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QUANTITATIVE UNCERTAINTY PRINCIPLES FOR THE CANONICAL FOURIER BESSEL TRANSFORM

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Abstract. The aim of this paper is to prove new uncertainty principles for the Canonical Fourier Bessel transform. To do so we prove a quantitative uncertainty inequality about the essential supports of a nonzero function for this transformation.

1. Introduction

The classical linear canonical transform (LCT) is considered as a generalization of the Fourier transform, and was first proposed in the 1970s by Collins [5] and Moshinsky and Quesne [26]. Very recently, many works have been devoted the LCT under many different names and in different contexts. Namely, in [22] the LCT is known as the generalized Fresnel transform, in [4] is called ABCD transform and in [1] is also called the special affine Fourier transform. Also, the LCT has been studied by many authors for various Fourier transforms, for examples [11,23,34]. In [11], the authors introduced the Dunkl linear canonical transform (DLCT) which is a generalization of the LCT in the framework of Dunkl transform [7]. DLCT includes many well-known transforms such as the Dunkl transform [7, 10] and the canonical Fourier Bessel transform [8, 11]. The LCT plays an important role in many fields of optics, radar system analysis, GRIN medium system analysis, filter design, phase retrieval, pattern recognition and many others [3, 28, 29]. In [8] the authors established some important properties of the Canonical Fourier Bessel transform (QFBT) such as Riemann-Lebesgue lemma, inversion formula, Plancherel theorem and some uncertainty principles.

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On the other hand, the uncertainty principle plays one important role in signal processing. It describes a function and its Fourier transform, which cannot both be simultaneously sharply localized. If we try to limit the behaviour of one we lose control of the other. Many of these uncertainty principles have already been studied from several points of view for the Fourier transform, such as Heisenberg-Pauli-Weyl inequality [6] and local uncertainty inequality [30]. Uncertainty principles have implications in two main areas: quantum physics and signal analysis. In quantum physics, they tell us that a particle's speed and position cannot both be measured with arbitrary precision. In signal analysis, they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies the signal consists of. Timelimited functions and bandlimited functions are basic tools of signal and image processing. Unfortunately, the simplest form of the uncertainty principle tells us that a signal cannot be simultaneously time and bandlimited. This leads to the investigation of the set of almost time and almost bandlimited functions, which has been initially carried through Landau, Pollak [24, 25] and then by Donoho, Stark [9]. In recent past, many works have been devoted to establish some uncertainty principles in different setting and for various transforms (see for example [2, 12–21, 31]) and others.

The purpose of this paper is to obtain uncertainty principle similar to Donoho-Stark's principle for the QFBT.

In order to describe our results, we first need to introduce some facts about harmonic analysis related to Canonical Fourier Bessel transform. For more details, see [8].

Throughout this paper, α denotes a real number such that $\alpha \geqslant -\frac{1}{2}$. We use the following notation.

• $C_{e,0}(\mathbb{R})$ denotes the space of even continuous functions on $\mathbb{R}$ and vanishing at infinity. We provide $C_{e,0}(\mathbb{R})$ with the topology of uniform convergence.

• $L^{p,\alpha}$ denotes the Lebesgue space of measurable functions f on $\mathbb{R}_+$, such that

$$\|f\|_{p,\alpha} = \left(\int_0^{+\infty} |f(y)|^p y^{2\alpha+1} dy\right)^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \leqslant p < +\infty,$$
$$\|f\|_{\infty,\alpha} = \operatorname*{ess\,sup}_{y\in\mathbb{R}_+} |f(y)| < +\infty, \quad \text{if } p = +\infty.$$

We provide $L^{p,\alpha}$ with the topology defined by the norm $\|\cdot\|_{p,\alpha}$.

• $L^{2,\alpha}$ denotes the Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ given by

$$\langle f, g\rangle = \int_0^{+\infty} f(y)\overline{g(y)}y^{2\alpha+1}dy.$$

• $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an arbitrary matrix in $SL(2,\mathbb{R})$, such that $b \neq 0$.

Definition 1.1. The canonical Fourier Bessel transform of a function $f \in L^{1,\alpha}$ is defined by

$$\mathscr{F}_\alpha^m(f)(x) = \frac{c_\alpha}{(ib)^{\alpha+1}} \int_0^{+\infty} K_\alpha^m(x,y)f(y)y^{2\alpha+1}dy,$$

where

$$c_\alpha = \frac{1}{2^\alpha \Gamma(\alpha+1)} \tag{1.1}$$

and

$$K_\alpha^m(x,y) = e^{\frac{i}{2}\left(\frac{dx^2}{b}+\frac{ay^2}{b}\right)} j_\alpha\left(\frac{xy}{b}\right).$$

Here j_α denotes the normalized Bessel function of order $\alpha \geqslant -\frac{1}{2}$ and defined by [33]

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+1+k)}\left(\frac{z}{2}\right)^{2k}, \quad z \in \mathbb{C}.$$

Proposition 1.1 ([8])**.** *We denote by* Δ_α^m *the differential operator*

$$\Delta_\alpha^m = \frac{d^2}{dx^2} + \left(\frac{2\alpha+1}{x} - 2i\frac{d}{b}x\right)\frac{d}{dx} - \left(\frac{d^2}{b^2}x^2 + 2i(\alpha+1)\frac{d}{b}\right).$$

(1) *For each* $y \in \mathbb{R}$, *the kernel* $K_\alpha^m(\cdot, y)$ *of the canonical Fourier Bessel transform* $\mathscr{F}_\alpha^m$ *is the unique solution of*

$$\begin{cases} \Delta_\alpha^m K_\alpha^m(\cdot,y) = \frac{-y^2}{b^2}K_\alpha^m(\cdot,y), \\ K_\alpha^m(0,y) = e^{\frac{iay^2}{2b}}, \\ \frac{d}{dx}K_\alpha^m(0,y) = 0. \end{cases}$$

(2) *For each* $x, y \in \mathbb{R}$ *the kernel* K_α^m *has the following integral representation*

$$K_\alpha^m(x,y) = \begin{cases} \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}e^{\frac{i}{2}\left(\frac{dx^2}{b}+\frac{ay^2}{b}\right)}\int_0^1(1-t^2)^{\alpha-\frac{1}{2}}\cos(\frac{xyt}{b})dt, & \text{if } \alpha > -\frac{1}{2}, \\ e^{\frac{i}{2}\left(\frac{dx^2}{b}+\frac{ay^2}{b}\right)}\cos(\frac{xy}{b}), & \text{if } \alpha = -\frac{1}{2}. \end{cases}$$

In particular, we have

$$|K_\alpha^m(x,y)| \leqslant 1 \quad \text{for all } x, y \in \mathbb{R}. \tag{1.2}$$

Theorem 1.1 ([8])**.** (1) (*Plancherel theorem*) *If* $f \in L^{1,\alpha} \cap L^{2,\alpha}$, *then* $\mathscr{F}_\alpha^m(f) \in L^{2,\alpha}$ *and*

$$\|\mathscr{F}_\alpha^m(f)\|_{2,\alpha} = \|f\|_{2,\alpha}. \tag{1.3}$$

(2) (*Orthogonality relation*) *For every* $f, g \in L^{2,\alpha}$, *we have*

$$\langle f, g\rangle = \langle \mathscr{F}_\alpha^m(f), \mathscr{F}_\alpha^m(g)\rangle. \tag{1.4}$$

(3) (*The reversibility property*) *For all* $f \in L^{1,\alpha}$, *with* $\mathscr{F}_\alpha^m \in L^{1,\alpha}$, *we have*

$$(\mathscr{F}_\alpha^m \circ \mathscr{F}_\alpha^{m^{-1}})(f) = (\mathscr{F}_\alpha^{m^{-1}} \circ \mathscr{F}_\alpha^m)(f) = f, \quad a.e. \tag{1.5}$$

Babenko-Beckner inequality. Let $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitrary matrix in $SL(2,\mathbb{R})$, such that $b \neq 0$. Let p and q be real numbers such that $1 < p \leqslant 2$

and $\frac{1}{p}+\frac{1}{q}=1$. Then, $\mathscr{F}_\alpha^m$ extends to a bounded linear operator on $L^{p,\alpha}$, $\alpha \geqslant -\frac{1}{2}$ and we have

$$\|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} \leqslant |b|^{(\alpha+1)(\frac{2}{q}-1)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1} \|f\|_{p,\alpha}, \tag{1.6}$$

where c_α is the constant given by (1.1).

Riemann-Lebesgue lemma. For all $f \in L^{1,\alpha}$, the canonical Fourier Bessel transform $\mathscr{F}_\alpha^m(f)$ belongs to $C_{e,0}(\mathbb{R})$ and verifies

$$\|\mathscr{F}_\alpha^m(f)\|_{\infty,\alpha} \leqslant c_\alpha |b|^{-(\alpha+1)} \|f\|_{1,\alpha}. \tag{1.7}$$

2. Donoho-Stark's Uncertainty Principle for the Canonical Fourier Bessel Transform

In this section, based on the techniques of Donoho-Stark [9], we will show uncertainty principle of concentration-type the canonical Fourier Bessel transform.

In the following, we consider a pair of orthogonal projections on $L^{2,\alpha}$. The first is the time-limiting operator defined

$$P_S f = \chi_S f, \tag{2.1}$$

and the second is the frequency-limiting operator defined by

$$\mathscr{F}_\alpha^m(Q_\Sigma f) = \chi_\Sigma \mathscr{F}_\alpha^m(f), \tag{2.2}$$

where S and Σ are two measurable subsets of $\mathbb{R}_+$ and χ_S and χ_Σ denote the characteristic functions of S and Σ.

Definition 2.1. Let $0 < \varepsilon_S, \varepsilon_\Sigma < 1$ and let $f \in L^{2,\alpha}$ be a nonzero function.

(1) We say that f is ε_S-concentrated on S if

$$\|P_{S^c} f\|_{2,\alpha} \leqslant \varepsilon_S \|f\|_{2,\alpha}. \tag{2.3}$$

(2) We say that f is ε_Σ-concentrated on Σ for the canonical Fourier Bessel transform if

$$\|Q_{\Sigma^c} f\|_{2,\alpha} \leqslant \varepsilon_\Sigma \|f\|_{2,\alpha}. \tag{2.4}$$

P_S and Q_Σ are projections. Indeed, let $f, g \in L^{2,\alpha}$. By relation (1.4), we have

$$\begin{aligned}
\langle P_S^2 f, g\rangle = \langle P_S f, P_S g\rangle &= \langle \mathscr{F}_\alpha^m(P_S f), \mathscr{F}_\alpha^m(P_S g)\rangle \\
&= \int_0^{+\infty} \mathscr{F}_\alpha^m(P_S f)(y)\overline{\mathscr{F}_\alpha^m(P_S g)(y)} y^{2\alpha+1} dy \\
&= \int_S \mathscr{F}_\alpha^m(f)(y)\overline{\mathscr{F}_\alpha^m(g)(y)} y^{2\alpha+1} dy \\
&= \int_0^{+\infty} \mathscr{F}_\alpha^m(P_S f)(y)\overline{\mathscr{F}_\alpha^m(g)(y)} y^{2\alpha+1} dy \\
&= \langle P_S f, g\rangle.
\end{aligned}$$

Thus, $P_S^2 = P_S$ and hence P_S is a projection.

By the same way,

$$\begin{aligned}\langle Q_\Sigma^2 f, g\rangle = \langle Q_\Sigma f, Q_\Sigma g\rangle &= \langle \mathscr{F}_\alpha^m(Q_\Sigma f), \mathscr{F}_\alpha^m(Q_\Sigma g)\rangle \\ &= \int_0^{+\infty} \mathscr{F}_\alpha^m(Q_\Sigma f)(y)\overline{\mathscr{F}_\alpha^m(Q_\Sigma g)(y)}y^{2\alpha+1}dy \\ &= \int_\Sigma \mathscr{F}_\alpha^m(f)(y)\overline{\mathscr{F}_\alpha^m(g)(y)}y^{2\alpha+1}dy \\ &= \int_0^{+\infty} \mathscr{F}_\alpha^m(Q_\Sigma f)(y)\overline{\mathscr{F}_\alpha^m(g)(y)}y^{2\alpha+1}dy \\ &= \langle Q_\Sigma f, g\rangle.\end{aligned}$$

Thus, $Q_\Sigma^2 = Q_\Sigma f$ and hence $Q_\Sigma f$ is a projection.

For all $f \in L^{2,\alpha}$, given the kernel N which satisfies the following two conditions: $f(\cdot)N(\cdot, y) \in L^{1,\alpha}$ for almost every $y \in \mathbb{R}_+$ and if

$$\mathcal{M}f(x) = \int_0^{+\infty} f(y)N(x,y)y^{2\alpha+1}dy,$$

then $\mathcal{M}f \in L^{2,\alpha}$. Then we define the norm of $\mathcal{M}$ to be

$$\|\mathcal{M}\| = \sup_{f\in L^{2,\alpha}} \frac{\|\mathcal{M}f\|_{2,\alpha}}{\|f\|_{2,\alpha}}, \quad f \neq 0,$$

and the Hilbert-Schmidt norm of $\mathcal{M}$ is given by

$$\|\mathcal{M}\|_{HS} = \left(\int_0^{+\infty}\int_0^{+\infty} |N(x,y)|^2 x^{2\alpha+1}y^{2\alpha+1}dxdy\right)^{\frac{1}{2}}.$$

It is clear that $\|P_S\| = \|Q_\Sigma\| = 1$ (see [9]). If $|\Sigma| < +\infty$, where Σ is a set of finite measure of $\mathbb{R}_+$, we have by [27]

$$|\Sigma| = \int_\Sigma x^{2\alpha+1}dx.$$

Lemma 2.1. *If S and Σ are two measurable sets of $\mathbb{R}_+$ such that $|S| < +\infty$ and $|\Sigma| < +\infty$, then*

$$\|P_S Q_\Sigma\|_{HS} = \|Q_\Sigma P_S\|_{HS}.$$

Proof. From relations (1.5), (2.1) and (2.2), we have

$$\begin{aligned}Q_\Sigma P_S(f)(x) &= \frac{c_\alpha}{(-ib)^{\alpha+1}}\int_\Sigma \overline{K_\alpha^m(y,x)}\mathscr{F}_\alpha^m(\chi_S f)(y)y^{2\alpha+1}dy \\ &= \frac{c_\alpha}{(-ib)^{\alpha+1}}\int_\Sigma \overline{K_\alpha^m(y,x)}\left(\frac{c_\alpha}{(ib)^{\alpha+1}}\int_S K_\alpha^m(y,z)f(z)z^{2\alpha+1}dz\right)y^{2\alpha+1}dy \\ &= \frac{c_\alpha^2}{b^{2\alpha+2}}\int_S f(z)\left(\int_\Sigma \overline{K_\alpha^m(y,x)}K_\alpha^m(y,z)y^{2\alpha+1}dy\right)z^{2\alpha+1}dz \\ &= \int_S f(z)k(x,z)z^{2\alpha+1}dz,\end{aligned}$$

where

$$k(x,z)=\frac{c_\alpha^2}{b^{2\alpha+2}}\int_\Sigma \overline{K_\alpha^m(y,x)}K_\alpha^m(y,z)y^{2\alpha+1}dy,\quad z\in S, x\in\mathbb{R}_+.$$

In the same way, we get

$$\begin{aligned}P_SQ_\Sigma(f)(x)&=\chi_S(x)Q_\Sigma(f)(x)\\&=\chi_S(x)\frac{c_\alpha}{(-ib)^{\alpha+1}}\int_\Sigma \overline{K_\alpha^m(y,x)}\mathscr{F}_\alpha^m(f)(y)y^{2\alpha+1}dy\\&=\chi_S(x)\frac{c_\alpha^2}{b^{2\alpha+2}}\int_\Sigma \overline{K_\alpha^m(y,x)}\left(\int_0^{+\infty}K_\alpha^m(y,z)f(z)z^{2\alpha+1}dz\right)y^{2\alpha+1}dy\\&=\chi_S(x)\frac{c_\alpha^2}{b^{2\alpha+2}}\int_0^{+\infty}f(z)\left(\int_\Sigma \overline{K_\alpha^m(y,x)}K_\alpha^m(y,z)y^{2\alpha+1}dy\right)z^{2\alpha+1}dz\\&=\chi_S(x)\int_0^{+\infty}f(z)k(x,z)z^{2\alpha+1}dz.\end{aligned}$$

Then, from the above results we can easily obtain that

$$\|Q_\Sigma P_S\|_{HS}=\left(\int_S\int_0^{+\infty}|k(x,z)|^2x^{2\alpha+1}z^{2\alpha+1}dxdz\right)^{\frac{1}{2}}$$

and

$$\|P_SQ_\Sigma\|_{HS}=\left(\int_0^{+\infty}\int_S|k(x,z)|^2x^{2\alpha+1}z^{2\alpha+1}dxdz\right)^{\frac{1}{2}},$$

which yields the desired result. □

Using Cauchy-Schwarz inequality, we can easily obtain that

$$\|P_SQ_\Sigma\|\leqslant\|P_SQ_\Sigma\|_{HS}. \tag{2.5}$$

Lemma 2.2. *If S and Σ are two measurable subsets of $\mathbb{R}_+$ such that $|S|<+\infty$ and $|\Sigma|<+\infty$, then*

$$\|P_SQ_\Sigma\|\leqslant\frac{c_\alpha}{|b|^{\alpha+1}}\sqrt{|S||\Sigma|},$$

where c_α is the constant given by relation (1.1).

Proof. For $x\in S$, let $g_x(t)=k(x,t)$. Note that

$$\mathscr{F}_\alpha^m(g_x)(y)=\frac{c_\alpha}{(ib)^{\alpha+1}}\chi_\Sigma(y)K_\alpha^m(x,y).$$

By relations (1.3) and (1.2), we have

$$\begin{aligned}\int_0^{+\infty}|g_x(t)|^2t^{2\alpha+1}dt&=\int_0^{+\infty}|\mathscr{F}_\alpha^m(g_x)(y)|^2y^{2\alpha+1}dy\\&=\frac{c_\alpha^2}{|b|^{2\alpha+2}}\int_\Sigma|K_\alpha^m(x,y)|^2y^{2\alpha+1}dy\\&\leqslant\frac{c_\alpha^2}{|b|^{2\alpha+2}}|\Sigma|.\end{aligned}$$

Hence,

$$\int_0^{+\infty}\int_0^{+\infty}|k(x,t)|^2x^{2\alpha+1}t^{2\alpha+1}dxdt \leqslant \frac{c_\alpha^2}{|b|^{2\alpha+2}}|\Sigma|\int_S x^{2\alpha+1}dx = \frac{c_\alpha^2}{|b|^{2\alpha+2}}|\Sigma||S|.$$

Therefore,

$$\|P_SQ_\Sigma\|_{HS}^2 \leqslant \frac{c_\alpha^2}{|b|^{2\alpha+2}}|\Sigma||S|.$$

And the proof is complete by (2.5). □

Proposition 2.1. *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and assume that $\varepsilon_S+\varepsilon_\Sigma<1$, f is ε_S-concentrated on S and $\mathscr{F}_\alpha^m$ is ε_Σ-concentrated on Σ, with $\|f\|_{2,\alpha}=1$. Then*

$$\frac{c_\alpha^2}{|b|^{2\alpha+2}}|\Sigma||S| \geqslant (1-\varepsilon_S-\varepsilon_\Sigma)^2.$$

Proof. Assume that $0<|S|,|\Sigma|<+\infty$. As $\|Q_\Sigma\|=1$, it follows that

$$\begin{aligned}\|f-Q_\Sigma P_S(f)\|_{2,\alpha} &\leqslant \|f-Q_\Sigma(f)\|_{2,\alpha}+\|Q_\Sigma(f)-Q_\Sigma P_S(f)\|_{2,\alpha}\\ &\leqslant \varepsilon_\Sigma+\|Q_\Sigma\|\|f-P_S(f)\|_{2,\alpha}\\ &\leqslant \varepsilon_\Sigma+\varepsilon_S.\end{aligned}$$

The triangle inequality gives

$$\|Q_\Sigma P_S(f)\|_{2,\alpha} \geqslant \|f\|_{2,\alpha}-\|f-Q_\Sigma P_S(f)\|_{2,\alpha} \geqslant 1-\varepsilon_\Sigma-\varepsilon_S.$$

Hence,

$$\|Q_\Sigma P_S\| \geqslant 1-\varepsilon_\Sigma-\varepsilon_S.$$

Then from lemmas 2.1 and 2.2, we get the desired result. □

Theorem 2.1 (Donoho-Stark uncertainty principle-type)**.** *Let $f\in L^{2,\alpha}$ and S,Σ be two measurable subsets of $\mathbb{R}_+$ such that $|S||\Sigma|<\frac{|b|^{2\alpha+2}}{c_\alpha^2}$ and let $\varepsilon_S,\varepsilon_\Sigma>0$ such that $\varepsilon_S^2+\varepsilon_\Sigma^2<1$. If f is ε_S-concentrated on S and ε_Σ-concentrated on Σ for the canonical Fourier Bessel transform, then*

$$\frac{c_\alpha^2}{|b|^{2\alpha+2}}|S||\Sigma| \geqslant \left(1-\sqrt{\varepsilon_S^2+\varepsilon_\Sigma^2}\right)^2.$$

Proof. Since $I=P_S+P_{S^c}=P_SQ_\Sigma+P_SQ_{\Sigma^c}+P_{S^c}$, then, using the orthogonality of P_S and P_{S^c}, we have

$$\begin{aligned}\|f-P_SQ_\Sigma(f)\|_{2,\alpha}^2 &= \|P_SQ_{\Sigma^c}(f)+P_{S^c}(f)\|_{2,\alpha}^2\\ &= \|P_SQ_{\Sigma^c}(f)\|_{2,\alpha}^2+\|P_{S^c}(f)\|_{2,\alpha}^2\\ &\leqslant \|P_S\|^2\|Q_{\Sigma^c}(f)\|_{2,\alpha}^2+\|P_{S^c}(f)\|_{2,\alpha}^2.\end{aligned}$$

From (2.1), we have

$$\|P_S\| \leqslant 1. \tag{2.6}$$

Since P_S is a projection on $L^{2,\alpha}$, then

$$\|P_S\| = \|P_S \circ P_S\| \leqslant \|P_S\|^2. \tag{2.7}$$

By (2.6) and (2.7), we deduce that $\|P_S\| = 1$. Thus,

$$\|f - P_S Q_\Sigma(f)\|_{2,\alpha} \leqslant \sqrt{\|Q_{\Sigma^c}(f)\|_{2,\alpha}^2 + \|P_{S^c}(f)\|_{2,\alpha}^2}. \tag{2.8}$$

On the other hand,

$$\|f - P_S Q_\Sigma(f)\|_{2,\alpha} \geqslant \|f\|_{2,\alpha} - \|P_S Q_\Sigma(f)\|_{2,\alpha} \geqslant (1 - \|P_S Q_\Sigma\|)\|f\|_{2,\alpha}.$$

Then, by (2.8), we have

$$(1 - \|P_S Q_\Sigma\|)\|f\|_{2,\alpha} \leqslant \sqrt{\|Q_{\Sigma^c}(f)\|_{2,\alpha}^2 + \|P_{S^c}(f)\|_{2,\alpha}^2}.$$

Since $\frac{c_\alpha}{|b|^{\alpha+1}}\sqrt{|S||\Sigma|} < 1$, it follows from Lemma 2.2 that

$$\|f\|_{2,\alpha}^2 \leqslant \left(1 - \frac{c_\alpha}{|b|^{\alpha+1}}\sqrt{|S||\Sigma|}\right)^{-2} \left(\|Q_{\Sigma^c}(f)\|_{2,\alpha}^2 + \|P_{S^c}(f)\|_{2,\alpha}^2\right). \tag{2.9}$$

Now, by relations (2.3) and (2.4), we get

$$\|Q_{\Sigma^c}(f)\|_{2,\alpha}^2 + \|P_{S^c}(f)\|_{2,\alpha}^2 \leqslant (\varepsilon_S^2 + \varepsilon_\Sigma^2)\|f\|_{2,\alpha}^2. \tag{2.10}$$

By combining relations (2.9) and (2.10), we obtain the desired result. □

3. $L^{p,\alpha}$-Uncertainty Principles for the Canonical Fourier Bessel Transform

In this section, building on the techniques of Donoho and Stark [9] and Soltani [32], we show a quantitative uncertainty inequality about the essential supports of a nonzero function $f \in L^{p,\alpha}$, $1 \leqslant p \leqslant 2$ and its canonical Fourier Bessel transform.

Proposition 3.1. *Let $f \in L^{1,\alpha} \cap L^{p,\alpha}$, $1 < p \leqslant 2$. Then*

$$\|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} \leqslant \frac{c_\alpha}{|b|^{\alpha+1}} |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{1}{q}} |\operatorname{supp} f|^{\frac{1}{q}} \|f\|_{p,\alpha},$$

with $q = \frac{p}{p-1}$.

Proof. Let $f \in L^{1,\alpha} \cap L^{p,\alpha}$, $1 \leqslant p \leqslant 2$. Then by Hölder's inequality and (1.7), we get

$$\begin{aligned}\|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} &\leqslant |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{1}{q}} \|\mathscr{F}_\alpha^m(f)\|_{\infty,\alpha} \\ &\leqslant \frac{c_\alpha}{|b|^{\alpha+1}} |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{1}{q}} \|f\|_{1,\alpha} \\ &\leqslant \frac{c_\alpha}{|b|^{\alpha+1}} |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{1}{q}} |\operatorname{supp} f|^{\frac{1}{q}} \|f\|_{p,\alpha},\end{aligned}$$

which gives the desired result. □

Proposition 3.2. *Let $f \in L^{2,\alpha} \cap L^{p,\alpha}$, $1 < p < 2$. Then*

$$1 \leqslant |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1} |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{q-2}{2q}} |\operatorname{supp} f|^{\frac{2-p}{2p}},$$

with $q = \frac{p}{p-1}$.

Proof. Let $f \in L^{2,\alpha} \cap L^{p,\alpha}$, $1 \leqslant p \leqslant 2$. Then by Hölder's inequality and (1.6), we get

$$\begin{aligned}
\|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} &\leqslant |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{q-2}{2q}} \|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} \\
&\leqslant |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1} |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{q-2}{2q}} \|f\|_{p,\alpha} \\
&\leqslant |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1} |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{q-2}{2q}} |\operatorname{supp} f|^{\frac{2-p}{2p}} \|f\|_{2,\alpha}.
\end{aligned}$$

Relation (1.3) completes the proof. □

Definition 3.1. Let $0 < \varepsilon_S, \varepsilon_\Sigma < 1$.

(1) We say that a function $f \in L^{p,\alpha}$, $1 \leqslant p \leqslant 2$ is ε_S-concentrated to S in $L^{p,\alpha}$-norm if and only if

$$\|f - P_S f\|_{p,\alpha} \leqslant \varepsilon_S \|f\|_{p,\alpha}. \tag{3.1}$$

(2) Let $f \in L^{p,\alpha}$, $1 \leqslant p \leqslant 2$. We say that $\mathscr{F}_\alpha^m(f)$ is ε_Σ-concentrated on Σ in $L^{q,\alpha}$-norm, $q = \frac{p}{p-1}$ if and only if

$$\|\mathscr{F}_\alpha^m(f) - \mathscr{F}_\alpha^m(Q_\Sigma f)\|_{q,\alpha} \leqslant \varepsilon_\Sigma \|\mathscr{F}_\alpha^m(f)\|_{q,\alpha}. \tag{3.2}$$

Lemma 3.1. *Let $f \in L^{p,\alpha}$, $1 < p \leqslant 2$. Then*

$$\|\mathscr{F}_\alpha^m(Q_\Sigma f)\|_{q,\alpha} \leqslant |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1} \|f\|_{p,\alpha},$$

with $q = \frac{p}{p-1}$.

Proof. Let $f \in L^{p,\alpha}$, $1 < p \leqslant 2$ and $q = \frac{p}{p-1}$. From relations (1.6) and (2.2), we get

$$\begin{aligned}
\|\mathscr{F}_\alpha^m(Q_\Sigma f)\|_{q,\alpha} &= \left(\int_\Sigma |\mathscr{F}_\alpha^m(f)(x)|^q x^{2\alpha+1} dx\right)^{\frac{1}{q}} \\
&\leqslant \|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} \\
&\leqslant |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1} \|f\|_{p,\alpha},
\end{aligned}$$

which yields the desired result. □

Lemma 3.2. *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and $f \in L^{p,\alpha}$, $1 < p \leqslant 2$, $q = \frac{p}{p-1}$. Then*

$$\|\mathscr{F}_\alpha^m(Q_\Sigma P_S f)\|_{q,\alpha} \leqslant \frac{c_\alpha}{|b|^{\alpha+1}} |S|^{\frac{1}{q}} |\Sigma|^{\frac{1}{q}} \|f\|_{p,\alpha}.$$

Proof. Assume that $|S| < +\infty$ and $|\Sigma| < +\infty$. From relation (2.2), we have

$$\|\mathscr{F}_\alpha^m(Q_\Sigma P_S f)\|_{q,\alpha} = \left(\int_\Sigma |\mathscr{F}_\alpha^m(\chi_S f)(x)|^q x^{2\alpha+1} dx\right)^{\frac{1}{q}}. \tag{3.3}$$

By (1.2) and Hölder's inequality it follows that

$$\begin{aligned} |\mathscr{F}_\alpha^m(\chi_S f)(x)| &\leqslant \frac{c_\alpha}{|b|^{\alpha+1}} \left(\int_S |f(y)|^p y^{2\alpha+1} dy\right)^{\frac{1}{p}} \left(\int_S |K_\alpha^m(x,y)|^q y^{2\alpha+1} dy\right)^{\frac{1}{q}} \\ &\leqslant \frac{c_\alpha}{|b|^{\alpha+1}} |S|^{\frac{1}{q}} \|f\|_{p,\alpha}. \end{aligned}$$

Then from (3.3), we obtain the desired result. □

Theorem 3.1. *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and $f \in L^{p,\alpha}$, $1 < p \leqslant 2$, $q = \frac{p}{p-1}$. If f is ε_S-concentration to S in $L^{p,\alpha}$-norm and $\mathscr{F}_\alpha^m(f)$ is ε_Σ-concentration to Σ in $L^{q,\alpha}$-norm, then*

$$\|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} \leqslant \frac{1}{1-\varepsilon_\Sigma} \left(\frac{c_\alpha}{|b|^{\alpha+1}} |S|^{\frac{1}{q}} |\Sigma|^{\frac{1}{q}} + \varepsilon_S |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}} \right)^{\alpha+1} \right) \|f\|_{p,\alpha}.$$

Proof. Assume that $|S| < +\infty$ and $|\Sigma| < +\infty$. From the triangle inequality, relations (1.6), (3.1), (3.2) and Lemma 3.2, we get

$$\begin{aligned} \|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} \leqslant & \|\mathscr{F}_\alpha^m(Q_\Sigma P_S f)\|_{q,\alpha} + \|\mathscr{F}_\alpha^m(f) - \mathscr{F}_\alpha^m(Q_\Sigma P_S f)\|_{q,\alpha} \\ \leqslant & \|\mathscr{F}_\alpha^m(Q_\Sigma P_S f)\|_{q,\alpha} + \|\mathscr{F}_\alpha^m(f) - \mathscr{F}_\alpha^m(Q_\Sigma f)\|_{q,\alpha} \\ & + \|\mathscr{F}_\alpha^m(Q_\Sigma f) - \mathscr{F}_\alpha^m(Q_\Sigma P_S f)\|_{q,\alpha} \\ \leqslant & \frac{c_\alpha}{|b|^{\alpha+1}} |S|^{\frac{1}{q}} |\Sigma|^{\frac{1}{q}} \|f\|_{p,\alpha} + \varepsilon_\Sigma \|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} \\ & + |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}} \right)^{\alpha+1} \|f - P_S f\|_{p,\alpha} \\ \leqslant & \left(\frac{c_\alpha}{|b|^{\alpha+1}} |S|^{\frac{1}{q}} |\Sigma|^{\frac{1}{q}} + \varepsilon_S |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}} \right)^{\alpha+1} \right) \|f\|_{p,\alpha} \\ & + \varepsilon_\Sigma \|\mathscr{F}_\alpha^m(f)\|_{q,\alpha}, \end{aligned}$$

which gives the desired result. □

Theorem 3.2 (Donoho-Stark's uncertainty principle-type)**.** *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and $f \in L^{p_1,\alpha} \cap L^{p_2,\alpha}$, $1 < p_1 < p_2 \leqslant 2$. If f is ε_S-concentration to S in $L^{p_1,\alpha}$-norm and $\mathscr{F}_\alpha^m(f)$ is ε_Σ-concentration to Σ in $L^{q_2,\alpha}$-norm,*

$q_2 = \frac{p_2}{p_2-1}$, *then*

$$\|\mathscr{F}_\alpha^m(f)\|_{q_2,\alpha} \leqslant \frac{|S|^{\frac{p_2-p_1}{p_1p_2}}|\Sigma|^{\frac{q_1-q_2}{q_1q_2}}}{(1-\varepsilon_\Sigma)(1-\varepsilon_S)}|b|^{(\alpha+1)(\frac{2}{q_1}-1)}\left(\frac{(c_\alpha p_1)^{\frac{1}{p_1}}}{(c_\alpha q_1)^{\frac{1}{q_1}}}\right)^{\alpha+1}\|f\|_{p_2,\alpha},$$

where $q_1 = \frac{p_1}{p_1-1}$.

Proof. Assume that $|S| < +\infty$ and $|\Sigma| < +\infty$. Let $f \in L^{p_1,\alpha} \cap L^{p_2,\alpha}$, $1 < p_1 < p_2 \leqslant 2$. Since $\mathscr{F}_\alpha^m(f)$ is ε_Σ-concentration to Σ in $L^{q_2,\alpha}$-norm, then, by Hölder's inequality, we obtain

$$\begin{aligned}\|\mathscr{F}_\alpha^m(f)\|_{q_2,\alpha} &\leqslant \varepsilon_\Sigma\|\mathscr{F}_\alpha^m(f)\|_{q_2,\alpha} + \|\chi_\Sigma\mathscr{F}_\alpha^m(f)\|_{q_2,\alpha}\\ &\leqslant \varepsilon_\Sigma\|\mathscr{F}_\alpha^m(f)\|_{q_2,\alpha} + |\Sigma|^{\frac{q_1-q_2}{q_1q_2}}\|\mathscr{F}_\alpha^m(f)\|_{q_1,\alpha}.\end{aligned}$$

Thus, by (1.6),

$$\|\mathscr{F}_\alpha^m(f)\|_{q_2,\alpha} \leqslant \frac{|\Sigma|^{\frac{q_1-q_2}{q_1q_2}}}{1-\varepsilon_\Sigma}|b|^{(\alpha+1)(\frac{2}{q_1}-1)}\left(\frac{(c_\alpha p_1)^{\frac{1}{p_1}}}{(c_\alpha q_1)^{\frac{1}{q_1}}}\right)^{\alpha+1}\|f\|_{p_1,\alpha}. \tag{3.4}$$

On the other hand, since f is ε_S-concentration to S in $L^{p_1,\alpha}$-norm, then by Hölder's inequality, we deduce that

$$\|f\|_{p_1,\alpha} \leqslant \varepsilon_S\|f\|_{p_1,\alpha} + \|\chi_S f\|_{p_1,\alpha} \leqslant \varepsilon_S\|f\|_{p_1,\alpha} + |S|^{\frac{p_2-p_1}{p_1p_2}}\|f\|_{p_2,\alpha}.$$

Thus,

$$\|f\|_{p_1,\alpha} \leqslant \frac{|S|^{\frac{p_2-p_1}{p_1p_2}}}{1-\varepsilon_S}\|f\|_{p_2,\alpha}. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain the result of this theorem. ☐

Corollary 3.1. *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and $f \in L^{2,\alpha} \cap L^{p,\alpha}$, $1 < p < 2$. If f is ε_S-concentration to S in $L^{p,\alpha}$-norm and $\mathscr{F}_\alpha^m(f)$ is ε_Σ-concentration to Σ in $L^{2,\alpha}$-norm, then*

$$(1-\varepsilon_\Sigma)(1-\varepsilon_S) \leqslant |S|^{\frac{2-p}{2p}}|\Sigma|^{\frac{q-2}{2q}}|b|^{(\alpha+1)(\frac{2}{q}-1)}\left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1},$$

where $q = \frac{p}{p-1}$.

Let $B^p(\Sigma)$, $1 \leqslant p \leqslant 2$, be the set of functions $g \in L^{p,\alpha}$ that are bandlimited to Σ, i.e., ($g \in B^p(\Sigma)$ implies $Q_\Sigma g = g$).

We say that f is ε_Σ-bandlimited to Σ in $L^{p,\alpha}$-norm if there is a $g \in B^p(\Sigma)$ with

$$\|f - g\|_{p,\alpha} \leqslant \varepsilon_\Sigma\|f\|_{p,\alpha}.$$

In the following, we state an $L^{p_1,\alpha} \cap L^{p_2,\alpha}$ bandlimited uncertainty principle of concentration-type.

Theorem 3.3 (Bandlimited principle-type)**.** *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and $f \in L^{p_1,\alpha} \cap L^{p_2,\alpha}$, $1 \leqslant p_1 < p_2 \leqslant 2$. If f is ε_S-concentration to S in $L^{p_1,\alpha}$-norm and ε_Σ-bandlimited to Σ in $L^{q_2,\alpha}$-norm, $q_2 = \frac{p_2}{p_2-1}$, then*

$$\begin{aligned}&\|f\|_{p_1,\alpha}\\ \leqslant&\frac{|S|^{\frac{p_2-p_1}{p_1p_2}}}{1-\varepsilon_S}\left[(1+\varepsilon_\Sigma)c_\alpha|\Sigma|^{\frac{1}{p_2}}|S|^{\frac{1}{p_2}}|b|^{(\alpha+1)(\frac{2}{q_2}-2)}\left(\frac{(c_\alpha p_2)^{\frac{1}{p_2}}}{(c_\alpha q_2)^{\frac{1}{q_2}}}\right)^{\alpha+1}+\varepsilon_\Sigma\right]\|f\|_{p_2,\alpha}.\end{aligned}$$

Proof. Assume that $|S| < +\infty$ and $|\Sigma| < +\infty$. Let $f \in L^{p_1,\alpha} \cap L^{p_2,\alpha}$, $1 \leqslant p_1 < p_2 \leqslant 2$. Since f is ε_S-concentration to S in $L^{p_1,\alpha}$-norm, then by Hölder's inequality, we deduce that

$$\|f\|_{p_1,\alpha} \leqslant \varepsilon_S\|f\|_{p_1,\alpha} + \|P_S f\|_{p_1,\alpha} \leqslant \varepsilon_S\|f\|_{p_1,\alpha} + |S|^{\frac{p_2-p_1}{p_1p_2}}\|P_S f\|_{p_2,\alpha}.$$

Thus,

$$\|f\|_{p_1,\alpha} \leqslant \frac{|S|^{\frac{p_2-p_1}{p_1p_2}}}{1-\varepsilon_S}\|P_S f\|_{p_2,\alpha}. \tag{3.6}$$

As f is ε_Σ-bandlimited to Σ in $L^{q_2,\alpha}$-norm, there is a $g \in B^{p_2}(\Sigma)$ with

$$\|f-g\|_{p_2,\alpha} \leqslant \varepsilon_\Sigma\|f\|_{p_2,\alpha}.$$

On the other hand, we have

$$\|P_S f\|_{p_2,\alpha} \leqslant \|P_S g\|_{p_2,\alpha} + \|P_S(f-g)\|_{p_2,\alpha} \leqslant \|P_S g\|_{p_2,\alpha} + \varepsilon_\Sigma\|f\|_{p_2,\alpha}.$$

But $g \in B^{p_2}(\Sigma)$, from (2.2), $g(x) = \mathscr{F}_\alpha^{m^{-1}}(\chi_\Sigma \mathscr{F}_\alpha^m(g))(x)$ and by (1.6) and Hölder's inequality, we deduce that

$$\begin{aligned}|g(x)| &\leqslant \frac{c_\alpha}{|b|^{\alpha+1}}|\Sigma|^{\frac{1}{p_2}}\|\mathscr{F}_\alpha^m(g)\|_{q_2,\alpha}\\ &\leqslant c_\alpha|\Sigma|^{\frac{1}{p_2}}|b|^{(\alpha+1)(\frac{2}{q_2}-2)}\left(\frac{(c_\alpha p_2)^{\frac{1}{p_2}}}{(c_\alpha q_2)^{\frac{1}{q_2}}}\right)^{\alpha+1}\|g\|_{p_2,\alpha}.\end{aligned}$$

Hence,

$$\begin{aligned}\|P_S g\|_{p_2,\alpha} &= \left(\int_S |g(x)|^{p_2}x^{2\alpha+1}dx\right)^{\frac{1}{p_2}}\\ &\leqslant c_\alpha|\Sigma|^{\frac{1}{p_2}}|S|^{\frac{1}{p_2}}|b|^{(\alpha+1)(\frac{2}{q_2}-2)}\left(\frac{(c_\alpha p_2)^{\frac{1}{p_2}}}{(c_\alpha q_2)^{\frac{1}{q_2}}}\right)^{\alpha+1}\|g\|_{p_2,\alpha}.\end{aligned}$$

Then by (3.6) and the fact that $\|g\|_{p_2,\alpha} \leqslant (1+\varepsilon_\Sigma)\|f\|_{p_2,\alpha}$, we get

$$\|f\|_{p_1,\alpha} \leqslant \frac{|S|^{\frac{p_2-p_1}{p_1p_2}}}{1-\varepsilon_S}\left[(1+\varepsilon_\Sigma)c_\alpha|\Sigma|^{\frac{1}{p_2}}|S|^{\frac{1}{p_2}}|b|^{(\alpha+1)(\frac{2}{q_2}-2)}\left(\frac{(c_\alpha p_2)^{\frac{1}{p_2}}}{(c_\alpha q_2)^{\frac{1}{q_2}}}\right)^{\alpha+1}+\varepsilon_\Sigma\right]\|f\|_{p_2,\alpha}.$$

This completes the desired result. ☐

Corollary 3.2. *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and $f \in L^{p,\alpha}$, $1 < p \leqslant 2$. If f is ε_S-concentration to S and ε_Σ-bandlimited to Σ in $L^{p,\alpha}$-norm, then*

$$\frac{1-\varepsilon_S-\varepsilon_\Sigma}{1+\varepsilon_\Sigma} \leqslant c_\alpha |\Sigma|^{\frac{1}{p}} |S|^{\frac{1}{p}} |b|^{(\alpha+1)(\frac{2}{q}-2)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}} \right)^{\alpha+1}.$$

Theorem 3.4 (Matolcsi-Szücs-type inequality)**.** *Let $f \in L^{p_1,\alpha} \cap L^{p_2,\alpha}$, $1 < p_1 \leqslant p_2 \leqslant 2$. Then*

$$\|\mathscr{F}^m_\alpha(f)\|_{q_2,\alpha} \leqslant |b|^{(\alpha+1)(\frac{2}{q_1}-1)} \left(\frac{(c_\alpha p_1)^{\frac{1}{p_1}}}{(c_\alpha q_1)^{\frac{1}{q_1}}} \right)^{\alpha+1} |\mathrm{supp}\, \mathscr{F}^m_\alpha(f)|^{\frac{q_1-q_2}{q_1 q_2}} |\mathrm{supp}\, f|^{\frac{p_2-p_1}{p_1 p_2}} \|f\|_{p_2,\alpha},$$

where $q_1 = \frac{p_1}{p_1-1}$ and $q_2 = \frac{p_2}{p_2-1}$.

Proof. Let $f \in L^{p_1,\alpha} \cap L^{p_2,\alpha}$, $1 < p_1 \leqslant p_2 \leqslant 2$, $q_1 = \frac{p_1}{p_1-1}$ and $q_2 = \frac{p_2}{p_2-1}$. Then, by relation (1.6) and Hölder's inequality, we obtain

$$\begin{aligned}
&\|\mathscr{F}^m_\alpha(f)\|_{q_2,\alpha} \\
\leqslant & |\mathrm{supp}\, \mathscr{F}^m_\alpha(f)|^{\frac{q_1-q_2}{q_1 q_2}} \|\mathscr{F}^m_\alpha(f)\|_{q_1,\alpha} \\
\leqslant & |b|^{(\alpha+1)(\frac{2}{q_1}-1)} \left(\frac{(c_\alpha p_1)^{\frac{1}{p_1}}}{(c_\alpha q_1)^{\frac{1}{q_1}}} \right)^{\alpha+1} |\mathrm{supp}\, \mathscr{F}^m_\alpha(f)|^{\frac{q_1-q_2}{q_1 q_2}} \|f\|_{p_1,\alpha} \\
\leqslant & |b|^{(\alpha+1)(\frac{2}{q_1}-1)} \left(\frac{(c_\alpha p_1)^{\frac{1}{p_1}}}{(c_\alpha q_1)^{\frac{1}{q_1}}} \right)^{\alpha+1} |\mathrm{supp}\, \mathscr{F}^m_\alpha(f)|^{\frac{q_1-q_2}{q_1 q_2}} |\mathrm{supp}\, f|^{\frac{p_2-p_1}{p_1 p_2}} \|f\|_{p_2,\alpha},
\end{aligned}$$

which yields the desired result. □

Corollary 3.3. *Let $f \in L^{2,\alpha} \cap L^{p,\alpha}$, $1 < p < 2$ and $q = \frac{p}{p-1}$. Then*

$$\left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}} \right)^{-(\alpha+1)} |b|^{(\alpha+1)(1-\frac{2}{q})} \leqslant |\mathrm{supp}\, f|^{\frac{2-p}{2p}} |\mathrm{supp}\, \mathscr{F}^m_\alpha(f)|^{\frac{q-2}{2q}}.$$

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