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$k-$FRACTIONAL OSTROWSKI TYPE INEQUALITIES VIA $(s,r)-$CONVEX

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Abstract. We introduce the generalized class named it the class of $(s,r)-$convex in mixed kind, this class includes $s-$convex in 1^{st} and 2^{nd} kind, $P-$convex, quasi convex and the class of ordinary convex. Also, we state the generalization of the classical Ostrowski inequality via $k-$fractional integrals, which is obtained for functions whose first derivative in absolute values is $(s,r)-$convex in mixed kind. Moreover, we establish some Ostrowski type inequalities via $k-$fractional integrals and their particular cases for the class of functions whose absolute values at certain powers of derivatives are $(s,r)-$convex in mixed kind by using different techniques including Hölder's inequality and power mean inequality. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means are given.

1. Introduction

In almost every field of science, inequalities play an important role. Although it is very vast discipline but our focus is mainly on Ostrowski type inequalities. In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives. This inequality is well known in the literature as Ostrowski inequality which is stated as follows.

Key words and phrases. Ostrowski inequality, convex function, power mean inequality, Hölder's inequality.

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Theorem 1.1 ([14]). *Let $f : [a, b] \to \mathbb{R}$ be differentiable function on (a, b), $|f'(t)| \le M$, for all $t \in (a, b)$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \le M(b-a) \left(\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right), \tag{1.1}$$

for all $x \in (a, b)$.

Also, one can see the numerous variants and applications in [5]-[11]. Nowadays, with the increasing demand of researchers for the study of natural phenomena, the use of fractional differential operators and fractional differential equations has become an effective means to achieve this goal. Compared with integer order operators, fractional operators, which can simulate natural phenomena better, are a class of operators developed in recent years. This kind of operators have expanded and have been widely used in modeling real-world phenomena such as biomathematics, electrical circuits, medicine, disease transmission and control.

On other hand convexity is very simple and ordinary concept. Due to its massive applications in industry and business, convexity has a great influence on our daily life. In the solution of many real world problems the concept of convexity is very decisive. The problems faced in constrained control and estimation are convex. Geometrically, a real valued function is said to be convex if the line segment segment joining any two of its points lies on or above the graph of the function in Euclidean space. First we present the important classes of convex functions from literature.

Definition 1.1 ([3]). The function $g : I \to \mathbb{R}$, $I \subset (0, \infty)$, is convex, if

$$g(tx + (1-t)y) \le tg(x) + (1-t)g(y),$$

for all $x, y \in I, t \in [0, 1]$.

Definition 1.2 ([15]). Let function $s \in (0, 1]$, the $g : I \to [0, \infty)$, $I \subset (0, \infty)$, is $s-$convex in 1^{st} kind, if

$$g(tx + (1-t)y) \le t^s g(x) + (1-t^s)g(y),$$

for all $x, y \in I$, $t \in [0, 1]$.

Definition 1.3 ([3]). The $g : I \to [0, \infty)$, $I \subset (0, \infty)$, is quasi convex, if

$$g(tx + (1-t)y) \le \max\{g(x), g(y)\},$$

for all $x, y \in I$, $t \in [0, 1]$.

Definition 1.4 ([15]). Let $s \in (0, 1]$, the function $g : I \to [0, \infty)$, $I \subset (0, \infty)$, is $s-$convex in 2^{nd} kind, if

$$g(tx + (1-t)y) \le t^s g(x) + (1-t)^s g(y),$$

for all $x, y \in I$, $t \in [0, 1]$.

Definition 1.5 ([3])**.** The function $g : I \to [0, \infty)$, $I \subset (0, \infty)$, is a $P-$convex, if $g(x) \geq 0$ and for all $x, y \in I$ and $t \in [0, 1]$,

$$g\left(tx + (1-t)y\right) \leq g(x) + g(y).$$

An important area in the field of applied and pure mathematics is the integral inequality. As it is known, inequalities aim to develop different mathematical methods. Nowadays, we need to seek accurate inequalities for proving the existence and uniqueness of the mathematical methods. The concept of convexity plays a strong role in the field of inequalities due to the behavior of its definition and its properties. Furthermore, there is a strong correlation between convexity and symmetry concepts.

Definition 1.6 ([12])**.** The Riemann-Liouville integrals $I^{\varepsilon}_{a+}f$ and $I^{\varepsilon}_{b-}f$ of $f \in L_1([a, b])$ having order $\varepsilon > 0$ with $0 \leq a < b$ are defined by

$$I^{\varepsilon}_{a+}f(x) = \frac{1}{\Gamma(\varepsilon)} \int_a^x \frac{f(t)}{(x-t)^{1-\varepsilon}} dt, \quad x > a,$$

and

$$I^{\varepsilon}_{b-}f(x) = \frac{1}{\Gamma(\varepsilon)} \int_x^b \frac{f(t)}{(t-x)^{1-\varepsilon}} dt, \quad x < b,$$

respectively. Here $\Gamma(\varepsilon) = \int_0^\infty e^{-u} u^{\varepsilon-1} du$ is the Gamma function and $I^0_{a+}f(x) = I^0_{b-}f(x) = f(x)$. We also make use of Euler's beta function, which is for $x, y > 0$ defined as

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Definition 1.7 ([12])**.** The $k-$fractional integrals ${}^kJ^{\varepsilon}_{a+}f$ and ${}^kJ^{\varepsilon}_{b-}f$ of $f \in L_1([a, b])$ having order $\varepsilon > 0$ with $0 \leq a < b$, $k > 0$ are defined by

$${}^kJ^{\varepsilon}_{a+}f(x) = \frac{1}{k\Gamma_k(\varepsilon)} \int_a^x \frac{f(t)}{(x-t)^{1-\frac{\varepsilon}{k}}} dt, \quad x > a,$$

and

$${}^kJ^{\varepsilon}_{b-}f(x) = \frac{1}{k\Gamma_k(\varepsilon)} \int_x^b \frac{f(t)}{(t-x)^{1-\frac{\varepsilon}{k}}} dt, \quad x < b,$$

respectively. Here $\Gamma_k(\varepsilon) = \int_0^\infty e^{-\frac{u^k}{k}} u^{\varepsilon-1} du$ is the generalized gamma function and ${}^1J^0_{a+}f(x) = {}^1J^0_{b-}f(x) = f(x)$.

Throught this paper, we denote

$$Y_f(\varepsilon, k, a, x, b) = \left(\frac{(x-a)^{\frac{\varepsilon}{k}} + (b-x)^{\frac{\varepsilon}{k}}}{(b-a)}\right) f(x) - \frac{k\Gamma_k(\varepsilon+1)}{b-a} \left({}^kJ^{\varepsilon}_{x-}f(a) + {}^kJ^{\varepsilon}_{x+}f(b)\right),$$

$$Z_f(\varepsilon, x, a, b) = \left(\frac{(x-a)^{\varepsilon} + (b-x)^{\varepsilon}}{b-a}\right) f(x) - \frac{\Gamma(\varepsilon+1)}{b-a} \left(I^{\varepsilon}_{x-}f(a) + I^{\varepsilon}_{x+}f(b)\right),$$

$${}^{\varepsilon}\kappa^b_a(x) = \left(\frac{(x-a)^{\varepsilon+1} + (b-x)^{\varepsilon+1}}{b-a}\right).$$

In order to prove our main results we need the following lemma.

Lemma 1.1 ([12]). *Let $f : I \to \mathbb{R}$, $I \subset (0, \infty)$, be an absolutely continuous function and $a, b \in I, a < b$. If $f' \in L_1[a, b]$, $\varepsilon, k > 0$, then*

$$\begin{aligned} Y_f(\varepsilon, k, a, x, b) =& \frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a} \int_0^1 t^{\frac{\varepsilon}{k}} f'\left(tx + (1-t)a\right) dt \\ &- \frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a} \int_0^1 t^{\frac{\varepsilon}{k}} f'\left(tx + (1-t)b\right) dt. \end{aligned}$$

Theorem 1.2 ([12]). *Let $f : I \to \mathbb{R}$ be differentiable mapping on I^0, with $a, b \in I, a < b$, $f' \in L_1[a, b]$ and for $\varepsilon, k > 1$, Montgomery identity for $k-$fractional integrals holds:*

$$f(x) = \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\, {}^kJ_a^{\varepsilon} f(b) - {}^kJ_a^{\varepsilon-1}(P_1(x, b)f(b)) + {}^kJ_a^{\varepsilon}(P_1(x, b)f'(b)),$$

where $P_1(x, t)$ is the fractional Peano Kernel defined by:

$$P_1(x, t) = \begin{cases} \frac{t-a}{b-a} \cdot \frac{k\Gamma_k(\varepsilon)}{(b-x)^{\frac{\varepsilon}{k}-1}}, & \text{if } t \in [a, x], \\ \frac{t-b}{b-a} \cdot \frac{k\Gamma_k(\varepsilon)}{(b-x)^{\frac{\varepsilon}{k}-1}}, & \text{if } t \in (x, b]. \end{cases}$$

Let $[a, b] \subseteq (0, +\infty)$, we may define special means as follows

(a) the arithmetic mean

$$A = A(a, b) := \frac{a+b}{2};$$

(b) the geometric mean

$$G = G(a, b) := \sqrt{ab};$$

(c) the harmonic mean

$$H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}};$$

(d) the logarithmic mean

$$L = L(a, b) := \begin{cases} a, & \text{if } a = b, \\ \frac{b-a}{\ln b - \ln a}, & \text{if } a \neq b; \end{cases}$$

(e) the identric mean

$$I = I(a, b) := \begin{cases} a, & \text{if } a = b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a \neq b; \end{cases}$$

(f) the $p-$logarithmic mean

$$L_p = L_p(a, b) := \begin{cases} a, & \text{if } a = b, \\ \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & \text{if } a \neq b, \end{cases}$$

where $p \in \mathbb{R} \setminus \{0, -1\}$.

2. $k-$Fractional Ostrowski Type Inequalities via $(s,r)-$Convex

In this section, we introduce the concept of $(s,r)-$convex in mixed kind. This class contains many classes of convex from literature of convex analysis. The main aim of this study is to reveal new generalized-Ostrowski-type inequalities via $(s,r)-$convex using $k-$fractional operator which generalizes Riemann-Liouville integral operator.

Definition 2.1. Let $(s,r)\in(0,1]^2$, the function $g:I\to[0,\infty)$, $I\subset(0,\infty)$, is $(s,r)-$convex in mixed kind, if

$$g\left(tx+(1-t)y\right)\leq t^{rs}g(x)+(1-t^r)^s g(y), \tag{2.1}$$

for all $x,y\in I$, $t\in[0,1]$.

Remark 2.1. In Definition 2.1, we can see the following.

(a) If $s=1$ and $r\in[0,1]$ in (2.1), we get $r-$convex in 1^{st} kind.
(b) If $r\to 0$ and $s=1$, in (2.1), we get quasi convex.
(c) If $r=1$ and $s\in[0,1]$ in (2.1), we get $s-$convex in 2^{nd} kind.
(d) If $s\to 0$ and $r=1$ in (2.1), we get $P-$convex.
(e) If $s=r=1$ in (2.1), gives us ordinary convex.

Now, we will generalize the Ostrowski type inequalities via $(s,r)-$convex by using $k-$fractional integral operator.

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be differentiable on (a,b), $f':[a,b]\to\mathbb{R}$ be integrable on $[a,b]$ and $g:I\subset\mathbb{R}\to\mathbb{R}$, be an $(s,r)-$convex function in mixed sense, then we have the inequalities*

$$\begin{aligned} &g\left(f(x)-\frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\ {}^kJ_a^{\varepsilon}f(b)+\ {}^kJ_a^{\varepsilon-1}(P_1(x,b)f(b))\right)\\ \leq&\frac{(b-x)^{1-\frac{\varepsilon}{k}}}{(b-a)^{rs}}\left((x-a)^{rs-1}\int_a^x g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt\right.\\ &\left.+\frac{((b-a)^r-(x-a)^r)^s}{b-x}\int_x^b g\left(\frac{(t-b)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt\right), \end{aligned} \tag{2.2}$$

for all $x\in(a,b)$.

Proof. Utilizing the Theorem 1.2, we get

$$\begin{aligned}
& f(x) - \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\ {}^kJ_a^{\varepsilon} f(b) + {}^kJ_a^{\varepsilon-1}(P_1(x,b)f(b)) \\
= & {}^kJ_a^{\varepsilon}(P_1(x,b)f'(b)) \\
= & \frac{1}{k\Gamma_k(\varepsilon)} \int_a^b P_1(x,t) \frac{f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}} dt \\
= & \left(\frac{x-a}{b-a}\right)\left(\frac{(b-x)^{1-\frac{\varepsilon}{k}}}{x-a} \int_a^x \frac{(t-a)\, f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}} dt\right) \\
& + \left(1 - \left(\frac{x-a}{b-a}\right)\right)\left(\frac{(b-x)^{1-\frac{\varepsilon}{k}}}{b-x} \int_x^b \frac{(t-b)\, f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}} dt\right),
\end{aligned}$$

for all $x \in (a,b)$. Next by using the $(s,r)-$convex function in mixed sense of $g : I \subset [0,\infty) \to \mathbb{R}$, we get

$$\begin{aligned}
& g\left(f(x) - \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\ {}^kJ_a^{\varepsilon} f(b) + {}^kJ_a^{\varepsilon-1}(P_1(x,b)f(b))\right) \\
\leq & \left(\frac{x-a}{b-a}\right)^{rs} g\left(\frac{(b-x)^{1-\frac{\varepsilon}{k}}}{x-a} \int_a^x \frac{(t-a)\, f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}} dt\right) \\
& + \left(1 - \left(\frac{x-a}{b-a}\right)^r\right)^s g\left(\frac{(b-x)^{1-\frac{\varepsilon}{k}}}{b-x} \int_x^b \frac{(t-b)\, f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}} dt\right),
\end{aligned}$$

for all $x \in (a,b)$. Applying Jensen's integral inequality [6], we get (2.2). □

Corollary 2.1. *In Theorem* 2.1, *one can see the following.*

(a) *If* $s = 1$ *and* $r \in (0,1]$ *in* (2.2), *then Ostrowski inequality for* $r-$*convex functions in* 1^{st} *kind:*

$$\begin{aligned}
& g\left(f(x) - \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\ {}^kJ_a^{\varepsilon} f(b) + {}^kJ_a^{\varepsilon-1}(P_1(x,b)f(b))\right) \\
\leq & \frac{(b-x)^{1-\frac{\varepsilon}{k}}}{(b-a)^r}\left((x-a)^{r-1} \int_a^x g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right) dt \right. \\
& \left. + \frac{(b-a)^r - (x-a)^r}{(b-x)} \int_x^b g\left(\frac{(t-b)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right) dt\right).
\end{aligned}$$

(b) *If* $s = 1$ *and* $r \to 0$ *in* (2.2), *we get quasi-convex function*

$$\begin{aligned}
& g\left(f(x) - \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\ {}^kJ_a^{\varepsilon} f(b) + {}^kJ_a^{\varepsilon-1}(P_1(x,b)f(b))\right) \\
\leq & \frac{(b-x)^{1-\frac{\varepsilon}{k}}}{(x-a)} \int_a^x g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right) dt.
\end{aligned}$$

(c) *If $r=1$ and $s\in[0,1)$ in (2.2), then fractional Ostrowski type inequality for $s-$convex functions in 2^{nd} kind:*

$$g\left(f(x)-\frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\ {}^kJ_a^\varepsilon f(b)+\ {}^kJ_a^{\varepsilon-1}(P_1(x,b)f(b))\right)$$
$$\leq\frac{(b-x)^{1-\frac{\varepsilon}{k}}}{(b-a)^s}\left((x-a)^{s-1}\int_a^x g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt+(b-x)^{s-1}\int_x^b g\left(\frac{(t-b)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt\right).$$

(d) *If $r=1$ and $s\to 0$ in (2.2), then fractional Ostrowski type inequality for $P-$convex functions:*

$$g\left(f(x)-\frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\ {}^kJ_a^\varepsilon f(b)+\ {}^kJ_a^{\varepsilon-1}(P_1(x,b)f(b))\right)$$
$$\leq(b-x)^{1-\frac{\varepsilon}{k}}\left(\frac{1}{x-a}\int_a^x g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt+\frac{1}{b-x}\int_x^b g\left(\frac{(t-b)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt\right).$$

(e) *If $s=r=1$ in (2.2), then fractional Ostrowski type inequality for convex functions:*

$$g\left(f(x)-\frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\ {}^kJ_a^\varepsilon f(b)+\ {}^kJ_a^{\varepsilon-1}(P_1(x,b)f(b))\right)$$
$$\leq\frac{(b-x)^{1-\frac{\varepsilon}{k}}}{b-a}\left(\int_a^x g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt+\int_x^b g\left(\frac{(t-b)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt\right).$$

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$, $[a,b]\subset(0,\infty)$, be an absolutely continuous, and $f'\in L_1[a,b]$. If $|f'|$ is $(s,r)-$convex, $|f'(x)|\leq M$, for all $x\in[a,b]$, and $\varepsilon,k>0$, then*

(2.3)
$$|Y_f(\varepsilon,k,a,x,b)|\leq M\left(\int_0^1 t^{\frac{\varepsilon}{k}}t^{rs}dt+\int_0^1 t^{\frac{\varepsilon}{k}}(1-t^r)^s dt\right)\left(\frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a}+\frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a}\right).$$

Proof. By using the Lemma 1.1,

$$|Y_f(\varepsilon,k,a,x,b)|\leq\frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a}\int_0^1 t^{\frac{\varepsilon}{k}}\left|f'\left(tx+(1-t)a\right)\right|dt$$
$$+\frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a}\int_0^1 t^{\frac{\varepsilon}{k}}\left|f'\left(tx+(1-t)b\right)\right|dt.$$

Since $|f'|$ is $(s,r)-$convex and by using $|f'(x)|\leq M$, we get

$$|Y_f(\varepsilon,k,a,x,b)|\leq\frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a}\int_0^1 t^{\frac{\varepsilon}{k}}\Big(t^{rs}|f'(x)|+(1-t^r)^s|f'(a)|\Big)dt$$
$$+\frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a}\int_0^1 t^{\frac{\varepsilon}{k}}\Big(t^{rs}|f'(x)|+(1-t^r)^s|f'(b)|\Big)dt.$$

Therefore,

$$|Y_f(\varepsilon,k,a,x,b)| \le \frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a}\left(|f'(x)|\int_0^1 t^{\frac{\varepsilon}{k}}t^{rs}dt + |f'(a)|\int_0^1 t^{\frac{\varepsilon}{k}}(1-t^r)^s dt\right)$$
$$+\frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a}\left(|f'(x)|\int_0^1 t^{\frac{\varepsilon}{k}}t^{rs}dt + |f'(b)|\int_0^1 t^{\frac{\varepsilon}{k}}(1-t^r)^s dt\right). \quad \square$$

Remark 2.2. In Theorem 2.2, one can also capture the inequalities for $s-$convex in 1st and 2nd kind, $P-$convex and convex via $k-$fractional integrals by using Remark 2.1.

Corollary 2.2. *In Theorem* 2.2, *one can see for* $k = 1$ *the following.*

(a) *The Ostrowski inequality for* $(s, r)-$*convex in mixed kind via fractional integrals:*

$$|Z_f(\varepsilon,x,a,b)| \le M\left(\frac{1}{\varepsilon+rs+1} + \frac{B\left(\frac{\varepsilon+1}{r}, s+1\right)}{r}\right)\ {}^{\varepsilon}\kappa_a^b(x).$$

(b) *If* $s = 1$ *and* $r \in (0, 1]$ *in inequality* (2.3), *then the Ostrowski inequality for* $r-$*convex in* 1*st* *kind via fractional integrals:*

$$|Z_f(\varepsilon,x,a,b)| \le M\left(\frac{1}{\varepsilon+r+1} + \frac{B\left(\frac{\varepsilon+1}{r}, 2\right)}{r}\right)\ {}^{\varepsilon}\kappa_a^b(x).$$

(c) *If* $r = 1$ *and* $s \in (0, 1]$ *in inequality* (2.3), *then the Ostrowski inequality for* $s-$*convex in* 2*nd* *kind via fractional integrals:*

$$|Z_f(\varepsilon,x,a,b)| \le M\left(\frac{1}{\varepsilon+s+1} + B(\varepsilon+1, s+1)\right)\ {}^{\varepsilon}\kappa_a^b(x).$$

(d) *If* $\varepsilon = r = 1$ *and* $s \in (0, 1]$ *in inequality* (2.3), *then the inequality* (2.1) *of Theorem* 2 *in* [1].

(e) *If* $r = 1$ *and* $s \in (0, 1]$ *in inequality* (2.3), *then the inequality* (2.6) *of Theorem* 7 *in* [15].

(f) *If* $s \to 0$ *and* $r = 1$, *in inequality* (2.3), *then the Ostrowski inequality for* $P-$*convex via fractional integrals:*

$$|Z_f(\varepsilon,x,a,b)| \le M\left(\frac{1}{\varepsilon+1} + B(\varepsilon+1, 1)\right)\ {}^{\varepsilon}\kappa_a^b(x).$$

(g) *If* $r = s = 1$, *in inequality* (2.3), *then the Ostrowski inequality for convex via fractional integrals:*

$$|Z_f(\varepsilon,x,a,b)| \le M\left(\frac{1}{\varepsilon+2} + B(\varepsilon+1, 2)\right)\ {}^{\varepsilon}\kappa_a^b(x).$$

(h) *If* $\varepsilon = r = s = 1$, *in inequality* (2.3), *then the Ostrowski inequality* (1.1) *for convex.*

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$, $[a, b] \subset (0, \infty)$, be an absolutely continuous, and $f' \in L[a, b]$. If $|f'|^q$ is $(s, r)-$convex for $q > 1$ and $|f'(x)| \leq M$, for all $x \in [a, b]$, and $\varepsilon, k > 0$, then*

$$|Y_f(\varepsilon, k, a, x, b)| \leq \frac{M}{L^{\frac{1}{q}-1}} \left(\frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a} + \frac{(b-x)^{\frac{\varepsilon}{k}+1}}{(b-a)} \right) \times \left(\int_0^1 t^{\frac{\varepsilon}{k}} t^{rs} dt + \int_0^1 t^{\frac{\varepsilon}{k}} (1-t^r)^s dt \right)^{\frac{1}{q}}, \tag{2.4}$$

where

$$L = \int_0^1 t^{\frac{\varepsilon}{k}} dt.$$

Proof. By using the Lemma 1.1, and Power mean inequality,

$$\begin{aligned} |Y_f(\varepsilon, k, a, x, b)| \leq & \frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a} \left(\int_0^1 t^{\frac{\varepsilon}{k}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\frac{\varepsilon}{k}} \left| f'\left(tx + (1-t)a\right) \right|^q dt \right)^{\frac{1}{q}} \\ & + \frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a} \left(\int_0^1 t^{\frac{\varepsilon}{k}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\frac{\varepsilon}{k}} \left| f'\left(tx + (1-t)b\right) \right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f'|^q$ is $(s, r)-$convex and $|f'(x)| \leq M$

$$\begin{aligned} |Y_f(\varepsilon, k, a, x, b)| \leq & \frac{M(x-a)^{\frac{\varepsilon}{k}+1}}{L^{\frac{1}{q}-1}(b-a)} \left(\int_0^1 t^{\frac{\varepsilon}{k}} t^{rs} dt + \int_0^1 t^{\frac{\varepsilon}{k}} (1-t^r)^s dt \right)^{\frac{1}{q}} \\ & + \frac{M(b-x)^{\frac{\varepsilon}{k}+1}}{L^{\frac{1}{q}-1}(b-a)} \left(\int_0^1 t^{\frac{\varepsilon}{k}} t^{rs} dt + \int_0^1 t^{\frac{\varepsilon}{k}} (1-t^r)^s dt \right)^{\frac{1}{q}}. \end{aligned}$$

□

Remark 2.3. In Theorem 2.3, one can also capture the inequalities for $s-$convex in 1st and 2nd kind, $P-$convex and convex via $k-$fractional integrals by using Remark 2.1.

Corollary 2.3. *In Theorem* 2.3, *one can see for $k = 1$ the following.*

(a) *The Ostrowski inequality for $(s, r)-$convex in mixed kind via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \leq \frac{M}{(\varepsilon+1)^{1-\frac{1}{q}}} \left(\frac{1}{\varepsilon + rs + 1} + \frac{B\left(\frac{\varepsilon+1}{r}, s+1\right)}{r} \right)^{\frac{1}{q}} {}^{\varepsilon}\kappa_a^b(x).$$

(b) *If $s = 1$ and $r \in (0, 1]$ in inequality* (2.4), *then the Ostrowski inequality for $r-$convex in 1st kind via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \leq \frac{M}{(\varepsilon+1)^{1-\frac{1}{q}}} \left(\frac{1}{\varepsilon + s + 1} + \frac{B\left(\frac{\varepsilon+1}{s}, 2\right)}{s} \right)^{\frac{1}{q}} {}^{\varepsilon}\kappa_a^b(x).$$

(c) *If* $r = 1$ *and* $s \in (0,1]$ *in inequality* (2.4), *then the Ostrowski inequality for* $s-$*convex in* 2^{nd} *kind via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \leq \frac{M}{(\varepsilon+1)^{1-\frac{1}{q}}} \left(\frac{1}{\varepsilon+s+1} + B(\varepsilon+1, s+1) \right)^{\frac{1}{q}} \ {}^{\varepsilon}\kappa_a^b(x).$$

(d) *If* $\varepsilon = r = 1$, *and* $s \in (0,1]$ *in inequality* (2.4), *then the inequality* (2.3) *of Theorem* 4 *in* [1].

(e) *If* $r = 1$ *and* $s \in (0,1]$ *in inequality* (2.4), *then the inequality* (2.8) *of Theorem* 9 *in* [15].

(f) *If* $r = 1$ *and* $s \to 0$ *in inequality* (2.4), *then the Ostrowski inequality for* $P-$*convex via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \leq \frac{M}{(\varepsilon+1)^{1-\frac{1}{q}}} \left(\frac{1}{\varepsilon+1} + B(\varepsilon+1, 1) \right)^{\frac{1}{q}} \ {}^{\varepsilon}\kappa_a^b(x).$$

(g) *If* $r = s = 1$, *in inequality* (2.4), *then the Ostrowski inequality for convex via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \leq \frac{M}{(\varepsilon+1)^{1-\frac{1}{q}}} \left(\frac{1}{\varepsilon+2} + B(\varepsilon+1, 2) \right)^{\frac{1}{q}} \ {}^{\varepsilon}\kappa_a^b(x).$$

Theorem 2.4. *Let* $f : [a,b] \to \mathbb{R}$, $[a,b] \subset (0,\infty)$, *be an absolutely continuous,* $f' \in L[a,b]$. *If* $|f'|^q$ *is* $(s,r)-$*convex,* $|f'(x)| \leq M$, *for all* $x \in [a,b]$, $\varepsilon, k > 0$, *and* $p, z > 1$ *with* $\frac{1}{z} + \frac{1}{q} = 1$, *then*

(2.5)
$$|Y_f(\varepsilon, k, a, x, b)| \leq \frac{MK^{\frac{1}{z}}}{b-a} \left(\frac{1}{rs+1} + \frac{1}{r} B\left(\frac{1}{r}, s+1\right) \right)^{\frac{1}{q}} \times \left((x-a)^{\frac{\varepsilon}{k}+1} + (b-x)^{\frac{\varepsilon}{k}+1} \right),$$

where

$$K = \int_0^1 t^{\frac{\varepsilon z}{k}} dt.$$

Proof. By using Lemma 1.1, and Hölder's inequality,

$$\begin{aligned}|Y_f(\varepsilon, k, a, x, b)| \leq{}& \frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a} \left(\int_0^1 t^{\frac{\varepsilon z}{k}} dt \right)^{\frac{1}{z}} \left(\int_0^1 |f'(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ &+ \frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a} \left(\int_0^1 t^{\frac{\varepsilon z}{k}} dt \right)^{\frac{1}{z}} \left(\int_0^1 |f'(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}.\end{aligned}$$

Since $|f'|^q$ is $(s,r)-$convex and $|f'(x)| \leq M$

$$\begin{aligned}|Y_f(\varepsilon, k, a, x, b)| \leq{}& \frac{K^{\frac{1}{z}}(x-a)^{\frac{\varepsilon}{k}+1}}{b-a} \left(\frac{M^q}{rs+1} + \frac{M^q}{r} B\left(\frac{1}{r}, s+1\right) \right)^{\frac{1}{q}} \\ &+ \frac{K^{\frac{1}{z}}(b-x)^{\frac{\varepsilon}{k}+1}}{b-a} \left(\frac{M^q}{rs+1} + \frac{M^q}{r} B\left(\frac{1}{r}, s+1\right) \right)^{\frac{1}{q}}.\end{aligned}$$

□

Remark 2.4. In Theorem 2.4, one can also capture the inequalities for $s-$convex in 1^{st} and 2^{nd} kind, $P-$convex and convex via $k-$fractional integrals by using Remark 2.1.

Corollary 2.4. *In Theorem* 2.4, *one can see for* $k = 1$ *the following.*

(a) *The Ostrowski inequality for* $(s, r)-$*convex in mixed kind via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \leq \frac{M}{(\varepsilon z + 1)^{\frac{1}{z}}} \left(\frac{1}{rs + 1} + \frac{B\left(\frac{1}{r}, s + 1\right)}{r} \right)^{\frac{1}{q}} \, {}^{\varepsilon}\kappa_a^b(x).$$

(b) *If* $s = 1$ *and* $r \in (0, 1]$ *in inequality* (2.5), *then the Ostrowski inequality for* $r-$*convex in* 1^{st} *kind via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \leq \frac{M}{(\varepsilon z + 1)^{\frac{1}{z}}} \left(\frac{1}{s + 1} + \frac{B\left(\frac{1}{s}, 2\right)}{s} \right)^{\frac{1}{q}} \, {}^{\varepsilon}\kappa_a^b(x).$$

(c) *If* $r = 1$ *and* $s \in (0, 1]$ *in inequality* (2.5), *then the Ostrowski inequality for* $s-$*convex in* 2^{nd} *kind via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \leq \frac{M}{(\varepsilon z + 1)^{\frac{1}{z}}} \left(\frac{1}{s + 1} + B\left(1, s + 1\right) \right)^{\frac{1}{q}} \, {}^{\varepsilon}\kappa_a^b(x).$$

(d) *If* $\varepsilon = r = 1$ *and* $s \in (0, 1]$ *in inequality* (2.5), *then the inequality* (2.2) *of Theorem* 3 *in* [1].

(e) *If* $r = 1$ *and* $s \in (0, 1]$ *in inequality* (2.5), *then the inequality* (2.7) *of Theorem* 8 *in* [15].

(f) *If* $r = 1$, *and* $s \to 0$ *in inequality* (2.5), *then the Ostrowski inequality for* $P-$*convex via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \leq \frac{(2)^{\frac{1}{q}} M}{(\varepsilon z + 1)^{\frac{1}{z}}} \, {}^{\varepsilon}\kappa_a^b(x).$$

(g) *If* $r = s = 1$, *in inequality* (2.5), *then the Ostrowski inequality for convex via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \leq \frac{M}{(\varepsilon z + 1)^{\frac{1}{z}}} \, {}^{\varepsilon}\kappa_a^b(x).$$

3. Applications to Special Means

If we replace f by $-f$ and $x = \frac{a+b}{2}$ in Theorem 2.1, we get the following.

Theorem 3.1. *Let $f:[a,b]\to\mathbb{R}$ be differentiable on (a,b), $f':[a,b]\to\mathbb{R}$ be integrable on $[a,b]$ and $g:I\to\mathbb{R}$, $I\subset\mathbb{R}$, be a $(s,r)-$convex function in mixed sense, then*

(3.1)
$$g\left(\frac{k\Gamma_k(\varepsilon)\left(\frac{b-a}{2}\right)^{1-\frac{\varepsilon}{k}}}{b-a}\ {}^kJ_a^{\varepsilon}f(b)-f\left(\frac{a+b}{2}\right)-\ {}^kJ_a^{\varepsilon-1}\left(P_1\left(\frac{a+b}{2},b\right)f(b)\right)\right)$$
$$\leq\frac{2^{\varepsilon-1}}{(b-a)^{\varepsilon}}\left(\frac{1}{2^{sr-1}}\int_{\frac{a+b}{2}}^{a}g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt+\frac{(2^r-1)^s}{2^{rs-1}}\int_b^{\frac{a+b}{2}}g\left(\frac{(t-b)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt\right).$$

Remark 3.1. In Theorem 3.1, if we put $\varepsilon=k=1$ in (3.1), we get

$$g\left(\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\right)$$
$$\leq\frac{1}{b-a}\left(\frac{1}{2^{sr-1}}\int_a^{\frac{a+b}{2}}g((a-t)f'(t))dt+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{a+b}{2}}^{b}g((b-t)f'(t))dt\right).$$

Remark 3.2. Assume that $g:I\to\mathbb{R}$, $I\subset[0,\infty)$, is an $(s,r)-$convex function in mixed kind.

(a) If $\varepsilon=k=1$, $f(t)=\frac{1}{t}$ in inequality (3.1), where $t\in[a,b]\subset(0,\infty)$, then we have

$$(b-a)g\left(\frac{A(a,b)-L(a,b)}{A(a,b)L(a,b)}\right)$$
$$\leq\frac{1}{2^{sr-1}}\int_a^{\frac{a+b}{2}}g\left(\frac{t-a}{t^2}\right)dt+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{a+b}{2}}^{b}g\left(\frac{t-b}{t^2}\right)dt.$$

(b) If $\varepsilon=k=1$, $f(t)=-\ln t$ in inequality (3.1), where $t\in[a,b]\subset(0,\infty)$, then we have

$$(b-a)g\left(\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right)$$
$$\leq\frac{1}{2^{sr-1}}\int_a^{\frac{a+b}{2}}g\left(\frac{t-a}{t}\right)dt+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{a+b}{2}}^{b}g\left(\frac{t-b}{t}\right)dt.$$

(c) If $\varepsilon=k=1$, $f(t)=t^p$, $p\in\mathbb{R}\setminus\{0,-1\}$ in inequality (3.1), where $t\in[a,b]\subset(0,\infty)$, then we have

$$(b-a)g\left(L_p^p(a,b)-A^p(a,b)\right)$$
$$\leq\frac{1}{2^{sr-1}}\int_a^{\frac{a+b}{2}}g\left(\frac{p\,(a-t)}{t^{1-p}}\right)dt+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{a+b}{2}}^{b}g\left(\frac{p\,(b-t)}{t^{1-p}}\right)dt.$$

Remark 3.3. In Theorem 2.3, one can see for $\varepsilon=k=1$ the following.

(a) Let $x = \frac{a+b}{2}$, $0 < a < b$, $q \geq 1$ and $f : \mathbb{R} \to \mathbb{R}^+$, $f(t) = t^n$ in (2.4). Then

$$\left|A^n(a,b) - L_n^n(a,b)\right| \leq \frac{M(b-a)}{(2)^{2-\frac{1}{q}}} \left(\frac{1}{rs+2} + \frac{B\left(\frac{2}{r}, s+1\right)}{r}\right)^{\frac{1}{q}}.$$

(b) Let $x = \frac{a+b}{2}$, $0 < a < b$, $q \geq 1$ and $f : (0,1] \to \mathbb{R}$, $f(t) = -\ln t$ in (2.4). Then

$$\left|\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right| \leq \frac{M(b-a)}{(2)^{2-\frac{1}{q}}} \left(\frac{1}{rs+2} + \frac{B\left(\frac{2}{r}, s+1\right)}{r}\right)^{\frac{1}{q}}.$$

Remark 3.4. In Theorem 2.4, one can see for $\varepsilon = k = 1$ the following.

(a) Let $x = \frac{a+b}{2}$, $0 < a < b$, $p^{-1} + q^{-1} = 1$ and $f : \mathbb{R} \to \mathbb{R}^+$, $f(t) = t^n$ in (2.5). Then

$$\left|A^n(a,b) - L_n^n(a,b)\right| \leq \frac{M(b-a)}{2(z+1)^{\frac{1}{z}}} \left(\frac{1}{rs+1} + \frac{B\left(\frac{1}{r}, s+1\right)}{r}\right)^{\frac{1}{q}}.$$

(b) Let $x = \frac{a+b}{2}$, $0 < a < b$, $p^{-1} + q^{-1} = 1$ and $f : (0,1] \to \mathbb{R}$, $f(t) = -\ln t$ in (2.5). Then

$$\left|\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right| \leq \frac{M(b-a)}{2(z+1)^{\frac{1}{z}}} \left(\frac{1}{rs+1} + \frac{B\left(\frac{1}{r}, s+1\right)}{r}\right)^{\frac{1}{q}}.$$

4. Conclusion

Ostrowski inequality is one of the most celebrated inequalities. We can find its various generalizations and variants in literature. In this paper, we presented the generalized notion of $(s,r)-$convex in mixed kind, this class of functions contains many important classes including class of $s-$convex in 1st and 2nd kind, $P-$convex, quasi convex and the class of convex. In this study, theorems are put forward to obtain new upper bounds by $k-$fractional operator for Ostrowski type inequalities. We have stated our first main result in Section 2, the generalization of Ostrowski inequality [14] via $k-$fractional integral and others results obtained by using different techniques including Hölder's inequality and power mean inequality. Also, various established results captured as special cases. Moreover, some applications in terms of special means was presented at the end.

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