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RELATION BETWEEN CONVERGENCE AND ALMOST CONVERGENCE OF COMPLEX UNCERTAIN SEQUENCES

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Abstract. In this paper, we introduce a new type of almost convergent complex uncertain sequence with respect to uniformly almost surely. We characterize the notion of almost convergence of sequences of complex uncertain variables further. We establish the interconnection between convergent complex uncertain sequence, bounded complex uncertain sequence and almost convergent complex uncertain sequence in all five aspects of uncertainty.

1. Introduction and Preliminaries

In the real world, often we face various types of indeterminacy. Frequency generated by samples plays important role in the study to deal with those indeterminate situations. Probability theory is an efficient tool to study the frequency. However, sometimes it is difficult to collect observed data when some unexpected events occur. In this case, decision maker have to invite experts to estimate the belief degree of each events occurrence. For dealing with belief degree legitimately, an axiomatic system named uncertainty theory satisfied normality, duality, and subadditivity was proposed by Liu [9]. As a fundamental concept in uncertainty theory, the uncertain variable was presented by Liu [9]. In order to describe an uncertain variable, Liu [9] introduced the concepts of uncertain measure, uncertain distribution and expected value of uncertain variable. The uncertain measure follows the axioms of normality, duality, subadditivity and product. In the year 2007, the notion of uncertain sequences and their four

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types of convergences, namely convergence in mean, measure, distribution, almost surely was introduced by Liu [9]. Then the same was extended by You [12] while he introduced a new type of convergent uncertain sequence with respect to uniformly almost surely. Thereafter, to describe the complex uncertain quantities, the notions of uncertain variable and uncertain distribution are presented by Peng [10] in that direction. Chen et al. [1] explored the work considering the sequence of complex uncertain variables due to Peng [10]. They reported five types of convergence concepts of uncertain sequences in complex environment by establishing interrelationships among them. Since its initiation, the study of complex uncertain sequences got the full attention of the researchers. These convergence concept of complex uncertain sequence has also been generalised by Datta and Tripathy [8], Das et al. [2–7]. Recently, Saha et al. [11] introduced the concept almost convergent complex uncertain sequence in a given uncertainty space. They have initiated almost convergence in four directions of uncertainty, namely almost convergence in mean, in measure, in distribution and in almost surely. Also, they established the interrelationships between each types of almost convergences upto some extent. In this article, at first we extend the study by introducing the fifth direction of uncertainty, i.e., almost convergent complex uncertain sequence with respect to uniformly almost surely. We show that every almost convergent complex uncertain sequence with respect to uniformly almost surely is almost convergent in almost surely. We further establish the interconnection between almost convergent, bounded and convergent sequences of complex uncertain variable.

We now present few concepts and related results in the following, which will be playing an important role in the whole study.

Definition 1.1 ([12]). Let us consider an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. Then a function ζ from Γ to the set of complex numbers which is measurable in the aspect of uncertainty is called a complex uncertain variable.

Definition 1.2 ([10]). Let us consider a sequence (ζ_n) of complex uncertain variables. Then (ζ_n) is said to be almost convergent to ζ in almost surely if there is such an uncertain event Λ with unit uncertain measure that

$$\lim_{m\to\infty} ||u_{n,m}(\gamma) - \zeta(\gamma)|| = 0,$$

uniformly in n and for all $\gamma \in \Lambda$, where $u_{n,m} = \frac{1}{m}\sum_{i=1}^{m} \zeta_{n+i-1}$.

Definition 1.3 ([10]). A sequence (ζ_n) of complex uncertain variables is called almost convergent in respect of measure to some finite limit ζ if the following condition is satisfied: for all positive integer n and a positive real ε

$$\lim_{m\to\infty} \mathcal{M}\{||u_{n,m} - \zeta|| \geq \varepsilon\} = 0, \quad \text{where } u_{n,m} = \frac{1}{m}\sum_{i=1}^{m} \zeta_{n+i-1}.$$

Definition 1.4 ([10])**.** Let us consider a sequence (ζ_n) of complex uncertain variables. The sequence is said to be almost convergent in respect of mean to a finite ζ if

$$\lim_{l\to\infty} E[||t_{n,l}-\zeta||]=0, \quad \text{where } t_{n,l}=\frac{1}{l}\sum_{i=1}^{l}\zeta_{n+i-1}.$$

Here n runs uniformly over $\mathbb{N}$.

Definition 1.5 ([10])**.** Let us consider infinite numbers of complex uncertain variables given by $\xi,\xi_1,\xi_2,\dots,$ and suppose $\Phi,\Phi_{1,m},\Phi_{2,m},\dots,$ are the distribution functions in respect of the complex uncertain variables ξ, $\frac{\xi_1+\xi_2+\cdots+\xi_m}{m}$, $\frac{\xi_2+\xi_3+\cdots+\xi_{m+1}}{m}$, $\dots$, respectively. Then the sequence (ξ_n) is said to be almost convergent to ξ in respect of distribution if

$$\lim_{m\to\infty}\Phi_{n,m}(c)=\Phi(c),$$

for all $n\in\mathbb{N}$, c being the complex point of continuity of the function Φ.

Theorem 1.1 ([10])**.** *If the real and imaginary part (ξ_n) and (η_n) of a sequence (ζ_n) almost converges to the finite limits ξ and η respectively with respect to measure, then $(\zeta_n)=(\xi_n+i\eta_n)$ almost converges in distribution to $\xi+i\eta$.*

Theorem 1.2 ([10])**.** *If (ζ_n) is an almost convergent sequence of complex uncertain variables in mean to some finite limit ζ, then it almost converges in respect of measure by preserving the limit.*

2. Main Results

At first our intend is to define almost convergent sequence of complex uncertain variables with respect to uniformly almost surely. We show existence of such sequence and establish the interrelationship with the almost convergent complex uncertain sequence in almost surely. Then, we initiate boundedness property of sequences of complex uncertain variables and prove the interconnection between convergent, bounded and almost convergent sequences of complex uncertain variables in all five aspects of uncertainty.

Definition 2.1. A complex uncertain sequence (ζ_n) is called almost convergent to a finite limit ζ in uniformly almost surely if there exists events $\{E_x\}$ with $\mathcal{M}\{E_x\}\to 0$ such that (ζ_n) almost converges to the same ζ uniformly in the domain $\Gamma - E_x$, where $x\in\mathbb{N}$, i.e.,

$$\lim_{p\to\infty}\left\|\frac{1}{p}\sum_{k=0}^{p-1}\zeta_{n+k}(\gamma)-\zeta(\gamma)\right\|=0,$$

for all $\gamma\in\Gamma-E_x$ and uniformly for all n.

Example 2.1. Let $\Gamma=\{\gamma_1,\gamma_2,\dots\}$ be an infinite set of uncertain events and $\mathcal{L}$ be the power set of Γ. Then $\mathcal{L}$ becomes σ-algebra on Γ.

Let the measurable set function $\mathcal{M}$ be defined as follows

$$\mathcal{M}\{\beta\} = \sum_{\gamma_j \in \beta} \frac{1}{2^j}.$$

Obviously, $\sum_{\gamma_j \in \beta} \frac{1}{2^j}$ is unity and $\mathcal{M}$ holds the other axioms of uncertain measure. So $\mathcal{M}$ becomes uncertain measure and thus, $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space.

Now, for a given $\varepsilon > 0$ (however small) exists $p \in \mathbb{N}$ such that $\frac{1}{2^p} < \varepsilon$.

Let (ζ_n) be a complex uncertain sequence, where the complex uncertain variable ζ_n is given by

$$\zeta_n(\gamma) = \begin{cases} \frac{1}{2}i, & \text{if } n \geq p, \\ 0, & \text{otherwise,} \end{cases}$$

for all $\gamma \in \Gamma$. Also, let ζ be the complex uncertain variable such that $\zeta(\gamma) = 0$ for all $\gamma \in \Gamma$. We have, $||\zeta_n(\gamma) - \zeta(\gamma)|| = \left|\left|\frac{1}{2}i\right|\right| = \frac{1}{2}$, whenever $n > N$ and $||\zeta_n(\gamma) - \zeta(\gamma)|| = 0$, for the remaining cases. Moreover, $\mathcal{M}\{\gamma_j\} \to 0$, as $j > p$. Then, from the above, one can see that $(\zeta_n(\gamma))$ almost converges uniformly to $\zeta(\gamma) = 0$, for all $\gamma \in \Gamma - \gamma_j,\ j > p$. Hence, (ζ_n) is almost convergent to ζ in uniformly almost surely.

The following theorem is due to Saha et al. [10].

Theorem 2.1 ([10]). *Suppose $(\zeta_n) = (\xi_n + i\eta_n)$ be a complex uncertain sequence. If the real uncertain sequences (ξ_n) and (η_n) almost converges to ξ and η respectively in respect of measure, then (ζ_n) is almost convergent to $\xi + i\eta$ in the same direction.*

We now establish the converse part of the same in the same context. This one result produces few more interrelationships between the other almost convergence concepts.

Theorem 2.2. *If a complex uncertain sequence (ζ_n), which is given by $\zeta_n = \xi_n + i\eta_n$, almost converges in measure to the finite limit $\xi + i\eta$, then the real part (ξ_n) and imaginary part (η_n) also almost converges to ξ and η in measure.*

Proof. Let (ζ_n), where $\zeta_n = \xi_n + i\eta_n$ is almost convergent to $\zeta = \xi + i\eta$ in measure. Then, for any $\delta > 0$, we have

$$\mathcal{M}\left\{\left|\left|\frac{1}{p}\sum_{k=0}^{p-1} \zeta_{n+k} - \zeta\right|\right| \geq \delta\right\} \to 0, \quad \text{as } n \to \infty$$

$$\Rightarrow \mathcal{M}\left\{\left|\left|\frac{1}{p}\sum_{k=0}^{p-1}(\xi_{n+k} + i\eta_{n+k}) - (\xi + i\eta)\right|\right| \geq \delta\right\} \to 0, \quad \text{as } n \to \infty$$

$$\Rightarrow \mathcal{M}\left\{\left|\left|\frac{1}{p}\sum_{k=0}^{p-1}(\xi_{n+k} - \xi) + i\frac{1}{p}\sum_{k=0}^{p-1}(\eta_{n+k} - \eta)\right|\right| \geq \delta\right\} \to 0, \quad \text{as } n \to \infty.$$

This implies that there exists $0 < \delta' < \frac{\delta}{2}$ such that

$$\mathcal{M}\left\{\gamma : \left|\left|\frac{1}{p}\sum_{k=0}^{p-1} \xi_{n+k}(\gamma) - \xi(\gamma)\right|\right| \geq \delta'\right\} \to 0, \quad \text{as } n \to \infty,$$

and

$$\mathcal{M}\left\{\gamma : \left\|\frac{1}{p}\sum_{k=0}^{p-1}\eta_{n+k}(\gamma) - \eta(\gamma)\right\| \geq \delta^{'}\right\} \to 0, \quad \text{as } n \to \infty.$$

Evidently, the uncertain sequences (ξ_n) and (η_n) are almost convergent in measure to ξ and η. □

In view of the above Theorem 2.2 and Theorem 1.1, we can deduce the following result.

Corollary 2.1. *Almost convergence in measure implies almost convergence in distribution.*

From Theorem 1.2 and Corollary 2.1, we can give the following.

Corollary 2.2. *An almost convergent sequence in mean almost converges with respect of distribution therein.*

Remark 2.1. The notion of almost convergence in almost surely and almost convergence in measure are the concepts no way related.

In the following two examples, we demonstrates the validity of the statement.

Example 2.2. We consider the space $(\Gamma, \mathcal{L}, \mathcal{M})$, with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and $\mathcal{L} = P(\Gamma)$. Define $\mathcal{M}$ as follows:

$$\mathcal{M}\{\Delta\} = \begin{cases} 0, & \text{if } \Delta = \phi, \\ 1, & \text{if } \Delta = \Gamma, \\ 0.6, & \text{if } \gamma_1 \in \Delta, \\ 0.4, & \text{if } \gamma_1 \notin \Delta. \end{cases}$$

We define ζ_n and ζ as follows:

$$\zeta_n(\alpha) = \begin{cases} i, & \text{if } \alpha = \gamma_1, \\ 2i, & \text{if } \alpha = \gamma_2, \\ 3i, & \text{if } \alpha = \gamma_3, \\ 4i, & \text{if } \alpha = \gamma_4, \\ 0, & \text{otherwise}, \end{cases}$$

for $n \in \mathbb{N}$ and $\zeta(\gamma) = 0$ for all $\gamma \in \Gamma$.

Observe that $\zeta_n \to \zeta$, except only for $\gamma = \gamma_1, \gamma_2, \gamma_3, \gamma_4$ and so (ζ_n) is almost convergent to ζ in almost surely. However, for some $\delta > 0$, we have

$$\mathcal{M}\{||\zeta_n - \zeta|| \geq \delta\} = \mathcal{M}\{\gamma : ||\zeta_n(\gamma) - \zeta(\gamma)|| \geq \delta\} = \mathcal{M}\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} = \mathcal{M}\{\Gamma\} = 1.$$

Consequently, the complex uncertain sequence (ζ_n) is not almost convergent in measure.

Example 2.3. Let us consider the space $(\Gamma, \mathcal{L}, \mathcal{M})$, with $\Gamma = [0,1]$, $\mathcal{L} = P[0,1]$. Here the uncertain measure is the Lebesgue measure.

Suppose ζ_n and ζ be given by

$$\zeta_n(\alpha) = \begin{cases} i, & \text{if } \frac{p}{2^t} \leq \alpha \leq \frac{1+p}{2^t}, \\ 0, & \text{elsewhere,} \end{cases}$$

and $\zeta(\alpha) = 0$ for all $\gamma \in \Gamma,\ n = 2^t + p \in \mathbb{N}$, where p, t are integers. Then,

$$\begin{aligned} \mathcal{M}\left\{\left\|\frac{1}{p}\sum_{k=0}^{p-1}\zeta_{n+k} - \zeta\right\| \geq \delta\right\} =& \mathcal{M}\left\{\gamma : \left\|\frac{1}{p}\sum_{k=0}^{p-1}\zeta_{n+k}(\gamma) - \zeta(\gamma)\right\| \geq \delta\right\} \\ =& \frac{1+p}{2^t} - \frac{p}{2^t} = \frac{1}{2^t} \end{aligned}$$

and hence,

$$\lim_{n\to\infty} \mathcal{M}\left\{\gamma : \left\|\tfrac{1}{p}\sum_{k=0}^{p-1}\zeta_{n+k}(\gamma) - \zeta(\gamma)\right\| \geq \delta\right\} = \lim_{t\to\infty} \tfrac{1}{2^t} = 0.$$

Thus, the complex uncertain sequence (ζ_n) almost converges to ζ in measure.

On the other hand, let $\gamma \in [0,1]$. Then, there are intervals of the form $[\frac{p}{2^t}, \frac{p+1}{2^t}]$ containing γ, for different values of p. Therefore, (ζ_n) does not converges to ζ in almost surely and hence (ζ_n) is not almost convergent to ζ in almost surely.

Remark 2.2. An almost convergent complex uncertain sequence in almost surely may not be almost convergent in distribution. The following example satisfies the same.

Example 2.4. Consider the uncertainty space and sequence taken in example 2.2. Let $\Phi_n(z)$ and $\Phi(z)$ be the uncertainty distribution functions of ζ_n and ζ, respectively. Then

$$\Phi_n(z) = \Phi_n(p+iq) = \begin{cases} 0, & \text{if } p < 0, q \in (-\infty, \infty), \\ 0, & \text{if } p \geq 0, q < 1, \\ 0.6, & \text{if } p \geq 0, 1 \leq q < 2, \\ 0.6, & \text{if } p \geq 0, 2 \leq q < 3, \\ 0.6, & \text{if } p \geq 0, 3 \leq q < 4, \\ 1, & \text{if } p \geq 0, q \geq 4, \end{cases}$$

and

$$\Phi(z = p+iq) = \begin{cases} 0, & \text{if } p < 0, q \in (-\infty, \infty), \\ 0, & \text{if } p \geq 0, q < 0, \\ 1, & \text{if } p \geq 0, q \geq 0. \end{cases}$$

Thus, (ζ_n) is not almost convergent in distribution to ζ.

Remark 2.3. An almost convergent complex uncertain sequence in mean may not be almost convergent in almost surely.

Example 2.5. From the Example 2.3,

$$E\left[\left\|\frac{1}{p}\sum_{k=0}^{p-1}\zeta_{n+k}(\gamma)-\zeta(\gamma)\right\|\right]=\frac{1}{2^t},$$

which tends to 0, as $n\to\infty$. Then, the sequence almost converges to ζ in mean also. But it was already proved that (ζ_n) is not almost convergent in almost surely.

Remark 2.4. A sequence (ζ_n) which is almost convergent in almost surely may not be almost convergent in mean. Explanation is provided in the following example.

Example 2.6. Consider the space $(\Gamma,\mathcal{L},\mathcal{M})$ with $\Gamma=\{\gamma_1,\gamma_2,\gamma_3,\dots\}$, $\mathcal{L}=P(\Gamma)$ and

$$\mathcal{M}\{\Lambda\}=\sum_{\gamma_j\in\Lambda}\frac{2}{3}\cdot\frac{1}{3^{(j-1)}}.$$

Define ζ_n and ζ respectively by

$$\zeta_n(\alpha)=\begin{cases}3^n i, & \text{if } \alpha=\gamma_n,\\ 0, & \text{elsewhere,}\end{cases}$$

for $n\in\mathbb{N}$ and $\zeta\equiv 0$.

One can easily observe that the sequence (ζ_n) almost converges to ζ in almost surely.

Now, for the uncertain variable $||\zeta_n||$, its uncertainty distribution function is given by

$$\Phi_n(p)=\begin{cases}0, & \text{if } p<0,\\ 1-\frac{1}{3^n}, & \text{if } 0\le p<3^n,\\ 1, & \text{elsewhere,}\end{cases}$$

for $n\in\mathbb{N}$.

Now, integration to the above distribution function for expected value gives us

$$E\left[\left\|\frac{1}{p}\sum_{k=0}^{p-1}\zeta_{n+k}-\zeta\right\|\right]=1.$$

Therefore, the complex uncertain sequence (ζ_n) is not almost convergent in mean to ζ.

Theorem 2.3. *The sequence (ζ_n) is almost convergent in almost surely to ζ if and only if for any $\varepsilon>0$ exists $N\in\mathbb{N}$ in such a way that*

$$\mathcal{M}\left\{\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1}\zeta_{n+x}(\gamma)-\zeta(\gamma)\right\|\ge\varepsilon\right\}\right\}=0.$$

Proof. The definition of almost convergence in almost surely leads us to the existence of such uncertain event Δ with $\mathcal{M}\{\Delta\}=1$, such that

$$\lim_{n\to\infty}\left\|\frac{1}{p}\sum_{r=0}^{p-1}\zeta_{n+r}(\alpha)-\zeta(\alpha)\right\|=0,\quad \text{for all } \alpha\in\Delta.$$

Let ε be a preassigned positive number. Then, there exists $N \in \mathbb{N}$ such that for any $\alpha \in \Delta$, we have

$$\mathcal{M}\left\{\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x}(\alpha) - \zeta(\alpha)\right\| < \varepsilon\right\}\right\} = 1,$$

where $n > N$.

Applying the duality axiom of uncertain measure to the above, we get

$$\mathcal{M}\left\{\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x}(\alpha) - \zeta(\alpha)\right\| \geq \varepsilon\right\}\right\} = 0, \quad \text{for all } \alpha \in \Delta.$$

Hence, the theorem is proved. □

Theorem 2.4. *The necessary and sufficient condition for a complex uncertain sequence (ζ_n) to almost converges in uniformly almost surely to ζ is that for any $\varepsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that*

$$\mathcal{M}\left\{\bigcup_{n=N}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x} - \zeta\right\| \geq \delta\right\}\right\} < \varepsilon.$$

Proof. Let the sequence (ζ_n) of complex uncertain variable almost converges to ζ in uniformly almost surely. Then for $\varepsilon > 0$, there exists $\delta > 0$ and an event B with measure less than ν, $\nu \to 0^+$, such that the sequence (ζ_n) converges uniformly to ζ on $\Gamma - B$. That means, there exists $n_0 \in \mathbb{N}$ so that

$$\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x}(\gamma) - \zeta(\gamma)\right\| < \varepsilon, \quad \text{for all } n \geq n_0 \text{ and all } \gamma \in \Gamma - B.$$

Also, $\nu < \varepsilon$. Thus, we have

$$\bigcup_{n=N}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x}(\gamma) - \zeta(\gamma)\right\| \geq \delta\right\} \subseteq B.$$

Applying the subadditivity axiom of uncertain measure, we get

$$\mathcal{M}\left\{\bigcup_{n=N}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x} - \zeta\right\| \geq \delta\right\}\right\} \leq \mathcal{M}\{B\} < \nu < \varepsilon.$$

Conversely, let

$$\mathcal{M}\left\{\bigcup_{n=N}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x} - \zeta\right\| \geq \delta\right\}\right\} < \varepsilon.$$

We take $\delta > 0$. Then for any $\nu > 0$, $a \geq 1$, there exists a positive integer a_s such that

$$\mathcal{M}\left\{\bigcup_{n=a_s}^{\infty} \left\{\left\|\frac{1}{p}\sum_{x=0}^{p-1} \zeta_{n+x} - \zeta\right\| \geq \frac{1}{a}\right\}\right\} < \frac{\nu}{2^a}.$$

Consider $B = \bigcup_{a=1}^{\infty} \bigcup_{n=a_s}^{\infty} \left\{ \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x} - \zeta \right\| \geq \frac{1}{a} \right\}$. Then

$$\mathcal{M}\{B\} \leq \sum_{a=1}^{\infty} \mathcal{M} \left\{ \bigcup_{n=a_s}^{\infty} \left\{ \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x} - \zeta \right\| \geq \frac{1}{a} \right\} \right\} \leq \sum_{a=1}^{\infty} \frac{\nu}{2^a} = \nu.$$

Moreover, $\sup_{\gamma \in \Gamma - B} \frac{1}{p} \sum_{x=0}^{p-1} ||\zeta_{n+x}(\gamma) - \zeta(\gamma)|| < \frac{1}{n}$, where $m = 1, 2, 3, \dots,$ and $n > a_s$. Therefore, the result is established. □

Theorem 2.5. *Let the sequence (ζ_n) be almost convergent in uniformly almost surely to ζ. Then, the sequence (ζ_n) is almost convergent in almost surely to ζ.*

Proof. Taking Theorem 2.3 into consideration, we have if the complex uncertain sequence (ζ_n) almost converges to ζ in uniformly almost surely, then

$$\mathcal{M} \left\{ \bigcup_{n=N}^{\infty} \left\{ \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x} - \zeta \right\| \geq \delta \right\} \right\} < \varepsilon.$$

Now, since

$$\mathcal{M} \left\{ \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x} - \zeta \right\| \geq \delta \right\} \right\} \leq \mathcal{M} \left\{ \bigcup_{n=N}^{\infty} \left\{ \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x} - \zeta \right\| \geq \delta \right\} \right\} < \varepsilon,$$

hence, (ζ_n) almost converges in almost surely to ζ. □

Theorem 2.6. *A complex uncertain sequence (ζ_n), which almost converges with respect to uniformly almost surely to ζ is also almost convergent in measure therein.*

Proof. Let (ζ_n) be almost convergent in uniformly almost surely to ζ. Then, for $\varepsilon > 0$ and $\delta > 0$ there exists $n_0 \in \mathbb{N}$ so that

$$\mathcal{M} \left\{ \bigcup_{n=n_0}^{\infty} \left\{ \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x} - \zeta \right\| \geq \delta \right\} \right\} < \varepsilon, \quad \text{for all } n \geq n_0.$$

Then

$$\begin{aligned} &\mathcal{M} \left\{ \gamma : \left\| \frac{1}{pq} \sum_{x=0}^{p-1} \sum_{y=0}^{q-1} \zeta_{m+x,n+y}(\gamma) - \zeta(\gamma) \right\| \geq \delta \right\} \\ \leq &\mathcal{M} \left\{ \bigcup_{n=n_0}^{\infty} \left\{ \gamma : \left\| \frac{1}{p} \sum_{x=0}^{p-1} \zeta_{n+x}(\gamma) - \zeta(\gamma) \right\| \geq \delta \right\} \right\} < \varepsilon. \end{aligned}$$

Hence, the sequence (ζ_n) almost converges in measure to ζ. □

Theorem 2.7. *Almost convergence in uniformly almost surely of a complex uncertain sequence implies its almost convergence in distribution with preservation of limit.*

Proof. It is straightforward from the Theorem 2.6 and Corollary 2.1. □

Remark 2.5. From the above discussion, a more complete version of interrelationships between different almost convergence in an uncertainty space can be depicted in the Figure 1 given in the top of the following page.

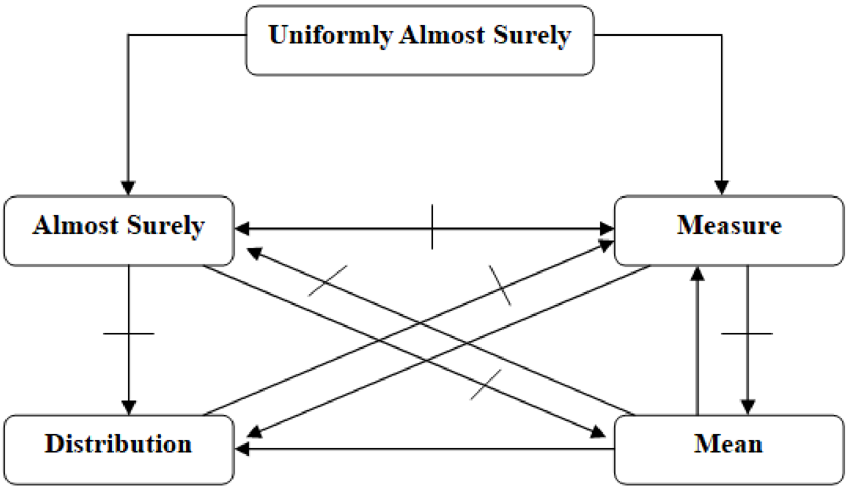

FIGURE 1. Interrelationships among five types of almost convergence

Saha et al. in [10] stated that in a given uncertainty space every convergent complex uncertain sequence is almost convergent to the same limit therein. The statement holds true for all the four aspects (in mean, measure, almost surely, distribution) introduced in [10] and in the fifth direction of uncertainty in uniformly almost surely, also. In this context, we give the detailed proof of the same below.

Theorem 2.8. *A convergent complex uncertain sequence which converges in uniformly almost surely to a finite limit, is also almost convergent to the same limit therein.*

Proof. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and (ζ_n) be a complex uncertain sequence which converges to ζ in uniformly almost surely. That means for any given $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and a sequence (E_x) of uncertain events with uncertain measure of each of the events tending to zero such that

$$||\zeta_n(\gamma) - \zeta(\gamma)|| < \varepsilon, \quad \text{for all } n \geq n_0.$$

Now, for every positive integer p, $n \geq n_0$, $\gamma \in \Gamma - E_x$ and any $\varepsilon > 0$, we have

$$\begin{aligned}&\left|\left|\frac{1}{p}\sum_{k=0}^{p-1}\zeta_{n+k}(\gamma) - \zeta(\gamma)\right|\right| \\ &= \left|\left|\frac{\zeta_n(\gamma) + \zeta_{n+1}(\gamma) + \cdots + \zeta_{n+p-1}(\gamma)}{p} - \zeta(\gamma)\right|\right|\end{aligned}$$

$$= \left\| \frac{\zeta_n(\gamma) + \zeta_{n+1}(\gamma) + \cdots + \zeta_{n+p-1}(\gamma) - p\zeta(\gamma)}{p} \right\|$$
$$= \left\| \frac{\{\zeta_n(\gamma) - \zeta(\gamma)\} + \{\zeta_{n+1}(\gamma) - \zeta(\gamma)\} + \cdots + \{\zeta_{n+p-1}(\gamma) - \zeta(\gamma)\}}{p} \right\|$$
$$\leq \left\{ \frac{||\zeta_n(\gamma) - \zeta(\gamma)||}{p} + \frac{||\zeta_{n+1}(\gamma) - \zeta(\gamma)||}{p} + \cdots + \frac{||\zeta_{n+p-1}(\gamma) - \zeta(\gamma)||}{p} \right\}$$
$$< \frac{\varepsilon}{p} + \frac{\varepsilon}{p} + \cdots + \frac{\varepsilon}{p} = \frac{p\varepsilon}{p} = \varepsilon,$$

uniformly for all n.

Since ε is chosen arbitrary, the obvious conclusion is that

$$\lim_{p\to\infty} \left\| \frac{1}{p} \sum_{k=0}^{p-1} \zeta_{n+k}(\gamma) - \zeta(\gamma) \right\| = 0.$$

Hence, (ζ_n) is an almost convergent complex uncertain sequence in uniformly almost surely to ζ. □

Remark 2.6. In the above theorem if we replace the sub-collection $\Gamma - E_x$, by Λ, which is a subset of Γ with $\mathcal{M}\{\Lambda\} = 1$, then we can easily prove that every convergent almost surely complex uncertain sequence is almost convergent in the same direction.

Theorem 2.9. *A convergent complex uncertain sequence in mean almost converges in the same aspect. Also, limits of the both cases are identical.*

Proof. Suppose (ζ_n) converges to ζ in mean. Then $\lim_{n\to\infty} E[||\zeta_n - \zeta||] = 0$. This implies, for a preassigned $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$E[||\zeta_n - \zeta||] < \varepsilon, \quad \text{for all } n \geq n_0. \tag{2.1}$$

Suppose $p \in \mathbb{N}$ be given. Then

$$\left\| \frac{1}{p} \sum_{k=0}^{p-1} \zeta_{n+k} - \zeta \right\| = \left\| \frac{\zeta_n + \zeta_{n+1} + \cdots + \zeta_{n+p-1}}{p} - \zeta \right\|$$
$$= \left\| \frac{\zeta_n + \zeta_{n+1} + \cdots + \zeta_{n+p-1} - p\zeta}{p} \right\|$$
$$= \left\| \frac{\{\zeta_n - \zeta\} + \{\zeta_{n+1} - \zeta\} + \cdots + \{\zeta_{n+p-1} - \zeta\}}{p} \right\|$$
$$\leq \left\| \frac{\zeta_n - \zeta}{p} \right\| + \left\| \frac{\zeta_{n+1} - \zeta}{p} \right\| + \cdots + \left\| \frac{\zeta_{n+p-1} - \zeta}{p} \right\|.$$

Applying the expected value operator to both sides, we get for any $n \geq n_0$

$$E\left[\left\| \frac{\{\zeta_n - \zeta\} + \{\zeta_{n+1} - \zeta\} + \cdots + \{\zeta_{n+p-1} - \zeta\}}{p} \right\| \right]$$

$$\leq \frac{1}{p} E\left[||\zeta_n - \zeta|| + ||\zeta_{n+1} - \zeta|| + \cdots + ||\zeta_{n+p-1} - \zeta||\right]$$
$$= \frac{1}{p}\{E[||\zeta_n - \zeta||] + E[||\zeta_{n+1} - \zeta||] + \cdots + E[||\zeta_{n+p-1} - \zeta||]\}$$
$$< \frac{1}{p}(\varepsilon + \varepsilon + \cdots + \varepsilon) = \frac{p\varepsilon}{p} = \varepsilon.$$

Consequently, (ζ_n) is an almost convergent complex uncertain sequence in mean to ζ. □

Remark 2.7. Using the complex uncertainty distribution operator, instead of expected value operator in the above Theorem 2.9, one can verify that convergence in distribution of a complex uncertain sequence implies its almost convergence.

Theorem 2.10. *For a complex uncertain sequence*

convergence in measure ⇒ almost convergence in measure.

Proof. Let (ζ_n) converges to ζ in measure. Then for any given $\varepsilon > 0$

$$\lim_{n\to\infty} \mathcal{M}\{||\zeta_n - \zeta|| > \varepsilon\} = 0.$$

Then for any $p \in \mathbb{N}$

$$\mathcal{M}\left\{\left\|\frac{1}{p}\sum_{k=0}^{p-1} \zeta_{n+k} - \zeta\right\| > \varepsilon\right\}$$
$$= \mathcal{M}\left\{\left\|\frac{\zeta_n + \zeta_{n+1} + \cdots + \zeta_{n+p-1}}{p} - \zeta\right\| > \varepsilon\right\}$$
$$= \mathcal{M}\left\{\left\|\frac{\zeta_n + \zeta_{n+1} + \cdots + \zeta_{n+p-1} - p\zeta}{p}\right\| > \varepsilon\right\}$$
$$= \mathcal{M}\left\{\left\|\frac{\{\zeta_n - \zeta\} + \{\zeta_{n+1} - \zeta\} + \cdots + \{\zeta_{n+p-1} - \zeta\}}{p}\right\| > \varepsilon\right\}$$
$$\leq \mathcal{M}\left\{\left\|\frac{\zeta_n - \zeta}{p}\right\| > \varepsilon'\right\} + \mathcal{M}\left\{\left\|\frac{\zeta_{n+1} - \zeta}{p}\right\| > \varepsilon'\right\} + \cdots + \mathcal{M}\left\{\left\|\frac{\zeta_{n+p-1} - \zeta}{p}\right\| > \varepsilon'\right\}$$
$$= \mathcal{M}\{||\zeta_n - \zeta|| > p\varepsilon'\} + \mathcal{M}\{||\zeta_{n+1} - \zeta|| > p\varepsilon'\} + \cdots + \mathcal{M}\{\|\zeta_{n+p-1} - \zeta|| > p\varepsilon'\},$$

for some $\varepsilon' < \frac{\varepsilon}{p}$. Taking limiting case of $n \geq n_0$ to infinity, we get

$$\lim_{n\to\infty} \mathcal{M}\left\{\left\|\frac{1}{p}\sum_{k=0}^{p-1} \zeta_{n+k} - \zeta\right\| > \varepsilon\right\} = 0.$$

Consequently, (ζ_n) is an almost convergent complex uncertain sequence in measure to ζ. □

Theorem 2.11. *If a complex uncertain sequence (ζ_n) is almost convergent in mean then (ζ_n) is bounded in mean also.*

Proof. Since (ζ_n) converges to ζ in mean for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$E\left[\left|\left|\frac{1}{p}\sum_{x=0}^{p-1}\zeta_{n+x} - \zeta\right|\right|\right] < \varepsilon, \quad \text{for all } p > n_0 \text{ and uniformly for all } n \in \mathbb{N}$$

$$\Rightarrow E\left[\left|\left|\frac{1}{p}\sum_{k=n}^{n+p-1}\zeta_k - \zeta\right|\right|\right] < \varepsilon, \quad \text{for all } p > n_0 \text{ and uniformly for all } n \in \mathbb{N}.$$

This holds valid for $p = p+1$ and so $E\left[\left|\left|\sum_{k=n}^{n+p-1}\zeta_k\right|\right|\right]$ is finite. Thus, there exists a finite real number M such that

$$E\left[\left|\left|\sum_{k=n}^{n+p-1}\zeta_k\right|\right|\right] \leq \frac{M}{2}.$$

The above inequality can be established if we take $p = p+1$ and $q = q+1$. Now,

$$E[||\zeta_n||] = E\left[\left|\left|\sum_{k=n}^{n+p}\zeta_k - \sum_{k=n+1}^{n+p}\zeta_k\right|\right|\right] \leq E\left[\left|\left|\sum_{k=n}^{n+p}\zeta_k\right|\right|\right] + E\left[\left|\left|\sum_{k=n+1}^{n+p}\zeta_k\right|\right|\right] \leq \frac{M}{2} + \frac{M}{2} = M.$$

Therefore, $\sup_n E[||\zeta_n||] \leq M$ and hence the complex uncertain sequence is bounded in mean. □

Remark 2.8. The above theorem holds good for the remaining cases of uncertainty. That is, almost convergence of complex uncertain sequences implies its boundedness in measure, distribution, almost surely and uniformly almost surely too.

References

[1] B. Das, B. C. Tripathy, P. Debnath and B. Bhattacharya, *Almost convergence of complex uncertain double sequences*, Filomat **35**(1) (2021), 61–78. https://doi.org/10.2298/FIL2101061D

[2] B. Das, B. C. Tripathy, P. Debnath and B. Bhattacharya, *Characterization of statistical convergence of complex uncertain double sequence*, Anal. Math. Phys. **10**(4) (2020), 1–20. https://doi.org/10.1007/s13324-020-00419-7

[3] B. Das, B. C. Tripathy, P. Debnath and B. Bhattacharya, *Statistical convergence of complex uncertain triple sequence*, Comm. Statist. Theory Methods (2021), 1–13. https://doi.org/10.1080/03610926.2020.1871016

[4] B. Das, B. C. Tripathy, P. Debnath and B. Bhattacharya, *Study of matrix transformation of uniformly almost surely convergent complex uncertain sequences*, Filomat **34**(14) (2020), 4907–4922. https://doi.org/10.2298/FIL2014907D

[5] B. Das, B. C. Tripathy, P. Debnath, J. Nath and B. Bhattacharya, *Almost convergence of complex uncertain triple sequences*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences **91**(2) (2021), 245–256. https://doi.org/10.1007/s40010-020-00721-w

[6] B. Das, P. Debnath and B. C. Tripathy, *Characterization of matrix classes transforming between almost sure convergent sequences of complex uncertain variables*, Journal of Uncertain Systems **14**(3) (2021), 1–12. https://doi.org/10.1142/S1752890921500197

[7] B. Liu, *Uncertainty Theory*, 2nd Edition, Springer-Verlag, Berlin, 2007.

[8] C. You, *On the convergence of uncertain sequences*, Math. Comput. Model. **49** (2009), 482–487. https://doi.org/10.1016/j.mcm.2008.07.007

[9] D. Datta and B. C. Tripathy, *Convergence of complex uncertain double sequences*, New Mathematics and Natural Computation **16**(3) (2020), 447–459. https://doi.org/10.1142/S1793005720500271

[10] S. Saha, B. C. Tripathy and S. Roy, *On almost convergent of complex uncertain sequences*, New Mathematics and Natural Computation **16**(3) (2020), 573–580. https://doi.org/10.1142/S1793005720500349

[11] X. Chen, Y. Ning and X. Wang, *Convergence of complex uncertain sequences*, Journal of Intelligent & Fuzzy Systems **30** (2016), 3357–3366. https://doi.org/10.3233/IFS-152083

[12] Z. Peng, *Complex uncertain variable*, PhD Thesis, Tsinghua University, 2012.

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