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NUMERICAL METHOD FOR SOLUTION OF FOURTH-ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS BY GREEN'S FUNCTION

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Abstract. In this paper, we generalize Picard-Green's Embedded method for solving fourth-order Volterra integro-differential equations. We prove the existence and uniqueness theorems. Moreover, we illustrate some numerical examples to present the better approximation with a minimum error. We use MATLAB for numerical solutions.

1. Introduction

Several authors have been interested in differential equations since they are widely used in applications in the technical field as well as in the science and engineering sciences. Particularly elastic theory, biomechanics, electromagnetics, fluids models in physics and biology such as dynamics, heat transfer, population dynamics, and the spread of infectious diseases are frequently encountered.

Studies for the solution of integral and integro-differential equations (IDEs) have continued since Volterra [1, 9, 19]. Although studies on these equations include linear equations, it is often not possible to find their analytical solutions to these equations. For this reason, numerical approaches [2] find more place in the literature. Various algorithms for finding the approximate numerical values are introduced and implemented to find the best results.

Some of these are Wavelet-Galerkin method [6], monotone iterative methods [5, 20], homotopy perturbation method reproducing kernel [4], Adomian decomposition

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method [8], Picard-Green's method [7, 18], Tau method [11], spectral collocation methods [12], Taylor polynomials [14], Lagrange interpolation [16], exponential spline method [17] and the references therein. Furthermore, higher-order boundary value problems (BVPs) for IDEs have been researched by Agarwal [3] and Morchalo [15].

Consider the following boundary value problem

$$\begin{aligned} L[y] &= p_0(t)y''''(t) + p_1(t)y'''(t) + p_2(t)y''(t) + p_3(t)y'(t) + p_4(t)y(t) \\ &= f(t) + \int_0^t K(t,s)g(y(s))ds, \end{aligned} \tag{1.1}$$

with the boundary conditions

$$\begin{aligned} B_a[y] &= \alpha_1 y(a) + \alpha_2 y'(a) + \alpha_3 y''(a) + \alpha_4 y'''(a) = \zeta_1, \\ B_b[y] &= \beta_1 y(b) + \beta_2 y'(b) + \beta_3 y''(b) + \beta_4 y'''(b) = \zeta_2, \\ B_c[y] &= \gamma_1 y(c) + \gamma_2 y'(c) + \gamma_3 y''(c) + \gamma_4 y'''(c) = \zeta_3, \\ B_d[y] &= \omega_1 y(d) + \omega_2 y'(d) + \omega_3 y''(d) + \omega_4 y'''(d) = \zeta_4, \end{aligned} \tag{1.2}$$

where $t \in (a,b)$, ζ_i, $i = 1, \dots, 4$, are constants and either $c = a$ or $c = b$ and either $d = a$ or $d = b$. The existence and uniqueness results for (1.1)–(1.2) are given in [10].

The Green's function $G(t,s)$ of problem (1.1) and (1.2) is;

$$G(t,s) = \begin{cases} a_1 y_1 + a_2 y_2 + a_3 y_3 + a_4 y_4, & a < t < s, \\ b_1 y_1 + b_2 y_2 + b_3 y_3 + b_4 y_4, & s < t < b, \end{cases}$$

where $t \neq s$, y_i are linearly independent solutions of $L[y]$ and a_i, b_i are constants for $i = 1, \dots, 4$.

To implement the proposed methodology, we denote the linear integral operator

$$T[y] = y_h + \int_a^b G(t,s)(p_0(s)y''''(s) + p_1(s)y'''(s) + p_2(s)y''(s) + p_3(s)y'(s) + p_4(s)y(s))ds, \tag{1.3}$$

where y_h is the homogeneous solution of (1.1)–(1.2). From (1.3), we get

$$\begin{aligned} T[y] =& y_h + \int_a^b G(t,s)\Big[p_0(s)y''''(s) + p_1(s)y'''(s) + p_2(s)y''(s) + p_3(s)y'(s) + p_4(s)y(s) \\ & - f(s) - \int_0^s K(t,s)g(y(t))dt\Big]ds + \int_a^b G(t,s)\left(f(s) + \int_0^s K(t,s)g(y(t))dt\right)ds. \end{aligned} \tag{1.4}$$

Let y_p be the particular solution of (1.1), then

$$y_p = \int_a^b G(t,s)\left(f(s) + \int_0^s K(t,s)g(y(t))dt\right)ds. \tag{1.5}$$

By applying $y = y_p + y_h$, from (1.4) and (1.5), we obtain

$$\begin{aligned} T[y] =& y + \int_a^b G(t,s)\Big[p_0(s)y''''(s) + p_1(s)y'''(s) + p_2(s)y''(s) + p_3(s)y'(s) + p_4(s)y(s) \\ & - f(s) - \int_0^s K(t,s)g(y(t))dt\Big]ds. \end{aligned} \tag{1.6}$$

Let the starting function y_0 be the homogeneous solution of $L[y] = 0$ and $y_{n+1} = T[y_n]$, for all $n \geq 0$, then Picard-Green's fixed point iteration method for (1.1) is defined as

$$\begin{aligned} y_{n+1} =& y_n + \int_a^b G(t,s)\Big[p_0(s){y_n}''''(s) + p_1(s){y_n}'''(s) + p_2(s){y_n}''(s) + p_3(s){y_n}'(s) \\ &+ p_4(s)y_n(s) - f(s) - \int_0^s K(t,s)g(y_n(t))dt\Big]ds. \end{aligned} \tag{1.7}$$

In this paper, we generalize Picard-Green's Embedding method (PGEM) for the fourth-order BVPs of Volterra IDEs. We show convergence and prove the convergence theorem. We demonstrate that the developed method offers a better approach than the existing methods by numerical examples.

2. Convergence Analysis and Convergence Rate

In this section, we will introduce convergence analysis using nonlinear differential equations and the contraction principle and determine the convergence rate.

Consider the fourth-order BVP

$$y^{iv}(t) = f(t, y(t), y'(t), y''(t), y'''(t)) + \int_0^t K(t,s)g(y(s))ds, \tag{2.1}$$

with the boundary conditions

$$y(0) = y'(0) = y(1) = y'(1) = 0. \tag{2.2}$$

The solution of the problem (2.1)–(2.2) is as follows

$$y_p = \int_0^1 G(t,s)\left[f(s, y_p, {y'}_p, {y''}_p, {y'''}_p) + \int_0^t K(t,s)g(y_p(s))ds\right]ds \tag{2.3}$$

and

$$\begin{aligned} T[y_p] = \int_0^1 G(t,s)\Big[&p_0(s){y_p}''''(s) + p_1(s){y_p}'''(s) \\ &+ p_2(s){y_p}''(s) + p_3(s){y_p}'(s) + p_4(s)y_p(s)\Big]ds, \end{aligned} \tag{2.4}$$

where $G(t,s)$ is

$$G(t,s) = \begin{cases} t^3\left(\frac{-2s^3+3s^2-1}{6}\right) + t^2\left(\frac{s^3-2s^2+s}{2}\right), & 0 < t < s, \\ s^3\left(\frac{-2t^3+3t^2-1}{6}\right) + s^2\left(\frac{t^3-2t^2+t}{2}\right), & s < t < 1. \end{cases}$$

From (2.3) and (2.4), we get

$$\begin{aligned} T[y_p] =& y_p \\ &+ \int_a^b G(t,s)\Big[p_0(s)y''''(s) + p_1(s)y'''(s) + p_2(s)y''(s) + p_3(s)y'(s) + p_4(s)y(s) \\ &- f(s, y_p, {y'}_p, {y''}_p, {y'''}_p) + \int_0^s K(t,s)g(y_p(t)dt\Big]ds. \end{aligned}$$

By applying PGEM to the problem (2.1)–(2.2), we obtain the following iterative scheme.

$$y_{n+1} = y_n + \int_a^b G^*(t,s)\left[{y_n}''''(s) - f(s, y_n(s), {y_n}'(s), {y_n}''(s), {y_n}'''(s)) - \int_0^s K(t,s)g(y_n(t))dt\right] ds.$$

In particular, we have

$$\begin{aligned} y_{n+1} =& y_n - \int_0^t \left(s^3\left(\frac{-2t^3+3t^2-1}{6}\right) + s^2\left(\frac{t^3-2t^2+t}{2}\right)\right) \quad (2.5)\\ &\times \left[{y_n}''''(s) - f(s, y_n(s), {y_n}'(s), {y_n}''(s), {y_n}'''(s)) - \int_0^s K(t,s)g(y_n(t))dt\right] ds \quad (2.6)\\ &- \int_t^1 \left(t^3\left(\frac{-2s^3+3s^2-1}{6}\right) + t^2\left(\frac{s^3-2s^2+s}{2}\right)\right)\\ &\times \left[{y_n}''''(s) - f(s, y_n(s), {y_n}'(s), {y_n}''(s), {y_n}'''(s)) - \int_0^s K(t,s)g(y_n(t))dt\right] ds. \end{aligned}$$

Theorem 2.1. *Let $X = C[0,1]$ be a Banach space with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$, $x \in X$. Assume that the function g satisfies the Lipschitz condition such that $|g(y) - g(v)| \le L|y - v|$, $L \in (0,1]$. Then operator T defined in (1.6) is a Banach's contraction and the sequence y_n converges strongly to the solution of the problem (2.1) and (2.2) under the following conditions*

$$Q = \left(\frac{1}{98}\right) A < 1,$$

where

$$A = \max_{[0,1]\times R^4} \left|\frac{\partial f(t,y,y',y'',y''')}{\partial y}\right| + \|K\| L \left(\frac{1}{2}\right).$$

Proof. Integrating (2.5) by parts, we get

$$y_{n+1} = y_n(t) + \int_0^1 G^*(t,s)\left[f(s, y_n, {y_n}', {y_n}'', {y_n}''') + \int_0^s K(t,s)g(y_n(t))dt\right] ds. \quad (2.7)$$

Let $T_G : [0,1] \to [0,1]$ be the right side of (2.7), then

$$\begin{aligned} \|T_G(y_n) - T_G(y_m)\| = \Big\| \int_0^1 G^*(t,s) \Big[& f(s,y,y',y'',y''') + \int_0^s K(t,s)g(y_n(t))dt) \\ & - f(s, y_m, y'_m, y''_m, y'''_m) + \int_0^s K(t,s)g(y_m(t))dt \Big] ds \Big\|. \end{aligned}$$

By using the fact that

$$\|G\| = \max_{0\le t,s\le 1} |G^*(t,s)| = \frac{1}{98},$$

we get

$$\|T_G(y_n) - T_G(y_m)\| \leq \frac{1}{98}\int_0^1 \Big\| \Big[f(s, y_n, y_n', y_n'', y_n''') + \int_0^s K(t,s) g(y_n(t)) dt \\ - f(s, y_m, y_m', y_m'', y_m''') + \int_0^t K(t,s) g(y_m(s)) dt \Big] \Big\| ds.$$

Implementing Mean Value Theorem, we obtain

$$\|T_G(y_n) - T_G(y_m)\| \leq \frac{1}{98} A \|y_n - y_m\|.$$

Therefore, we get

$$\|T_G(y_n) - T_G(y_m)\| \leq Q \|y_n(t) - y_m(t)\|, \tag{2.8}$$

where $Q \in (0,1)$. From (2.8) we have

$$\begin{aligned} \|y_n - y_m\| &= \|(y_n - y_{n-1}) + (y_{n-1} - y_{n-2}) + \cdots + (y_{m+1} - y_m)\| \\ &\leq \|y_n - y_{n-1}\| + \|y_{n-1} - y_{n-2}\| + \cdots + \|y_{m+1} - y_m\| \\ &\leq (Q^{n-1} + Q^{n-2} + \cdots + Q^m)\|y_1 - y_0\| \\ &\leq Q^m (1 + Q + Q^2 + \cdots + Q^{n-m-1})\|y_1 - y_0\| \\ &= Q^m \left(\frac{1 - Q^{n-m}}{1 - Q} \right) \|y_1 - y_0\|. \end{aligned}$$

Since $Q \in (0,1)$, we have

$$\|y_n - y_m\| \leq \frac{Q^m}{1 - Q} \|y_1 - y_0\|, \tag{2.9}$$

which converges to zero, i.e., $\|y_n - y_m\| \to 0$, while $m \to 0$. Thus, $T_G(y)$ is a contraction mapping. □

Let y^* be the solution of problem (2.1) and (2.2). Then $T(y^*) = y^*$. From (2.8) and (2.9), we have

$$\|y_{n+1} - y^*\| = \|T(y_n) - y^*\| \leq Q\|y_n - y^*\| \leq \cdots \leq Q^{n+1}\|y_0 - y^*\|.$$

Since $0 < Q < 1$, it concludes that y_n converges strongly to y^*. The rest proof can be completed from the proof of [13, Proposition 1].

3. Numerical Examples

In this section, we give numerical examples to confirm the applicability of the main results.

Example 3.1. Consider the fourth order BVP

$$y^{iv}(t) = f(t) + \int_0^t y(s) ds, \tag{3.1}$$

with the boundary conditions

$$y(0) = y'(0) = 1, \quad y(1) = 1 + e, \quad y'(1) = 2e, \tag{3.2}$$

where $f(t) = -t + 5e^t - 1$ and the exact solution $y(t) = 1 + te^t$ and the Green's function is

$$G(t,s) = \begin{cases} t^3\left(\frac{-2s^3+3s^2-1}{6}\right) + t^2\left(\frac{s^3-2s^2+s}{2}\right), & 0 < t < s, \\ s^3\left(\frac{-2t^3+3t^2-1}{6}\right) + s^2\left(\frac{t^3-2t^2+t}{2}\right), & s < t < 1. \end{cases}$$

By applying PGEM, we get

$$\begin{aligned} y_{n+1} = & y_n - \int_0^t \left[s^3\left(\frac{-2t^3+3t^2-1}{6}\right) + s^2\left(\frac{t^3-2t^2+t}{2}\right)\right] \\ & \times \left[y_n^{iv}(s) + s - 5e^s + 1 - \int_0^s y_n(t)dt\right] ds \\ & - \int_t^1 \left[t^3\left(\frac{-2s^3+3s^2-1}{6}\right) + t^2\left(\frac{s^3-2s^2+s}{2}\right)\right] \\ & \times \left[y_n^{iv}(s) + s - 5e^s + 1 - \int_0^s y_n(t)dt\right] ds, \end{aligned} \tag{3.3}$$

where the starting function is $y_0 = t^3 + (e-2)t^2 + t + 1$. The absolute error of the problem is estimated by

$$Err = |y(t) - y_n(t)|.$$

Table 1 gives the maximum errors of the problem (3.1)–(3.2) to demonstrate the high accuracy of the proposed method. Considering the values in the table, the margin of error decreases considerably and approaches zero as the number of iterations increases.

Table 1. The maximum errors of Example 1

No. of iterations	6	8	10	12
Max Error(n)	2.96E-18	1.23E-24	5.13E-31	2.13E-37

Table 2 shows the absolute errors for the second and third iterations solved by two different methods. The table shows that PGEM has a better convergence rate than Adomian Decomposition Method (MADM). Meanwhile, the chart 1 represents the line graphs of the absolute errors of both methods for the third iteration. Therefore, it is clear that PGEM approaches 0 faster than MADM.

Table 2. The absolute errors (n) of Example 1

t	Numerical Solution	PGEM Error (2)	PGEM Error(3)	MADM Error(2)	MADM Error(3)
0.1	1.1111924502842667426690545607791	1.68E-06	1.08E-09	4.54E-05	2.29E-08
0.3	1.4086348815636588244756957461284	1.05E-05	6.75E-09	4.23E-05	4.74E-07
0.5	1.8295725879184859388029560326160	1.70E-05	1.10E-08	6.63E-05	3.37E-07
0.7	2.4133047634097381621411544034260	1.35E-05	8.81E-09	6.92E-05	4.78E-07
0.9	3.2143183796746349483923129747728	2.74E-06	1.80E-09	7.97E-06	5.81E-08

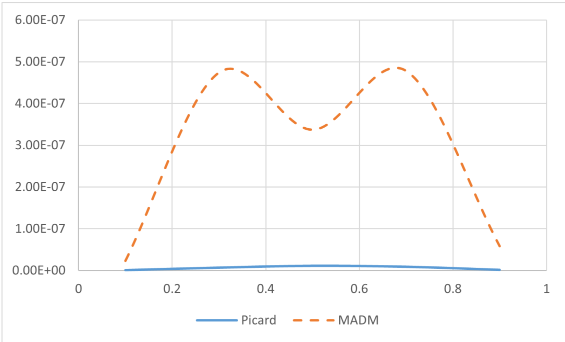

FIGURE 1. The relative absolute errors of Example 1

Example 3.2. Consider non-linear BVP

$$y^{iv} = 1 + \int_0^t e^{-s} y^2(s) ds, \tag{3.4}$$

corresponding to boundary conditions

$$y(0) = y'(0) = 1, \quad y(1) = y'(1) = e. \tag{3.5}$$

The exact solution of the problem given above is $y(t) = e^x$, and the Green's function of (3.4)–(3.5) is

$$G(t,s) = \begin{cases} t^3\left(\frac{-2s^3+3s^2-1}{6}\right) + t^2\left(\frac{s^3-2s^2+s}{2}\right), & 0 < t < s, \\ s^3\left(\frac{-2t^3+3t^2-1}{6}\right) + s^2\left(\frac{t^3-2t^2+t}{2}\right), & s < t < 1, \end{cases}$$

where the starting function is $y_0 = (-e-3)t^3 + (2e-5)t^2 + t + 1$.

By applying PGEM, we get

$$\begin{aligned} y_{n+1} = & y_n - \int_0^t \left[s^3\left(\frac{-2t^3+3t^2-1}{6}\right) + s^2\left(\frac{t^3-2t^2+t}{2}\right)\right] \\ & \times \left[y_n^{iv}(s) - 1 - \int_0^s e^{-s} y_n^2(t) dt \right] ds \\ & - \int_t^1 \left[t^3\left(\frac{-2s^3+3s^2-1}{6}\right) + t^2\left(\frac{s^3-2s^2+s}{2}\right)\right] \\ & \times \left[y_n^{iv}(s) - 1 - \int_0^s e^{-s} y_n^2(t) dt \right] ds. \end{aligned} \tag{3.6}$$

Table 3 demonstrates the high accuracy of the proposed method for the problem given in Example 2. It presents second iteration errors for PGEM, MADM, and MDMGF (Modified Decomposition Method with Green function). The results of recommended method PGEM converge to the exact solution faster.

Table 4 shows the third iteration errors for the methods discussed in Table 3. When we examine these results, it is clear that the results of the PGEM method decrease faster as the number of iterations increases and converge to zero faster than the other

TABLE 3. The absolute errors (n) of Example 2

t	Numerical Solution Error(2)	PGEM	MADM	MDMGF
0.1	1.1051709173255609245824473770908	5.85E-07	8.48E-05	1.43E-05
0.3	1.3498588028660124806131472805192	3.66E-06	9.16E-05	9.17E-05
0.5	1.6487212630265689433471827882745	5.93E-06	3.66E-04	1.56E-04
0.7	2.0137527013290129690745570786289	4.71E-06	4.54E-04	1.34E-04
0.9	2.4596031099034457029111001120299	9.55E-07	3.00E-05	3.00E-05

TABLE 4. The other absolute errors (n) of Example 2

t	Numerical Solution Error(3)	MADM	PGEM	MDMGF
0.1	1.1051709173255609245824473770908	2.32E-06	7.50E-10	4.32E-08
0.3	1.3498588028660124806131472805192	7.72E-05	4.71E-09	2.75E-07
0.5	1.6487212630265689433471827882745	7.52E-05	7.67E-09	4.54E-07
0.7	2.0137527013290129690745570786289	4.72E-05	6.14E-09	3.72E-07
0.9	2.4596031099034457029111001120299	7.61E-06	1.25E-09	7.61E-08

methods, as in Table 3. These results clearly show that PGEM is more effective, as we tried to demonstrate.

While the Figure 2 shows the comparisons of the values in the tables 3 and 4, Fig. 3 depicts the comparisons between the exact solutions and the numerical solutions obtained in the third iteration. Overall, it is clear from the first graph that the values obtained via PGEM tend to approach zero faster than other methods. Moreover, as shown by the second graph, the numerical solutions got by PGEM are very close to the exact values.

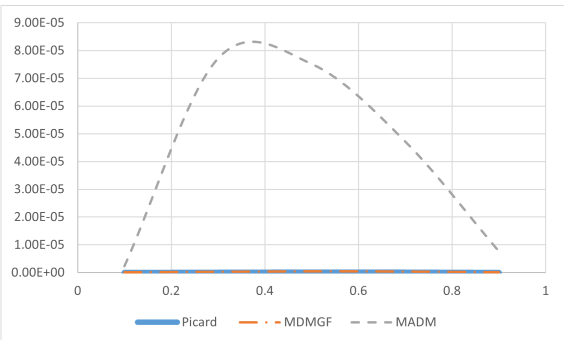

FIGURE 2. The absolute errors of Example 2

4. CONCLUSION

In this study, we generalize Picard-Green's fixed-point iteration method, one of the most popular methods for fourth-order nonlinear and linear IVPs, by embedding

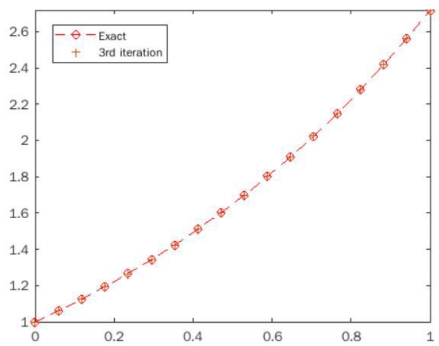

Figure 3. Exact and numerical solutions

Green's function. We proved the convergence and got the convergence rate. We solve some examples to show the correctness and generality of the proposed scheme. We compared the numerical results obtained by the determined method with the results of the methods well known in the literature. For comparison, we considered the MADM and MDMGF methods. We used MATLAB to calculate numerical results. We presented the obtained results with the help of tables and figures. Our method gives better results than other methods when comparing numerical results, exact results, and calculated values. Therefore, the aim of our study has been revealed.

There are many iteration methods in the literature to find the best approach. This study compared the results obtained for the fourth-order Volterra integro-differential equations with the Adomian decomposition methods. However, solving higher order linear and nonlinear differential and integro-differential equations with a better approach than other existing methods is still a problem to be developed. We believe its solution will lead to many studies.

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