \left(\sqrt{ab}\right)^{n-1+2r-2k}. \qquad \square
\end{aligned}$$

Theorem 3.1. *The generating function of the bi-periodic hyper-Fibonacci sequence is given by*

$$\sum_{n\geq 0} q_n^{(r)} z^n =$$
$$z\frac{\left(1+\sqrt{\frac{a}{b}}\right)\left(1+\sqrt{ab}z - z^2\right)\left(1+\sqrt{ab}z\right)^r + \left(1-\sqrt{\frac{a}{b}}\right)\left(1-\sqrt{ab}z - z^2\right)\left(1-\sqrt{ab}z\right)^r}{2\left(1-(ab+2)z^2+z^4\right)\left(1-abz^2\right)^r}.$$

Proof. Using Lemma 3.1 and (1.5), we get

$$\begin{aligned}
\sum_{n\geq 0} q_n^{(r)} z^n &= \frac{1}{2}\left(1+\sqrt{\frac{a}{b}}\right) \sum_{n\geq 0} F_n^{(r)}\left(\sqrt{ab}\right) z^n - \frac{1}{2}\left(1-\sqrt{\frac{a}{b}}\right) \sum_{n\geq 0} F_n^{(r)}\left(\sqrt{ab}\right)(-z)^n \\
&= \frac{1}{2}\left(1+\sqrt{\frac{a}{b}}\right) \frac{z}{\left(1-\sqrt{ab}z - z^2\right)\left(1-\sqrt{ab}z\right)^r} \\
&\quad - \frac{1}{2}\left(1-\sqrt{\frac{a}{b}}\right) \frac{-z}{\left(1+\sqrt{ab}z - z^2\right)\left(1+\sqrt{ab}z\right)^r},
\end{aligned}$$

which gives the desired result. $\square$

Note that, if we take $r = 0$, we obtain the generating function of the bi-periodic Fibonacci sequence (1.7). If we take $a = b = x$, we obtain the generating function of hyper-Fibonacci polynomials (1.5) with $y = 1$.

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