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WEAVING g-FRAMES FOR OPERATORS

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Abstract. Bemrose et al. introduced weaving frames and later, Deepshikha et al. generalized them to weaving K-frames. In this note, as a generalization of these notions, we introduce approximate K-duals and investigate the properties of K-g-frames and weaving K-g-frames. We show that woven K-g-frames and weakly woven K-g-frames coincide. We also study perturbation and erasure of woven K-g-frames and we show that they are stable under small perturbations. Also we generalize some of the known results in frame theory to K-g-frames and weaving K-g-frames.

1. Introduction and Preliminaries

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [7] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [5] and popularized from then on. Frames are generalizations of bases in Hilbert spaces. A frame as well as an orthonormal basis allows that each element in the underlying Hilbert space to be written as an unconditionally convergent series in linear combinations of the frame elements; however, in contrast to the situation for a basis, the coefficients might not be unique. Frames are very useful in characterization of function spaces and other fields of applications such as filter bank theory, sigma-delta quantization, signal and image processing and wireless communications.

Sun in [14] introduced g-frames as another generalization of frames. He showed that frames, oblique frames, pseudo frames and fusion frames are special cases of g-frames see also [9] and [10]. Weaving frames were introduced in [1] and investigated in [2,3,12]. In [13] we have generalized weaving frames to the Banach spaces. This concept

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is motivated on distributed signal processing, see [1]. A potentional application of weaving frames together is dealing with wireless sensor networks which may be subjected to distributed processing under different frames. The theory can be used in the processing of signals using Gabor frames.

Frames for operators, which are also called K-frames are more general than ordinary frames, where K is a bounded linear operator in a separable Hilbert space H. K-frames were introduced by Găvruta [8] and investigated in [15]. Because of the higher generality of K-frames, many properties for ordinary frames may not hold for K-frames (for example, the corresponding synthesis operator for K-frames is not surjective). Deepshikha et .al in [6] generalized weaving frames to weaving K-frames.

Throughout this paper H denotes a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathfrak{I}$ is a finite or countable subset of $\mathbb{Z}$ and $\{H_i : i \in \mathfrak{I}\}$ is a sequence of separable Hilbert spaces. Also, for every $i \in \mathfrak{I}$, $L(H, H_i)$ is the set of all bounded linear operators from H to H_i, and $L(H, H)$ is denoted by $L(H)$. Also, $GL(H) = \{T \in L(H) : T$ is invertible $\}$. Also throughout this paper we let $K \in L(H)$, with closed range.

A family $\{\varphi_i\}_{i\in\mathfrak{I}} \subseteq H$ is a *frame* for H, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in\mathfrak{I}} |\langle f, \varphi_i\rangle|^2 \leq B\|f\|^2,$$

for each $f \in H$. A family $\{\varphi_i\}_{i\in\mathfrak{I}} \subseteq H$ is a *K-frame* for H, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|K^*f\|^2 \leq \sum_{i\in\mathfrak{I}} |\langle f, \varphi_i\rangle|^2 \leq B\|f\|^2,$$

for each $f \in H$. A sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ is called a *g-frame* for H with respect to $\{H_i : i \in \mathfrak{I}\}$ if there exist $0 < A \leq B < \infty$ such that for every $f \in H$

$$A\|f\|^2 \leq \sum_{i\in\mathfrak{I}} \|\Lambda_i f\|^2 \leq B\|f\|^2,$$

A, B are called g-frame bounds. In this case we say that $\{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ is an (A, B) g-frame. We call Λ a *tight g-frame* if $A = B$ and a *Parseval g-frame* if $A = B = 1$. If only the right hand side inequality is required, Λ is called a *g-Bessel sequence* see [4].

For every sequence $\{H_i\}_{i\in\mathfrak{I}}$, the space

$$\left(\sum_{i\in\mathfrak{I}} \bigoplus H_i\right)_{\ell^2} = \left\{(f_i)_{i\in\mathfrak{I}} : f_i \in H_i, i \in \mathfrak{I}, \sum_{i\in\mathfrak{I}} \|f_i\|^2 < \infty\right\},$$

with pointwise operations and the following inner product is a Hilbert space

$$\langle (f_i)_{i\in\mathfrak{I}}, (g_i)_{i\in\mathfrak{I}}\rangle = \sum_{i\in\mathfrak{I}} \langle f_i, g_i\rangle.$$

If Λ is a g-Bessel sequence, then *the synthesis operator* for Λ is the linear operator

$$T_\Lambda : \left(\sum_{i\in\mathfrak{I}} \bigoplus H_i\right)_{\ell^2} \mapsto H, \quad T_\Lambda(f_i)_{i\in\mathfrak{I}} = \sum_{i\in\mathfrak{I}} \Lambda_i^* f_i.$$

The adjoint of the synthesis operator is called *the analysis operator* and is defined by

$$T_\Lambda^* : H \mapsto \left(\sum_{i\in\mathfrak{I}} \bigoplus H_i\right)_{\ell^2}, \quad T_\Lambda^* f = (\Lambda_i f)_{i\in\mathfrak{I}}.$$

We call $S_\Lambda = T_\Lambda T_\Lambda^*$ the *g-frame operator* of Λ and $S_\Lambda f = \sum_{i\in\mathfrak{I}} \Lambda_i^*\Lambda_i f$, $f \in H$.

If $\Lambda = (\Lambda_i)_{i\in\mathfrak{I}}$ is a g-frame with lower and upper g-frame bounds A, B, respectively, then the g-frame operator of Λ is a bounded, positive and invertible operator on H and

$$A\langle f, f\rangle \le \langle S_\Lambda f, f\rangle \le B\langle f, f\rangle, \quad f \in H,$$

so

$$A \cdot I \le S_\Lambda \le B \cdot I.$$

Let $K \in L(H)$. A sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ is called a *K-g-frame*, if there exist constants $0 < A \le B < \infty$ such that

$$A\|K^* f\|^2 \le \sum_{i\in\mathfrak{I}} \|\Lambda_i f\|^2 \le B\|f\|^2, \quad f \in H.$$

Remark 1.1. Plainly, every g-frame is a K-g-frame, $K \neq 0$, since

$$\frac{A}{\|K^*\|^2}\|K^* f\|^2 \le A\|f\|^2 \le \sum_{i\in\mathfrak{I}} \|\Lambda_i f\|^2 \le B\|f\|^2.$$

Conversly, if K^* is bounded from below (equivalently if K is surjective), then every K-g-frame is an ordinary g-frame.

Găvruta showed that every K-frame in H is a frame for $R(K)$ and so every element of $R(K)$ can be reconstructed see [8, 15]. We generalize this result to K-g-frames.

Lemma 1.1. *Let $K \in L(H)$ with closed range $R(K)$. Then*

(a) *$K\,|_{R(K^*)}: R(K^*) \to R(K)$ and $K^*\,|_{R(K)}: R(K) \to R(K^*)$ are isomorphisms.*
(b) *If $\{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ is a K-g-frame with g-frame operator S, then $S\,|_{R(K)}: R(K) \to S(R(K))$ is an isomorphism, i.e., $\{\Lambda_i \in L(R(K), H_i) : i \in \mathfrak{I}\}$ is a g-frame.*

Proof. (a) Since $R(K)$ is closed, then $R(K^*)$ is also closed and $(\ker(K))^\perp = R(K^*)$, $(\ker(K^*))^\perp = R(K)$. Hence, $K\,|_{R(K^*)}: R(K^*) \to R(K)$ is a bounded bijective linear map. Now, by Banach isomorphism theorem $K\,|_{R(K^*)}$ is an isomorphism and similarly $K^*\,|_{R(K)}: R(K) \longrightarrow R(K^*)$ is an isomorphism. Therefore, there exist $A, B > 0$ such that for each $y \in R(K)$

$$A\|y\| \le \|K^* y\| \le B\|y\|.$$

(b) Since $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a K-g-frame, there exist $0 < A' < B' < \infty$ such that for each $x \in H$

$$A'\|K^*(x)\|^2 \leq \langle Sx, x\rangle \leq B'\|x\|^2,$$

specially for each $x \in H$, we have

$$A'A^2\|K(x)\|^2 \leq \langle SKx, Kx\rangle \leq \|S(K(x))\| \cdot \|Kx\|,$$

so by (2.1) for each $x \in H$, we have $A'A^2\|K(x)\| \leq \|S(K(x))\|$. Therefore, $S\mid_{R(K)}$ is one-to-one and $S(R(K))$ is closed. Now, again by Banach isomorphism theorem, we have the result. □

A small modification in [14] gains the following result.

Lemma 1.2. *Let for each $i \in \mathcal{I}$, $\{e_{i,j} : j \in \mathcal{I}_i\}$ be an orthonormal basis for H_i. Then $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a K-g-frame if and only if $\{\Lambda_i^*(e_{i,j})\}_{i\in\mathcal{I},j\in\mathcal{I}_i}$ is a K-frame.*

In [15] the authors defined the atomic system for K and by using this idea we introduce the following definition.

Definition 1.1. Let $K \in L(H)$. A sequence $\{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ is called an *atomic g-system* for K, if the following conditions are satisfied:

(a) $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a g-Bessel sequence;
(b) for any $x \in H$, there exists $\mathfrak{g}_x = (g_i)_i \in (\sum_{i\in\mathcal{I}} \bigoplus H_i)_{\ell^2}$ such that $Kx = \sum_{i\in\mathcal{I}} \Lambda_i^*(g_i)$, where $\|\mathfrak{g}_x\| \leq C\|x\|$, C is a positive constant.

We recall some definitions from [12].

Definition 1.2. Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ be two g-frames for H. We call $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_i\}_{i\in\mathcal{I}}$ *woven g-frames* if there exist $0 < A \leq B < \infty$ such that for every $\sigma \subset \mathcal{I}$ and every $f \in H$

$$A\|f\|^2 \leq \sum_{i\in\sigma} \|\Lambda_i f\|^2 + \sum_{i\in\sigma^c} \|\Gamma_i f\|^2 \leq B\|f\|^2.$$

In this case, for convenience we say that $\{\Lambda_i\}_{i\in\mathcal{I}}$, $\{\Gamma_i\}_{i\in\mathcal{I}}$ are an (A, B)-woven g-frame.

Proof of the following lemma is similar to [15, Theorem 3.5] which we reaffirm.

Lemma 1.3. *Let $\{\Lambda_i\}_{i\in\mathcal{I}}$ be a g-Bessel sequence in H. Then $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a K-g-frame for H, if and only if there exists $A > 0$ such that $S \geq AKK^*$, where S is the g-frame operator for $\{\Lambda_i\}_{i\in\mathcal{I}}$.*

Remark 1.2. Since $S^{\frac{1}{2}}S^{\frac{1}{2}} = S \geq AKK^*$, by Douglas theorem, there exists $C \in L(H)$ such that $K = S^{\frac{1}{2}}C$.

Definition 1.3. Let $K \in L(H)$ and $\{\Lambda_i \in L(H, H_i) : i \in \mathcal{I}\}$ and $\{\Gamma_i \in L(H, H_i) : i \in \mathcal{I}\}$ be K-g-frames. We say that $\{\Lambda_i\}_{i\in\mathcal{I}}$, $\{\Gamma_i\}_{i\in\mathcal{I}}$ are *woven K-g-frames* if there exist constants $0 < A \leq B < \infty$ such that for every $\sigma \subset \mathcal{I}$ and every $f \in H$

$$A\|K^*f\|^2 \leq \sum_{i\in\sigma} \|\Lambda_i f\|^2 + \sum_{i\in\sigma^c} \|\Gamma_i f\|^2 \leq B\|f\|^2.$$

In this case we say that $\{\Lambda_i\}_{i\in\mathcal{I}}$, $\{\Gamma_i\}_{i\in\mathcal{I}}$ are (A, B) woven K-g-frames.

Example 1.1. Let H be a Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{N}\}$ and let $\Lambda_n, \Gamma_n, K : H \to H$ be defined by

$$\Lambda_n(x) = \langle x, e_{5n}\rangle e_{5n} + \langle x, e_{5n-1}\rangle e_{5n-1},$$
$$\Gamma_n(x) = \langle x, e_{5n}\rangle e_{5n} + \langle x, e_{5n+1}\rangle e_{5n+1},$$

and $K(x) = \sum_{n\in\mathbb{N}}\langle x, e_{5n}\rangle e_{5n}$ for every $x \in H$.

Then $\{\Gamma_n : n \in \mathbb{N}\}$ and $\{\Lambda_n : n \in \mathbb{N}\}$ are woven K-g-frames.

Since K is the orthogonal projection of H onto M, the closed subspace of H generated by $\{e_{5n} : n \in \mathbb{N}\}$, then $K = K^*$. Now for every $x \in H$ and $\sigma \subseteq I$ we have

$$\begin{aligned}\|K^*(x)\|^2 &= \sum_{n\in\mathbb{N}} |\langle x, e_{5n}\rangle|^2 \le \sum_{n\in\sigma} \|\Lambda_n(x)\|^2 + \sum_{n\in\sigma^c} \|\Gamma_n(x)\|^2 \\ &= \sum_{n\in\sigma} |\langle x, e_{5n}\rangle|^2 + \sum_{n\in\sigma} |\langle x, e_{5n-1}\rangle|^2 + \sum_{n\in\sigma^c} |\langle x, e_{5n}\rangle|^2 + \sum_{n\in\sigma^c} |\langle x, e_{5n+1}\rangle|^2 \\ &\le 3\sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2 = 3\|x\|^2,\end{aligned}$$

and we have the result.

As we have in [12, Remark 3.2] if $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$ are g-Bessel sequences with bounds B and B' and g-frame operators S and S', respectively, then for every $\sigma \subset \mathfrak{I}$, $0 \le S_\sigma \le S \le B \cdot I$ and $0 \le S'_{\sigma^c} \le S' \le B' \cdot I$. Therefore, $0 \le S_\sigma + S'_{\sigma^c} \le (B + B') \cdot I$. Hence, $\{\Lambda_i\}_{i\in\sigma} \bigcup \{\Gamma_i\}_{i\in\sigma^c}$ is a g-Bessel sequence with bound $B + B'$ and g-frame operator $S_\sigma + S'_{\sigma^c}$, where $S_\sigma f = \sum_{i\in\sigma} \Lambda_i^*\Lambda_i f$ and $S'_{\sigma^c} f = \sum_{i\in\sigma^c} \Gamma_i^*\Gamma_i f$.

In this paper we try to generalize some of the known results in K-frames, weaving frames and weaving g-frames to K-g-frames.

2. Weaving K-g-frame

In [1], the authors introduced the concept of weaving frames. In this section we also study weaving K-g-frames.

Definition 2.1. Let $K \in L(H)$. The sequences $\{\Lambda_i\}_{i\in\mathfrak{I}}$, $\{\Gamma_i\}_{i\in\mathfrak{I}}$ are called a *woven atomic g-system* for K, if the following conditions are satisfied:

(a) $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$ are g-Bessel sequences;
(b) there exist positive constants C_1, C_2 such that for any $x \in H$, and any $\sigma \subset \mathfrak{I}$ there exist $\mathfrak{g}_x = (g_i)_i, \mathfrak{g}'_x = (g'_i)_i \in (\sum_{i\in\mathfrak{I}} \bigoplus H_i)_{\ell^2}$ such that $Kx = \sum_{i\in\sigma} \Lambda_i^*(g_i) + \sum_{i\in\sigma^c} \Gamma_i^*(g'_i)$ with $\|\mathfrak{g}_x\| \le C_1\|x\|$ and $\|\mathfrak{g}'_x\| \le C_2\|x\|$.

Theorem 2.1. *Let $\{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ and $\{\Gamma_i \in L(H, H_i) : i \in \mathfrak{I}\}$ be a woven atomic g-system for K. Then $\{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ and $\{\Gamma_i \in L(H, H_i) : i \in \mathfrak{I}\}$ are woven K-g-frames.*

Proof. Let $x \in H$. For every $y \in H$ with $\|y\| = 1$ and every $\sigma \subset \mathfrak{I}$, there exist $(g_i)_i, (g'_i)_i \in (\sum_{i\in\mathfrak{I}} \bigoplus H_i)_{\ell^2}$, such that $Ky = \sum_{i\in\sigma} \Lambda_i^* g_i + \sum_{i\in\sigma^c} \Gamma_i^* g'_i$, then

$$\|K^*x\| = \sup_{\|y\|=1} |\langle K^*x, y\rangle| = \sup_{\|y\|=1} \left|\left\langle x, \sum_{i\in\sigma} \Lambda_i^* g_i + \sum_{i\in\sigma^c} \Gamma_i^* g'_i \right\rangle\right|$$

$$\leq \sup_{\|y\|=1}\left|\left\langle x, \sum_{i\in\sigma}\Lambda_i^* g_i\right\rangle\right| + \sup_{\|y\|=1}\left|\left\langle x, \sum_{i\in\sigma^c}\Gamma_i^* g_i'\right\rangle\right|$$
$$= \sup_{\|y\|=1}\left|\sum_{i\in\sigma}\langle\Lambda_i x, g_i\rangle\right| + \sup_{\|y\|=1}\left|\sum_{i\in\sigma^c}\langle\Gamma_i x, g_i'\rangle\right|$$
$$\leq \sup_{\|y\|=1}\left(\sum_{i\in\sigma}\|\Lambda_i x\|^2\right)^{\frac{1}{2}}\left(\sum_{i\in\sigma}\|g_i\|^2\right)^{\frac{1}{2}} + \sup_{\|y\|=1}\left(\sum_{i\in\sigma^c}\|\Gamma_i x\|^2\right)^{\frac{1}{2}}\left(\sum_{i\in\sigma^c}\|g_i'\|^2\right)^{\frac{1}{2}}$$
$$\leq \sup_{\|y\|=1}\left(\sum_{i\in\sigma}\|\Lambda_i x\|^2 + \sum_{i\in\sigma^c}\|\Gamma_i x\|^2\right)^{\frac{1}{2}}\left[\left(\sum_{i\in\mathcal{I}}\|g_i\|^2\right)^{\frac{1}{2}} + \left(\sum_{i\in\mathcal{I}}\|g_i'\|^2\right)^{\frac{1}{2}}\right]$$
$$\leq (C_1 + C_2)\sup_{\|y\|=1}\|y\|\left(\sum_{i\in\sigma}\|\Lambda_i x\|^2 + \sum_{i\in\sigma^c}\|\Gamma_i x\|^2\right)^{\frac{1}{2}}.$$

Therefore, $\sum_{i\in\sigma}\|\Lambda_i x\|^2 + \sum_{i\in\sigma^c}\|\Gamma_i x\|^2 \geq \frac{1}{(C_1+C_2)^2}\|K^* x\|^2$. □

Definition 2.2. We call $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_i\}_{i\in\mathcal{I}}$ *weakly woven K-g-frames,* if for every $\sigma \subset \mathcal{I}$, $\{\Lambda_i\}_{i\in\sigma}\bigcup\{\Gamma_i\}_{i\in\sigma^c}$ is a K-g-frame.

Lemma 2.1. *Let $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_i\}_{i\in\mathcal{I}}$ be K-g-frames. Suppose that for every $\epsilon > 0$ and every two disjoint finite sets $I_1, J_1 \subset \mathcal{I}$ there exists a subset $\sigma \subset \mathcal{I}\backslash(I_1 \cup J_1)$ such that for $\delta = \mathcal{I}\backslash(I_1 \cup J_1 \cup \sigma)$ the lower K-g-frame bound of $\{\Lambda_i\}_{i\in I_1\cup\sigma}\bigcup\{\Gamma_i\}_{i\in J_1\cup\delta}$ is less than ϵ. Then there exists $\mathcal{Q} \subset \mathcal{I}$ such that $\{\Lambda_i\}_{i\in\mathcal{Q}}\bigcup\{\Gamma_i\}_{i\in\mathcal{I}\backslash\mathcal{Q}}$ is not a K-g-frame, i.e, $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_i\}_{i\in\mathcal{I}}$ are not weakly woven K-g-frames.*

Proof. Let $\epsilon > 0$ and for each $p \in \mathbb{N}$, $A_p = [-p, p] \cap \mathcal{I}$ where $[-p, p] \cap \mathbb{Z} = \{-p, \dots, 0, 1, \dots, p\}$. We prove that there exist an increasing sequence $\{f_n\}_{n=1}^{\infty} \subset \mathbb{N}$, a sequence $\{h_n\}_{n=1}^{\infty} \subset H$ with $\|h_n\| = 1$, and sequences $\{\sigma_n\}, \{\delta_n\}$ of subsets $\mathcal{I}$ with $\sigma_n \subset A_{n-1}^c = \mathcal{I}\backslash A_{n-1}$, $\delta_n = A_{n-1}^c\backslash\sigma_n$, such that $I_n = I_{n-1} \cup (\sigma_n \cap A_n)$, $J_n = J_{n-1} \cup (\delta_n \cap A_n)$ satisfy both

$$\sum_{i\in I_{n-1}\cup\sigma_n}\|\Lambda_i(h_n)\|^2 + \sum_{i\in J_{n-1}\cup\delta_n}\|\Gamma_i(h_n)\|^2 < \frac{\epsilon}{n}\|K^*\|^2,$$
$$\sum_{i\in\mathcal{I},|i|\geq f_n+1}\|\Lambda_i(h_n)\|^2 + \sum_{i\in\mathcal{I},|i|\geq f_n+1}\|\Gamma_i(h_n)\|^2 < \frac{\epsilon}{n}\|K^*\|^2.$$

We proceed by induction. By taking $I_0 = J_0 = \emptyset$, we can choose $\sigma_1 \subset \mathcal{I}$ such that for $\delta_1 = \sigma_1^c = \mathcal{I}\backslash\sigma_1$ the lower K-g-frame bound of $\{\Lambda_i\}_{i\in\sigma_1}\bigcup\{\Gamma_i\}_{i\in\delta_1}$ is less than ϵ. Therefore there is some $h_1 \in H$ with $\|h_1\| = 1$ such that

$$\sum_{i\in\sigma_1}\|\Lambda_i(h_1)\|^2 + \sum_{i\in\delta_1}\|\Gamma_i(h_1)\|^2 < \epsilon\|K^*\|^2.$$

Since

$$\sum_{i\in\mathcal{I}}\|\Lambda_i(h_1)\|^2 + \sum_{i\in\mathcal{I}}\|\Gamma_i(h_1)\|^2 < +\infty,$$

there is $f_1 \in \mathbb{N}$ such that

$$\sum_{i\in\mathfrak{I},|i|\geq f_1+1}\|\Lambda_i(h_1)\|^2+\sum_{i\in\mathfrak{I},|i|\geq f_1+1}\|\Gamma_i(h_1)\|^2<\epsilon\|K^*\|^2.$$

Let σ_i, δ_i, h_i and f_i for $i = 1, 2, \ldots, n-1$ with the above conditions are given. Then $J_{n-1} \cap I_{n-1} = \emptyset$ and $I_{n-1} \cup J_{n-1} = A_{n-1}$. By the hypothesis there is $\sigma_n \subset \mathfrak{I}\backslash A_{n-1}$ with $\delta_n = \mathfrak{I}\backslash(A_{n-1} \cup \sigma_n)$ such that $\{\Lambda_i\}_{i\in I_{n-1}\cup\sigma_n} \bigcup\{\Gamma_i\}_{i\in J_{n-1}\cup\delta_n}$ has lower K-g-frame bound less than ϵ. Hence, there exist $h_n \in H$ with $\|h_n\| = 1$ such that

$$\sum_{i\in I_{n-1}\cup\sigma_n}\|\Lambda_i(h_n)\|^2+\sum_{i\in J_{n-1}\cup\delta_n}\|\Gamma_i(h_n)\|^2<\frac{\epsilon}{n}\|K^*\|^2.$$

Similar to the above argument there is $f_n > f_{n-1}$ such that

$$\sum_{i\in\mathfrak{I},|i|\geq f_n+1}\|\Lambda_i(h_n)\|^2+\sum_{i\in\mathfrak{I},|i|\geq f_n+1}\|\Gamma_i(h_n)\|^2<\frac{\epsilon}{n}\|K^*\|^2.$$

By taking $I_n = I_{n-1} \cup (\sigma_n \cap A_n)$, $J_n = J_{n-1} \cup (\delta_n \cap A_n)$ for each n, $J_n \cap I_n = \emptyset$ and $I_n \cup J_n = A_n$. Therefore,

$$\left(\bigcup_{i=1}^{\infty} I_i\right)\sqcup\left(\bigcup_{j=1}^{\infty} J_j\right)=\mathfrak{I},$$

where $\sqcup$ denotes a disjoint union. For

$$\mathfrak{Q}=\bigcup_{i=1}^{\infty} I_i \quad \text{and} \quad \mathfrak{Q}^c=\bigcup_{j=1}^{\infty} J_j,$$

we have

$$\begin{aligned}\sum_{i\in\mathfrak{Q}}\|\Lambda_i(h_n)\|^2+\sum_{i\in\mathfrak{I}\backslash\mathfrak{Q}}\|\Gamma_i(h_n)\|^2&=\left(\sum_{i\in I_n}\|\Lambda_i(h_n)\|^2+\sum_{j\in J_n}\|\Gamma_i(h_n)\|^2\right)\\&\quad+\left(\sum_{i\in\mathfrak{Q}\cap A_n^c}\|\Lambda_i(h_n)\|^2+\sum_{i\in\mathfrak{Q}^c\cap A_n^c}\|\Gamma_i(h_n)\|^2\right)\\&\leq\left(\sum_{i\in I_{n-1}\cup\sigma_n}\|\Lambda_i(h_n)\|^2+\sum_{i\in J_{n-1}\cup\delta_n}\|\Gamma_i(h_n)\|^2\right)\\&\quad+\left(\sum_{i\in\mathfrak{I},|i|\geq f_n+1}\|\Lambda_i(h_n)\|^2+\sum_{i\in\mathfrak{I},|i|\geq f_n+1}\|\Gamma_i(h_n)\|^2\right)\\&<\frac{\epsilon}{n}\|K^*\|^2+\frac{\epsilon}{n}\|K^*\|^2.\end{aligned}$$

So that the lower K-g-frame bound of $\{\Lambda_i\}_{i\in\mathfrak{Q}} \bigcup\{\Gamma_i\}_{i\in\mathfrak{I}\backslash\mathfrak{Q}}$ is zero. Then, it is not a K-g-frame and the two original K-g-frames are not weakly woven. □

Corollary 2.1. *Let $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$ be K-g-frames. If they are weakly woven, then there exist $A > 0$ and finite disjoint subsets $J, Q \subset \mathfrak{I}$ such that for each $\sigma \subset \mathfrak{I} \setminus (J \cup Q)$*

and $\delta = \mathfrak{I}\backslash(J \cup Q \cup \sigma)$ *the sequence* $\{\Lambda_i\}_{i\in J\cup\sigma} \bigcup \{\Gamma_i\}_{i\in Q\cup\delta}$ *has lower* K*-*g*-frame bound* A*.*

In the proof of [2, Theorem 4.5], Casazza et al. dealt with frames, but their proof also works for K-g-frames and by a modification in their proof, we can get the following results.

Theorem 2.2. *Let* $\{\Lambda_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma_i\}_{i\in\mathfrak{I}}$ *be* K*-*g*-frames. Then the following are equivalent:*

(a) $\{\Lambda_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma_i\}_{i\in\mathfrak{I}}$ *are woven* K*-*g*-frames;*
(b) $\{\Lambda_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma_i\}_{i\in\mathfrak{I}}$ *are weakly woven* K*-*g*-frames.*

Definition 2.3. Let $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$ be g-Bessel sequences, with bounds B, B', respectively. Then the *operator* $S_{\Gamma,\Lambda} : H \to H$ defined by

$$S_{\Gamma,\Lambda}(f) = T_\Gamma T_\Lambda^*(f) = \sum_{i\in\mathfrak{I}} \Gamma_i^* \Lambda_i(f), \quad f \in H,$$

is a bounded linear operator with $\|S_{\Gamma,\Lambda}\| \le \sqrt{BB'}$. Also, $S_{\Gamma,\Lambda}^* = S_{\Lambda,\Gamma}$ and $S_{\Gamma,\Gamma} = S_\Gamma$, see [11].

The proof of [11, Lemma 2.11] also works for K-g-frames and we have the following result.

Lemma 2.2. *Let* $\{\Lambda_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma_i\}_{i\in\mathfrak{I}}$ *be* g*-Bessel sequences. If there exists* $\lambda > 0$ *such that* $\|S_{\Lambda,\Gamma}(f)\| \ge \lambda\|K^*f\|$*, then* $\{\Lambda_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma_i\}_{i\in\mathfrak{I}}$ *are* K*-*g*-frames.*

Example 2.1. Let H be a Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{N}\}$ and let Λ_n, Γ_n and $K : H \to H$ be defined by $\Lambda_n(x) = \langle x, e_{2n}\rangle e_{2n}$, $\Gamma_n(x) = \langle x, e_{2n}\rangle e_{2n} + \langle x, e_{2n+1}\rangle e_{2n+1}$, $K(x) = \sum_{n\in\mathbb{N}} \langle x, e_{2n}\rangle e_{2n}$, for every $x \in H$. Then $\{\Lambda_n : n \in \mathbb{N}\}$ and $\{\Gamma_n : n \in \mathbb{N}\}$ are woven K-g-frames for H with universal bounds 1 and 3. The reason is similar to Example 1.1.

Proposition 2.1. *Let* $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$*,* $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathfrak{I}\}$*,* $\Lambda' = \{\Lambda'_i \in L(H, H'_i) : i \in \mathfrak{I}\}$ *and* $\Gamma' = \{\Gamma'_i \in L(H, H'_i) : i \in \mathfrak{I}\}$ *be* g*-Bessel sequences with bounds* D_1*,* D_2*,* D_3*,* D_4*, respectively. If there exists* $\lambda > 0$ *such that* $\|(S^\sigma_{\Lambda,\Lambda'} + S^{\sigma^c}_{\Gamma,\Gamma'})f\| \ge \lambda\|K^*f\|$ *for each* $\sigma \subset \mathfrak{I}$ *and* $f \in H$*, then* $\{\Lambda'_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ *are woven* K*-*g*-frames and also* $\{\Lambda_i\}_{i\in\mathfrak{I}}$*,* $\{\Gamma_i\}_{i\in\mathfrak{I}}$ *are woven* K*-*g*-frames.*

Proof. As we saw before, they are woven g-Bessel sequences. Suppose that $\lambda > 0$ such that for all $\sigma \subset \mathfrak{I}$ and $f \in H$

$$\lambda\|K^*f\| \le \|(S^\sigma_{\Lambda,\Lambda'} + S^{\sigma^c}_{\Gamma,\Gamma'})f\|,$$

then,

$$\begin{aligned}\|(S^\sigma_{\Lambda,\Lambda'} + S^{\sigma^c}_{\Gamma,\Gamma'})f\| &\le \|S^\sigma_{\Lambda,\Lambda'}f\| + \|S^{\sigma^c}_{\Gamma,\Gamma'}f\| = \|(T_\Lambda T^*_{\Lambda'})^\sigma(f)\| + \|(T_\Gamma T^*_{\Gamma'})^{\sigma^c}(f)\| \\ &\le \|T_\Lambda\| \left(\sum_{i\in\sigma} \|\Lambda'_i f\|^2\right)^{\frac{1}{2}} + \|T_\Gamma\| \left(\sum_{i\in\sigma^c} \|\Gamma'_i f\|^2\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq \sqrt{D_1}\left(\sum_{i\in\sigma}\|\Lambda'_i f\|^2\right)^{\frac{1}{2}} + \sqrt{D_2}\left(\sum_{i\in\sigma^c}\|\Gamma'_i f\|^2\right)^{\frac{1}{2}}$$
$$\leq \left(\sqrt{D_1}+\sqrt{D_2}\right)\left(\sum_{i\in\sigma}\|\Lambda'_i f\|^2 + \sum_{i\in\sigma^c}\|\Gamma'_i f\|^2\right)^{\frac{1}{2}}.$$

Hence,

$$\sum_{i\in\sigma}\|\Lambda'_i f\|^2 + \sum_{i\in\sigma^c}\|\Gamma'_i f\|^2 \geq \frac{\lambda^2\|K^* f\|^2}{(\sqrt{D_1}+\sqrt{D_2})^2}.$$

On the other hand, since $S^*_{\Gamma,\Lambda} = S_{\Lambda,\Gamma}$, then $(S^{\sigma}_{\Lambda,\Lambda'} + S^{\sigma^c}_{\Gamma,\Gamma'})^* = S^{\sigma}_{\Lambda',\Lambda} + S^{\sigma^c}_{\Gamma',\Gamma}$ and we have the result. □

Theorem 2.3. *Let* $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ *and* $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathfrak{I}\}$ *be* (A, B) *woven* K*-*g*-frames and* $\Lambda' = \{\Lambda'_i \in L(H', H'_i) : i \in \mathfrak{I}\}$ *and* $\Gamma' = \{\Gamma'_i \in L(H', H'_i) : i \in \mathfrak{I}\}$ *be* (A', B') *woven* K*-*g*-frames.*

(i) *Then* $\{\Lambda_i \oplus \Lambda'_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma_i \oplus \Gamma'_i\}_{i\in\mathfrak{I}}$ *are* $(\min\{A, A'\}, \max\{B, B'\})$ *woven* K*-*g*-frames.*

(ii) *If* $H = H', H_i = H'_i$ *for each* $i \in \mathfrak{I}$*, and for every* $\sigma \subset \mathfrak{I}$
$$S^{\sigma}_{\Lambda,\Lambda'} + S^{\sigma}_{\Lambda',\Lambda} + S^{\sigma^c}_{\Gamma,\Gamma'} + S^{\sigma^c}_{\Gamma',\Gamma} \geq 0,$$
then $\{\Lambda_i + \Lambda'_i\}_{i\in\mathfrak{I}}$ *and* $\{\Gamma_i + \Gamma'_i\}_{i\in\mathfrak{I}}$ *are woven* K*-*g*-frames, where* $S^{\sigma^c}_{\Gamma,\Gamma'} = \sum_{i\in\sigma^c}\Gamma^*_i\Gamma'_i$.

Proof. (i) With a proof similar to the proof of [11, Proposition 2.16], $\{\Lambda_i \oplus \Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i \oplus \Gamma'_i\}_{i\in\mathfrak{I}}$ are K-g-frames. For every $\sigma \subset \mathfrak{I}$ and every $(f, g) \in H \oplus H'$

$$\begin{aligned}
&\sum_{i\in\sigma}\|(\Lambda_i \oplus \Lambda'_i)(f,g)\|^2 + \sum_{i\in\sigma^c}\|(\Gamma_i \oplus \Gamma'_i)(f,g)\|^2 \\
=&\sum_{i\in\sigma}\|(\Lambda_i f, \Lambda'_i g)\|^2 + \sum_{i\in\sigma^c}\|(\Gamma_i f, \Gamma'_i g)\|^2 \\
=&\sum_{i\in\sigma}\langle(\Lambda_i f, \Lambda'_i g), (\Lambda_i f, \Lambda'_i g)\rangle + \sum_{i\in\sigma^c}\langle(\Gamma_i f, \Gamma'_i g), (\Gamma_i f, \Gamma'_i g)\rangle \\
=&\sum_{i\in\sigma}(\|\Lambda_i f\|^2 + \|\Lambda'_i g\|^2) + \sum_{i\in\sigma^c}(\|\Gamma_i f\|^2 + \|\Gamma'_i g\|^2) \\
\leq& B\|f\|^2 + B'\|g\|^2 \leq \max\{B, B'\}\|(f,g)\|^2,
\end{aligned}$$

similarly for the lower bound.

(ii) It is clear that $S^{\sigma}_{\Lambda,\Lambda'} + S^{\sigma}_{\Lambda',\Lambda} + S^{\sigma^c}_{\Gamma,\Gamma'} + S^{\sigma^c}_{\Gamma',\Gamma}$ is a self-adjoint operator. For every $\sigma \subset \mathfrak{I}$ we have

$$\begin{aligned}
S^{\sigma}_{\Lambda+\Lambda'} + S^{\sigma^c}_{\Gamma+\Gamma'} =& \sum_{i\in\sigma}(\Lambda_i + \Lambda'_i)^*(\Lambda_i + \Lambda'_i) + \sum_{i\in\sigma^c}(\Gamma_i + \Gamma'_i)^*(\Gamma_i + \Gamma'_i) \\
=& \sum_{i\in\sigma}{\Lambda_i}^*\Lambda_i + \sum_{i\in\sigma}{\Lambda'_i}^*\Lambda'_i + \sum_{i\in\sigma^c}{\Gamma_i}^*\Gamma_i + \sum_{i\in\sigma^c}{\Gamma'_i}^*\Gamma'_i \\
&+ \sum_{i\in\sigma}({\Lambda_i}^*\Lambda'_i + {\Lambda'_i}^*\Lambda_i) + \sum_{i\in\sigma^c}({\Gamma_i}^*\Gamma'_i + {\Gamma'_i}^*\Gamma_i)
\end{aligned}$$

$$
\begin{aligned}
&= S_\Lambda^\sigma + S_\Gamma^{\sigma^c} + S_{\Lambda'}^\sigma + S_{\Gamma'}^{\sigma^c} + S_{\Lambda,\Lambda'}^\sigma + S_{\Lambda',\Lambda}^\sigma + S_{\Gamma,\Gamma'}^{\sigma^c} + S_{\Gamma',\Gamma}^{\sigma^c} \\
&\geq AKK^* + A'KK^* = (A+A')KK^*.
\end{aligned}
$$

Also, plainly $\{\Lambda_i + \Lambda'_i\}_{i\in\sigma} \cup \{\Gamma_i + \Gamma'_i\}_{i\in\sigma^c}$ is a g-Bessel sequence. □

Definition 2.4. Let $\Lambda = \{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\Gamma = \{\Gamma_i\}_{i\in\mathfrak{I}}$ be g-Bessel sequences. Then

(a) Γ is a K-dual of Λ if for each $f \in H$, we have $Kf = S_{\Gamma,\Lambda}(f) = \sum_{i\in\mathfrak{I}} \Gamma_i^*\Lambda_i(f)$;

(b) Γ is an approximate K-dual of Λ if there exists $0 < r < 1$ such that for every $f \in H$,

$$\|K(f) - S_{\Gamma,\Lambda}(f)\| \leq r\|K(f)\|.$$

Plainly, every K-dual is an approximate K-dual, and for the converse we have the following result.

Proposition 2.2. *Let $\Gamma = \{\Gamma_i\}_{i\in\mathfrak{I}}$ be an approximate K-dual of Λ. Then Λ has a K-dual and every element $K(f)$ of $R(K)$ can be reconstructed from $\{\Gamma_i^* \circ \Lambda_i(f)\}_{i\in\mathfrak{I}}$.*

Proof. Since Γ is an approximate K-dual of Λ, there exists $0 < r < 1$ such that

$$\|K(f) - S_{\Gamma,\Lambda}(f)\| \leq r\|K(f)\|, \quad f \in H. \tag{2.1}$$

Now, from (2.1) it follows that $S_{\Gamma,\Lambda}(f) = 0$ if and only if $K(f) = 0$. Therefore, we can define $U : R(K) \to R(S_{\Gamma,\Lambda})$ by $U(K(f)) = S_{\Gamma,\Lambda}(f)$ for every $f \in H$. Hence U is an injective bounded linear map and by using (2.1) we have

$$\|Kf - U(Kf)\| \leq r\|Kf\|, \quad f \in H. \tag{2.2}$$

So, for every $f \in H$

$$(1-r)\|Kf\| \leq \|U(Kf)\| \leq (1+r)\|Kf\|.$$

Hence, U has a closed range, $R(U) = R(S_{\Gamma,\Lambda})$. Now by Banach isomorphism theorem $U^{-1} : R(S_{\Gamma,\Lambda}) \to R(K)$ is a bounded linear map, which can be extended to $V : H \to H$, by $V = U^{-1} \circ \pi_{R(K)}$, where $\pi_{R(K)}$ is the orthogonal projection of H onto $R(U)$. It is clear that

$$K(f) = V \circ S_{\Gamma,\Lambda}(f) = \sum_{i\in\mathfrak{I}} (V \circ \Gamma_i^*) \circ \Lambda_i(f), \quad f \in H.$$

Therefore, $\{\Gamma_i \circ V^*\}_{i\in\mathfrak{I}}$ is a K-dual of $\{\Lambda_i\}_{i\in\mathfrak{I}}$. □

Remark 2.1. If in the above Proposision $R(S_{\Gamma,\Lambda}) \subseteq R(K)$, then we can regard $U : R(K) \to R(K)$ and from (2.2) it follows that

$$\|g - U(g)\| \leq r\|g\|, \quad g \in R(K).$$

Then $\|I_{R(K)} - U\| \leq r < 1$ and consequently U is invertible and the above inequality is similar to the inequality for approximate K-dual.

A small modification in the proofs of [3, Proposition 15] and [12, Theorem 3.14] shows that these properties hold for K-g-frames.

3. Perturbation

In this section we study the behaviour of K-g-frames under some perturbations.

The following result shows that approximate K-duals are stable under small perturbation.

Theorem 3.1. *Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ be a g-Bessel sequence and $\Psi = \{\psi_i \in L(H, H_i) : i \in \mathfrak{I}\}$ be an approximate K-dual (resp. K-dual) of Λ with $0 < r < 1$ and upper bound C. If $\Gamma = \{\Gamma_i \in L(H, H_i) : i \in \mathfrak{I}\}$ is a sequence such that*

$$\left(\sum_{i\in\mathfrak{I}} \| (\Lambda_i - \Gamma_i)(f) \|^2\right)^{\frac{1}{2}} \leq F\|K(f)\|, \quad f \in H,$$

and $\sqrt{CF} < 1 - r$ (resp. $CF < 1$), then Ψ is an approximate K-dual of Γ.

Proof. Let B be an upper bound for Λ. Then for any $f \in H$, we have

$$\left(\sum_{i\in\mathfrak{I}} \|\Gamma_i f\|^2\right)^{\frac{1}{2}} \leq \|\{\Lambda_i f\}_{i\in\mathfrak{I}}\|_2 + \|\{\Gamma_i f - \Lambda_i f\}_{i\in\mathfrak{I}}\|_2 \leq (\sqrt{B} + \sqrt{F}\|K\|)\|f\|,$$

so, Γ is a g-Bessel sequence. For any $f \in H$,

$$\|S_{\Psi,\Lambda} f - S_{\Psi,\Gamma} f\| \leq \sup_{\|g\|=1} \left\{ \left(\sum_{i\in\mathfrak{I}} \|(\Lambda_i - \Gamma_i) f\|^2\right)^{\frac{1}{2}} \left(\sum_{i\in\mathfrak{I}} \|\psi_i g\|^2\right)^{\frac{1}{2}} \right\} \leq \sqrt{CF}\|Kf\|.$$

Hence, for every $f \in H$

$$\|Kf - S_{\Psi,\Gamma} f\| \leq \|Kf - S_{\Psi,\Lambda} f\| + \|S_{\Psi,\Lambda} f - S_{\Psi,\Gamma} f\| \leq (r + \sqrt{CF})\|Kf\|.$$

Since $r + \sqrt{CF} < 1$ we have the result. If Ψ is a K-dual of Λ, then $S_{\Psi,\Lambda} f = Kf$ and we have $\|K - S_{\Psi,\Lambda}\| \leq \sqrt{CF} < 1$. ◻

Theorem 3.2. *Let $\{\Lambda_i \in L(H, H_i) : i \in \mathfrak{I}\}$ and $\{\Gamma_i \in L(H, H_i) : i \in \mathfrak{I}\}$ be (A, B) woven K-g-frames and let $T \in L(H)$ and $T_i, T'_i \in L(H_i)$ for each $i \in \mathfrak{I}$. If there exist $0 < m < M < \infty$ such that for each $i \in \mathfrak{I}$ and $f_i \in H_i$, $m\|f_i\| \leq \|T_i f\|, \|T'_i f_i\| \leq M\|f_i\|$, then $\{\Lambda'_i = T_i \Lambda_i T\}_{i\in\mathfrak{I}}$ and $\{\Gamma'_i = T'_i \Gamma_i T\}_{i\in\mathfrak{I}}$ are woven T^*K-g-frames, with universal bounds $m^2 A$ and $M^2 B\|T\|^2$. Moreover if $TK^* = K^*T$, $m\|f\| \leq \|Tf\|$, then $\{\Lambda'_i \in L(H, H_i) : i \in \mathfrak{I}\}$ and $\{\Gamma'_i \in L(H, H_i) : i \in \mathfrak{I}\}$ are woven K-g-frames, with universal bounds $m^4 A$ and $M^2 B\|T\|^2$.*

Proof. For every $\sigma \subset \mathfrak{I}$ and every $f \in H$

$$\begin{aligned}\sum_{i\in\sigma} \|\Lambda'_i f\|^2 + \sum_{i\in\sigma^c} \|\Gamma'_i f\|^2 &= \sum_{i\in\sigma} \|T_i \Lambda_i T f\|^2 + \sum_{i\in\sigma^c} \|T'_i \Gamma_i T f\|^2 \\ &\leq \sum_{i\in\sigma} \|T_i\|^2 \|\Lambda_i T f\|^2 + \sum_{i\in\sigma^c} \|T'_i\|^2 \|\Gamma_i T f\|^2 \\ &\leq M^2 \left(\sum_{i\in\sigma} \|\Lambda_i T f\|^2 + \sum_{i\in\sigma^c} \|\Gamma_i T f\|^2\right)\end{aligned}$$

$$\leq M^2B\|T\|^2\|f\|^2,$$

and similarly for every $\sigma \subset \mathcal{I}$ and every $f \in H$ we have,

$$\sum_{i\in\sigma}\|\Lambda_i' f\|^2 + \sum_{i\in\sigma^c}\|\Gamma_i' f\|^2 \geq m^2A\|K^*Tf\|^2 = m^2A\|(T^*K)^*f\|^2.$$

The rest of the proof is obvious. □

Corollary 3.1. *Let $\{\Lambda_i \in L(H, H_i)\}_{i\in\mathcal{I}}$ be a K-g-frame for H and $T \in L(H)$ be invertible. Then*

(i) *$\{\Lambda_i T\}_{i\in\mathcal{I}}$ is a K-g-frame, when $\Gamma K^* = K^*\Gamma$;*
(ii) *$\{T\Lambda_i\}_{i\in\mathcal{I}}$ is a K-g-frame, when $H_i \subseteq H$ for each $i \in \mathcal{I}$.*

Proof. Let $\{\Lambda_i\}_{i\in\mathcal{I}}$ be a K-g-frame with bounds A and B.

(i) For every $x \in H$, we have

$$\begin{aligned}\frac{A}{\|T^{-1}\|^2}\|K^*x\|^2 &\leq A\|TK^*x\|^2 = A\|K^*(Tx)\|^2\\ &\leq \sum_{i\in I}\|\Lambda_i Tx\|^2 \leq B\|Tx\|^2 \leq B\|T\|^2\|x\|^2.\end{aligned}$$

For (ii),

$$\begin{aligned}\frac{A}{\|T^{-1}\|^2}\|K^*(x)\|^2 &\leq \frac{1}{\|T^{-1}\|^2}\sum\|\Lambda_i x\|^2\\ &\leq \sum\|T\Lambda_i x\|^2 \leq \|T\|^2\sum\|\Lambda_i x\|^2 \leq B\|T\|^2\|x\|^2.\end{aligned}$$ □

For the erasure of K-g-frames, the following result shows that it is possible to remove some elements of a woven K-g-frame and still have a woven K-g-frame.

Proposition 3.1. *Suppose that $\{\Lambda_i\}_{i\in\mathcal{I}}$ and $\{\Gamma_i\}_{i\in\mathcal{I}}$ are (A, B) woven K-g-frames. If $\mathcal{J} \subset \mathcal{I}$ and*

$$\sum_{i\in\mathcal{J}}\|\Lambda_i f\|^2 \leq D\|K^*f\|^2,$$

for some, $0 < D < A$, then $\{\Lambda_i\}_{i\in\mathcal{I}\setminus\mathcal{J}}$ and $\{\Gamma_i\}_{i\in\mathcal{I}\setminus\mathcal{J}}$ are $(A - D, B)$ woven K-g-frames.

Proof. The proof is similar to the proof of [3, Proposition 16]. □

Corollary 3.2. *Let $\{\Lambda_i\}_{i\in\mathcal{I}}$ be a K-g-frame with lower frame bound A. If for some $\mathcal{J} \subset \mathcal{I}$ and $0 < D < A$,*

$$\sum_{i\in\mathcal{J}}\|\Lambda_i f\|^2 \leq D\|K^*f\|^2, \quad f \in H,$$

then $\{\Lambda_i\}_{i\in\mathcal{J}^c}$ is a K-g-frame with lower bound $A - D$.

Definition 3.1. Let $\{\Lambda_i\}_{i\in\mathcal{I}}$ be a K-g-frame and let $0 \leq \lambda_1, \lambda_2 < 1$. We say that the family $\{\Gamma_i\}_{i\in\mathcal{I}}$ is a (λ_1, λ_2)-perturbation of $\{\Lambda_i\}_{i\in\mathcal{I}}$ if we have

$$\|\Lambda_i f - \Gamma_i f\| \leq \lambda_1\|\Lambda_i f\| + \lambda_2\|\Gamma_i f\|, \quad \text{for all } f \in H.$$

Theorem 3.3. *Let $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$ be woven K-g-frames and $\{\Lambda'_i\}_{i\in\mathfrak{I}}$, $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ be (λ_1,λ_2), (μ_1,μ_2)-perturbations of $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$, respectively. Then $\{\Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ are woven K-g-frames.*

Proof. A simple calculation shows that $\{\Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ are K-g-frames. For each $f \in H$ we have

$$\|\Lambda'_i f\| - \|\Lambda_i f\| \le \|\Lambda_i f - \Lambda'_i f\| \le \lambda_1\|\Lambda_i f\| + \lambda_2\|\Lambda_i f\|,$$

hence

$$\frac{1-\lambda_1}{1+\lambda_2}\|\Lambda_i f\| \le \|\Lambda'_i f\| \le \frac{1+\lambda_1}{1-\lambda_2}\|\Lambda_i f\|.$$

Similarly, we have

$$\frac{1-\mu_1}{1+\mu_2}\|\Gamma_i f\| \le \|\Gamma'_i f\| \le \frac{1+\mu_1}{1-\mu_2}\|\Gamma_i f\|.$$

Now for every $\sigma \subset \mathfrak{I}$ and every $f \in H$

$$\begin{aligned} &\min\left\{\left(\frac{1-\lambda_1}{1+\lambda_2}\right)^2, \left(\frac{1-\mu_1}{1+\mu_2}\right)^2\right\}\left(\sum_{i\in\sigma}\|\Lambda_i f\|^2 + \sum_{i\in\sigma^c}\|\Gamma_i f\|^2\right)\\ \le&\sum_{i\in\sigma}\|\Lambda'_i f\|^2 + \sum_{i\in\sigma^c}\|\Gamma'_i f\|^2\\ \le&\max\left\{\left(\frac{1+\lambda_1}{1-\lambda_2}\right)^2, \left(\frac{1+\mu_1}{1-\mu_2}\right)^2\right\}\left(\sum_{i\in\sigma}\|\Lambda_i f\|^2 + \sum_{i\in\sigma^c}\|\Gamma_i f\|^2\right), \end{aligned}$$

and we have the result. □

Corollary 3.3. *Let $\{\Lambda_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma_i\}_{i\in\mathfrak{I}}$ be woven K-g-frames and $\{\Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ be sequences and $0 \le M_1, M_2$ such that for every $f \in H$, and every $i \in \mathfrak{I}$*

$$\begin{aligned} \|\Lambda_i f - \Lambda'_i f\| &\le M_1 \min\{\|\Lambda_i f\|, \|\Lambda'_i f\|\},\\ \|\Gamma_i f - \Gamma'_i f\| &\le M_2 \min\{\|\Gamma_i f\|, \|\Gamma'_i f\|\}, \end{aligned}$$

then $\{\Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ are woven K-g-frames.

Proof. It is clear that $\{\Lambda'_i\}_{i\in\mathfrak{I}}$ and $\{\Gamma'_i\}_{i\in\mathfrak{I}}$ are K-g-frames. From the hypothesis it follows that for each $i \in \mathfrak{I}, f \in H$, we have

$$\begin{aligned} \frac{1}{M_1+1}\|\Lambda_i f\| &\le \|\Lambda'_i f\| \le (M_1+1)\|\Lambda_i f\|,\\ \frac{1}{M_2+1}\|\Gamma_i f\| &\le \|\Gamma'_i f\| \le (M_2+1)\|\Gamma_i f\|. \end{aligned}$$

Now similar to the proof of the above theorem we have the result. □

Example 3.1. Let $\{\Lambda_n : n \in \mathbb{N}\}$, $\{\Gamma_n : n \in \mathbb{N}\}$, K and H be given as in Example 2.1 and $\Lambda'_n = \frac{1}{2}\Lambda_n$ and $\Gamma'_n = \frac{1}{3}\Gamma_n$. Then $\{\Lambda'_n : n \in \mathbb{N}\}$, $\{\Gamma'_n : n \in \mathbb{N}\}$ are a woven K-g-frame. It is enough to use Example 2.1 and Theorem 3.3.

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References

[1] T. Bemrose, P. G. Casazza, K. Gröchenig, M. C. Lammers and R. G. Lynch, *Weaving frames*, Oper. Matrices **10**(4) (2016), 1093–1116. https://doi.org/10.7153/oam-10-61

[2] P. G. Casazza, D. Freeman and R. G. Lynch, *Weaving Schauder frames*, J. Approx. Theory **211** (2016), 42–60. https://doi.org/10.1016/j.jat.2016.07.001

[3] P. G. Casazza and R. G. Lynch, *Weaving properties of Hilbert space frames*, 2015 International Conference on Sampling Theory and Applications (SampTA), 2015, 110–114. https://doi.org/10.1109/SAMPTA.2015.7148861

[4] O. Christensen, *An Introduction to Frames and Riesz Bases*, 2nd ed., Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, Cham, 2016. https://doi.org/10.1007/978-3-319-25613-9

[5] I. Daubechies, A. Grossmann and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys. **27** (1986), 1271–1283. https://doi.org/10.1063/1.527388

[6] Deepshikha and L. K. Vashisht, *Weaving K-frames in Hilbert spaces,* Results Math. **73** (2018), Article ID 81. https://doi.org/10.1007/s00025-018-0843-4

[7] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366. https://doi.org/10.1090/S0002-9947-1952-0047179-6

[8] L. Găvruta, *Frames for operators*, Appl. Comput. Harmon. Anal. **32** (2012), 139–144. https://doi.org/10.1016/j.acha.2011.07.006

[9] A. Khosravi and B. Khosravi, *Fusion frames and g-frames in Banach spaces*, Proc. Indian Acad. Sci. Math. Sci. **121**(2) (2011), 155–164. https://doi.org/10.1007/s12044-011-0020-0

[10] A. Khosravi and M. M. Azandaryani, *Approximate duality of g-frames in Hilbert spaces*, Acta Math. Sci. **34B**(3) (2014), 639–652.

[11] A. Khosravi and K. Musazadeh, *Fusion frames and g-frames*, J. Math. Anal. Appl. **342 (2)** (2008), 1068–1083. https://doi.org/10.1016/j.jmaa.2008.01.002

[12] A. Khosravi and J. S. Banyarani, *Weaving g-frames and weaving fusion frames,* Bull. Malays. Math. Sci. Soc. **42** (2019), 3111–3129. https://doi.org/10.1007/s40840-018-0647-4

[13] A. Khosravi and J. S. Banyarani, *Some properties of g-frames in Banach spaces,* Int. J. Wavelets Multiresolut. Inf. Process. **16**(6) (2018), Article ID 1850051. https://doi.org/10.1142/S0219691318500510

[14] W. Sun, */g-Frames and g-Riesz bases*, J. Math. Anal. Appl. **322**(1) (2006), 437–452. https://doi.org/10.1016/j.jmaa.2005.09.039

[15] X. Xiao, Y. Zhu and L. Găvruta, *Some properties of K-frames in Hilbert spaces*, Results Math. **63** (2013), 1243–1255. https://doi.org/10.1007/s00025-012-0266-6

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