

## HIGHER CODERIVATIONS ON COALGEBRAS AND CHARACTERIZATION

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**ABSTRACT.** In this paper we define higher coderivations on a coalgebra  $C$  and then we characterize them in terms of the coderivations on  $C$ . Indeed, we show that each higher coderivation is a combination of compositions of coderivations. Finally we prove a one to one correspondence between the set of all higher coderivations on  $C$  and all sequences of coderivations on  $C$ .

### 1. INTRODUCTION

A coalgebra  $(C, \Delta, \varepsilon)$  over a field  $\kappa$  is a  $\kappa$ -vector space  $C$  together with the  $\kappa$ -linear maps  $\Delta : C \rightarrow C \otimes C$  and  $\varepsilon : C \rightarrow \kappa$ , such that  $(I_C \otimes \Delta)\Delta = (\Delta \otimes I_C)\Delta$ , (coassociativity) and  $(I_C \otimes \varepsilon)\Delta = (\varepsilon \otimes I_C)\Delta$ , (counitary). The maps  $\Delta$  and  $\varepsilon$  are called, respectively, *coproduct* and *counit* of the coalgebra  $C$ . Given an element  $c$  of the coalgebra  $(C, \Delta, \varepsilon)$ , we know that there exist elements  $c_{1,i}$  and  $c_{2,i}$  in  $C$  such that  $\Delta(c) = \sum_i c_{1,i} \otimes c_{2,i}$ . In *Sweedlers notation*, this is abbreviated to  $\sum c_{(1)} \otimes c_{(2)}$ . Here, the subscripts “(1)” and “(2)” indicate the order of the factors in the tensor product. For more about basic definitions in coalgebras notion, you can see [1] and [3].

A  $\kappa$ -linear map  $f : C \rightarrow C$  on a  $\kappa$ -coalgebra  $(C, \Delta, \varepsilon)$  is called a *coderivation* if  $\Delta f = (I_C \otimes f + f \otimes I_C)\Delta$ . One can see examples and a general definition of coalgebras and coderivations in the sense of comodules in [2, 4, 6]. In this paper we define higher coderivations on a coalgebra  $C$  and then characterize them in terms of the coderivations on  $C$ . Indeed, we show that each higher coderivation is a combination of compositions of coderivations. As a corollary we characterize all higher coderivations which are ordinary. We have some nearly same properties for higher derivations, you

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can see in [5] and [7]. Throughout the paper, all coalgebras are assumed over a field of characteristic zero.

## 2. THE RESULTS

Throughout the paper,  $C$  denotes a coalgebra over a field of characteristic zero and  $I$  is the identity mapping on  $C$ . A *coalgebra*  $(C, \Delta, \varepsilon)$  over a field  $\kappa$  is a  $\kappa$ -vector space  $C$  together with the  $\kappa$ -linear maps  $\Delta : C \rightarrow C \otimes C$  and  $\varepsilon : C \rightarrow \kappa$ , such that  $(I_C \otimes \Delta)\Delta = (\Delta \otimes I_C)\Delta$ , (coassociativity), and  $(I_C \otimes \varepsilon)\Delta = (\varepsilon \otimes I_C)\Delta$ , (counitary). The maps  $\Delta$  and  $\varepsilon$  are called, respectively, *coproduct* and *counit* of the coalgebra  $C$ . A  $\kappa$ -linear map  $f : C \rightarrow C$  on a  $\kappa$ -coalgebra  $(C, \Delta, \varepsilon)$  is called a *coderivation* if  $\Delta f = (I_C \otimes f + f \otimes I_C)\Delta$ .

Now we define a new concept, named higher coderivation and then characterize this, but at first we prove some properties, following.

**Proposition 2.1.** *If  $f$  is a coderivation on coalgebra  $(C, \Delta, \varepsilon)$ , then we have*

$$(2.1) \quad \Delta f^n = \sum_{k=0}^n \binom{n}{k} (f^k \otimes f^{n-k})\Delta,$$

for each nonnegative integer  $n$ .

*Proof.* We use induction on  $n$ . For  $n = 1$  and  $a \in C$  we have

$$\Delta f(a) = \sum a_{(1)} \otimes f(a_{(2)}) + f(a_{(1)}) \otimes a_{(2)},$$

and its true, since  $f$  is a coderivation on  $C$ . Now suppose that the equality is true for  $n$ , then for  $n + 1$ , in the left side of equality, we have

$$\Delta f^{n+1}(a) = \Delta f^n(f(a)) = \sum_{k=0}^n \binom{n}{k} (f^k \otimes f^{n-k})\Delta(f(a)),$$

because of  $f$  being a coderivation, we have

$$\begin{aligned} \Delta f^{n+1}(a) &= \sum_{k=0}^n \binom{n}{k} (f^k \otimes f^{n-k})(I \otimes f + f \otimes I)\Delta(a) \\ &= \sum_{k=0}^n \sum \binom{n}{k} f^k(a_{(1)}) \otimes f^{n+1-k}(a_{(2)}) + f^{k+1}(a_{(1)}) \otimes f^{n-k}(a_{(2)}). \end{aligned}$$

On the other side we have

$$\begin{aligned} &\sum_{k=0}^{n+1} \binom{n+1}{k} (f^k \otimes f^{n+1-k})\Delta(a) \\ &= \sum_{k=0}^{n+1} \sum \binom{n+1}{k} f^k(a_{(1)}) \otimes f^{n+1-k}(a_{(2)}) \\ &= \left[ \sum_{k=0}^n \sum \left( \binom{n}{k} + \binom{n}{k-1} \right) (f^k(a_{(1)}) \otimes f^{n+1-k}(a_{(2)})) \right] + f^{n+1}(a_{(1)}) \otimes a_{(2)} \end{aligned}$$

$$\begin{aligned}
&= \left[ \sum_{k=0}^n \sum \binom{n}{k} f^k(a_{(1)}) \otimes f^{n+1-k}(a_{(2)}) \right. \\
&\quad \left. + \sum_{k=-1}^{n-1} \sum \binom{n}{k} (f^{k+1}(a_{(1)}) \otimes f^{n+1-(k+1)}(a_{(2)})) \right] + f^{n+1}(a_{(1)}) \otimes a_{(2)} \\
&= \sum_{k=0}^n \sum \binom{n}{k} f^k(a_{(1)}) \otimes f^{n+1-k}(a_{(2)}) + \sum_{k=-1}^n \sum \binom{n}{k} f^{k+1}(a_{(1)}) \otimes f^{n-k}(a_{(2)}),
\end{aligned}$$

and we have the result.  $\square$

We name the relation (2.1) *general coLeibnitz rule for coderivations*.

If we define a sequence  $\{f_n\}$  of linear mappings on  $C$  by  $f_0 = I$  and  $f_n = \frac{\lambda^n}{n!}$ , where  $I$  is the identity mapping on  $C$ , then general coLeibniz rule ensures us that  $f_n$ 's satisfy the condition

$$(2.2) \quad \Delta f_n = \sum_{k=0}^n (f_k \otimes f_{n-k}) \Delta,$$

for each nonnegative integer  $n$ . This motivates us to consider the sequences  $\{f_n\}$  of linear mappings on a coalgebra  $C$  satisfying (2.2). We call such a sequence a higher coderivation.

**Definition 2.1.** Let  $C$  be a coalgebra. We define a sequence  $\{f_n\}$  of linear mappings on  $C$  a *higher coderivation* if  $\Delta f_n(a) = \sum_{k=0}^n (f_k \otimes f_{n-k}) \Delta(a)$  for each  $a \in C$  and each nonnegative integer  $n$ .

Though, if  $\lambda : C \rightarrow C$  is a coderivation then  $f_n = \frac{\lambda^n}{n!}$  is a higher coderivation. We name this kind of higher coderivation an *ordinary higher coderivation*.

**Proposition 2.2.** Let  $\{f_n\}$  be a higher coderivation on a coalgebra  $C$  with  $f_0 = I$ . Then there is a sequence  $\{\lambda_n\}$  of coderivations on  $C$  such that

$$(n+1)f_{n+1} = \sum_{k=0}^n f_{n-k} \lambda_{k+1},$$

for each nonnegative integer  $n$ .

*Proof.* We use induction on  $n$ . Because of  $\{f_n\}$  being a higher coderivation, for  $n = 0$  we have

$$\begin{aligned}
\Delta f_1(a) &= [(f_0 \otimes f_1) + (f_1 \otimes f_0)] \Delta(a) \\
&= \sum f_0(a_{(1)}) \otimes f_1(a_{(2)}) + f_1(a_{(1)}) \otimes f_0(a_{(2)}) \\
&= \sum a_{(1)} \otimes f_1(a_{(2)}) + f_1(a_{(1)}) \otimes a_{(2)}.
\end{aligned}$$

Thus, if  $\lambda_0 = I$  and  $\lambda_1 = f_1$ , then  $\lambda_1$  is a coderivation on  $\mathcal{A}$  and

$$\Delta(f_0 \lambda_1)(a) = \Delta(\lambda_1(a)) = \sum \lambda_0(a_{(1)}) \otimes \lambda_1(a_{(2)}) + \lambda_1(a_{(1)}) \otimes \lambda_0(a_{(2)}).$$

Now suppose that  $\lambda_k$  is defined and is a coderivation for  $k \leq n$ . Putting  $\lambda_{n+1} = (n+1)f_{n+1} - \sum_{k=0}^{n-1} f_{n-k}\lambda_{k+1}$ , we show that the well-defined mapping  $\lambda_{n+1}$  is a coderivation on  $C$ . For  $a \in C$ , since  $\{f_n\}$  is a higher coderivation and  $\lambda_1, \dots, \lambda_n$  are coderivations, we have

$$\begin{aligned}
\Delta\lambda_{n+1}(a) &= (n+1)\Delta f_{n+1}(a) - \sum_{k=0}^{n-1} \Delta(f_{n-k}\lambda_{k+1})(a) \\
&= (n+1)\Delta f_{n+1}(a) \\
&\quad - \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \sum (f_l \otimes f_{n-k-l}) \left( a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)} \right) \\
&= (n+1) \sum_{k=0}^{n+1} (f_k \otimes f_{n+1-k}) \Delta(a) \\
&\quad - \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \sum (f_l \otimes f_{n-k-l}) \left( a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)} \right) \\
&= (n+1) \sum_{k=0}^{n+1} \sum f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) \\
&\quad - \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \sum (f_l \otimes f_{n-k-l}) \left( a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)} \right) \\
&= (n+1) \sum_{k=0}^{n+1} \sum f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) \\
&\quad - \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \sum f_l(a_{(1)}) \otimes f_{n-k-l}(\lambda_{k+1}(a_{(2)})) + f_l(\lambda_{k+1}(a_{(1)})) \otimes f_{n-k-l}(a_{(2)}).
\end{aligned}$$

Now, by properties of tensor product, we have

$$\begin{aligned}
\Delta\lambda_{n+1}(a) &= \sum_{k=0}^{n+1} \sum (k+n+1-k) \left( f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) \right) \\
&\quad - \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \sum (f_l \otimes f_{n-k-l}) \left( a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)} \right) \\
&= \sum_{k=0}^{n+1} \sum k f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) + f_k(a_{(1)}) \otimes (n+1-k) f_{n+1-k}(a_{(2)}) \\
&\quad - \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \sum (f_l \otimes f_{n-k-l}) \left( a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) + \lambda_{k+1}(a_{(1)}) \otimes a_{(2)} \right).
\end{aligned}$$

Writing

$$K = \sum_{k=0}^{n+1} \sum k f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) - \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k} \sum f_\ell \lambda_{k+1}(a_{(1)}) \otimes f_{n-k-\ell}(a_{(2)}),$$

$$L = \sum_{k=0}^{n+1} \sum f_k(a_{(1)}) \otimes (n+1-k)f_{n+1-k}(a_{(2)}) \\ - \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k} f_\ell(a_{(1)}) \otimes f_{n-k-\ell}\lambda_{k+1}(a_{(2)}),$$

we have  $\Delta\lambda_{n+1}(a) = K + L$ . Let us compute  $K$  and  $L$ . In the summation  $\sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k}$ , we have  $0 \leq k + \ell \leq n$  and  $k \neq n$ . Thus, if we put  $r = k + \ell$  then we can write it as the form  $\sum_{r=0}^n \sum_{k+\ell=r, k \neq n}$ . Putting  $\ell = r - k$  we indeed have

$$K = \sum_{k=0}^{n+1} \sum k f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) \\ - \sum_{r=0}^n \sum_{0 \leq k \leq r, k \neq n} \sum f_{r-k}\lambda_{k+1}(a_{(1)}) \otimes f_{n-r}(a_{(2)}) \\ = \sum_{k=0}^{n+1} \sum k f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) \\ - \sum \left( \sum_{r=0}^{n-1} \sum_{k=0}^r f_{r-k}\lambda_{k+1}(a_{(1)}) \otimes f_{n-r}(a_{(2)}) \right) - \sum_{k=0}^{n-1} f_{n-k}\lambda_{k+1}(a_{(1)}) \otimes a_{(2)}.$$

Putting  $r + 1$  instead of  $k$  in the first summation we have

$$K + \sum_{k=0}^{n-1} \sum f_{n-k}\lambda_{k+1}(a_{(1)}) \otimes a_{(2)} \\ = \sum_{r=0}^n \sum (r+1)f_{r+1}(a_{(1)}) \otimes f_{n-r}(a_{(2)}) - \sum_{r=0}^{n-1} \sum_{k=0}^r \sum f_{r-k}\lambda_{k+1}(a_{(1)}) \otimes f_{n-r}(a_{(2)}) \\ = \sum \left( \sum_{r=0}^{n-1} \left[ (r+1)f_{r+1}(a_{(1)}) - \sum_{k=0}^r f_{r-k}\lambda_{k+1}(a_{(1)}) \right] \otimes f_{n-r}(a_{(2)}) \right) \\ + (n+1)f_{n+1}(a_{(1)}) \otimes a_{(2)}.$$

By our assumption

$$(r+1)f_{r+1}(a) = \sum_{k=0}^r (f_{r-k}\lambda_{k+1})(a),$$

for  $r = 0, \dots, n-1$ . We can therefore deduce that

$$K = \sum \left[ (n+1)f_{n+1}(a_{(1)}) - \sum_{k=0}^{n-1} f_{n-k}\lambda_{k+1}(a_{(1)}) \right] \otimes a_{(2)} = \sum \lambda_{n+1}(a_{(1)}) \otimes a_{(2)}.$$

By a similar argument we have

$$L = \sum a_{(1)} \otimes \left[ (n+1)f_{n+1}(a_{(2)}) - \sum_{k=0}^{n-1} f_{n-k}\lambda_{k+1}(a_{(2)}) \right] = \sum a_{(1)} \otimes \lambda_{n+1}(a_{(2)}).$$

Thus,

$$\Delta\lambda_{n+1}(a) = K + L = (I \otimes \lambda_{n+1} + \lambda_{n+1} \otimes I)\Delta(a),$$

whence  $\lambda_{n+1}$  is a coderivation on  $C$ .  $\square$

To illustrate the recursive relation mentioned in Proposition 2.2, let us compute some terms of  $\{d_n\}$ .

*Example 2.1.* Using Proposition 2.2, the first five terms of  $\{f_n\}$  are

$$f_0 = I,$$

$$f_1(a) = f_0(\lambda_1(a)) = \lambda_1(a) \rightarrow f_1 = \lambda_1,$$

$$2f_2(a) = f_1(\lambda_1(a)) + f_0(\lambda_2(a)) = \lambda_1^2(a) + \lambda_2(a) \rightarrow 2f_2 = \lambda_1^2 + \lambda_2,$$

$$f_2 = \frac{1}{2}\lambda_1^2 + \frac{1}{2}\lambda_2,$$

$$3f_3 = f_2\lambda_1 + f_1\lambda_2 + f_0\lambda_3 = \left(\frac{1}{2}\lambda_1^2 + \frac{1}{2}\lambda_2\right)\lambda_1 + \lambda_1\lambda_2 + \lambda_3,$$

$$f_3 = \frac{1}{6}\lambda_1^3 + \frac{1}{6}\lambda_2\lambda_1 + \frac{1}{3}\lambda_1\lambda_2 + \frac{1}{3}\lambda_3,$$

$$4f_4 = f_3\lambda_1 + f_2\lambda_2 + f_1\lambda_3 + f_0\lambda_4$$

$$= \left(\frac{1}{6}\lambda_1^3 + \frac{1}{6}\lambda_2\lambda_1 + \frac{1}{3}\lambda_1\lambda_2 + \frac{1}{3}\lambda_3\right)\lambda_1 + \left(\frac{1}{2}\lambda_1^2 + \frac{1}{2}\lambda_2\right)\lambda_2 + \lambda_1\lambda_3 + \lambda_4,$$

$$f_4 = \frac{1}{24}\lambda_1^4 + \frac{1}{24}\lambda_2\lambda_1^2 + \frac{1}{12}\lambda_1\lambda_2\lambda_1 + \frac{1}{12}\lambda_3\lambda_1 + \frac{1}{8}\lambda_1^2\lambda_2 + \frac{1}{8}\lambda_2^2 + \frac{1}{4}\lambda_1\lambda_3 + \frac{1}{4}\lambda_4.$$

**Theorem 2.1.** *Let  $\{f_n\}$  be a higher coderivation on a coalgebra  $C$  with  $f_0 = I$ . Then there is a sequence  $\{\lambda_n\}$  of coderivations on  $C$  such that*

$$(n+1)f_{n+1} = \sum_{i=2}^{n+1} \left( \sum_{\sum_{j=1}^i r_j = n} \left( \prod_{j=1}^i \frac{1}{r_i + \dots + r_j} \right) \lambda_{r_i} \cdots \lambda_{r_1} \right),$$

where the inner summation is taken over all positive integers  $r_j$ , with  $\sum_{j=1}^i r_j = n$ .

*Proof.* We show that if  $f_n$  is of the above form then it satisfies the recursive relation of Proposition 2.2. Since the solution of the recursive relation is unique, this proves the theorem. Simplifying the notation we put  $a_{r_i, \dots, r_1} = \prod_{j=1}^i \frac{1}{r_i + \dots + r_j}$ . Note that if  $r_1 + \dots + r_i = n+1$  then  $(n+1)a_{r_i, \dots, r_1} = a_{r_i, \dots, r_2}$ . Moreover,  $a_{n+1} = \frac{1}{n+1}$ . Now we have

$$\begin{aligned} (n+1)f_{n+1} &= \sum_{i=2}^{n+1} \left( \sum_{\sum_{j=1}^i r_j = n+1} a_{r_i, \dots, r_1} (n+1) \lambda_{r_i} \cdots \lambda_{r_1} \right) + \lambda_{n+1} \\ &= \sum_{i=2}^{n+1} \left( \sum_{r_1=1}^{n+2-i} \sum_{\sum_{j=2}^i r_j = n+1-r_1} a_{r_i, \dots, r_2} \lambda_{r_i} \cdots \lambda_{r_2} \right) \lambda_{r_1} + \lambda_{n+1} \\ &= \sum_{r_1=1}^n \sum_{i=2}^{n-(r_1-1)} \left( \sum_{\sum_{j=2}^i r_j = n-(r_1-1)} a_{r_i, \dots, r_2} \lambda_{r_i} \cdots \lambda_{r_2} \right) \lambda_{r_1} + \lambda_{n+1} \end{aligned}$$

$$\begin{aligned}
&= \sum_{r_1=1}^n f_{n-(r_1-1)} \lambda_{r_1} + \lambda_{n+1} \\
&= \sum_{k=0}^n f_{n-k} \lambda_{k+1}.
\end{aligned}$$

□

*Example 2.2.* We evaluate the coefficients  $a_{r_i, \dots, r_1}$  for the case  $n = 4$ .

For  $n = 4$  we can write

$$4 = 1 + 3 = 3 + 1 = 2 + 2 = 1 + 1 + 2 = 1 + 2 + 1 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

By the definition of  $a_{r_i, \dots, r_1}$  we have

$$\begin{aligned}
a_4 &= \frac{1}{4}, \\
a_{1,3} &= \frac{1}{1+3} \cdot \frac{1}{3} = \frac{1}{12}, \\
a_{3,1} &= \frac{1}{3+1} \cdot \frac{1}{1} = \frac{1}{4}, \\
a_{2,2} &= \frac{1}{2+2} \cdot \frac{1}{2} = \frac{1}{8}, \\
a_{1,1,2} &= \frac{1}{1+1+2} \cdot \frac{1}{1+2} \cdot \frac{1}{2} = \frac{1}{24}, \\
a_{1,2,1} &= \frac{1}{1+2+1} \cdot \frac{1}{2+1} \cdot \frac{1}{1} = \frac{1}{12}, \\
a_{2,1,1} &= \frac{1}{2+1+1} \cdot \frac{1}{1+1} \cdot \frac{1}{1} = \frac{1}{8}, \\
a_{1,1,1,1} &= \frac{1}{1+1+1+1} \cdot \frac{1}{1+1+1} \cdot \frac{1}{1+1} \cdot \frac{1}{1} = \frac{1}{24}.
\end{aligned}$$

We can therefore deduce that

$$f_4 = \frac{1}{4} \lambda_4 + \frac{1}{12} \lambda_3 \lambda_1 + \frac{1}{4} \lambda_1 \lambda_3 + \frac{1}{8} \lambda_2 \lambda_2 + \frac{1}{24} \lambda_2 \lambda_1 \lambda_1 + \frac{1}{12} \lambda_1 \lambda_2 \lambda_1 + \frac{1}{8} \lambda_1 \lambda_1 \lambda_2 + \frac{1}{24} \lambda_1 \lambda_1 \lambda_1 \lambda_1.$$

**Theorem 2.2.** *Let  $C$  be a coalgebra,  $F$  be the set of all higher coderivations  $\{f_n\}_{n=0,1,\dots}$  on  $C$  with  $f_0 = I$  and  $\Lambda$  be the set of all sequences  $\{\lambda_n\}_{n=0,1,\dots}$  of coderivations on  $C$  with  $\lambda_0 = 0$ . Then there is a one to one correspondence between  $F$  and  $\Lambda$ .*

*Proof.* Let  $\{\lambda_n\} \in \Lambda$ . Define  $f_n : C \rightarrow C$  by  $f_0 = I$  and

$$f_n = \sum_{i=1}^n \left( \sum_{\sum_{j=1}^i r_j = n} \left( \prod_{j=1}^i \frac{1}{r_i + \dots + r_j} \right) \lambda_{r_i} \cdots \lambda_{r_1} \right).$$

We show that  $\{f_n\} \in F$ . By Theorem 2.1,  $\{f_n\}$  satisfies the recursive relation

$$(n+1)f_{n+1} = \sum_{k=0}^n f_{n-k} \lambda_{k+1}.$$

To show that  $\{f_n\}$  is a higher coderivation, we use induction on  $n$ . For  $n = 0$  we have

$$\Delta f_0(a) = \Delta(a) = \sum a_{(1)} \otimes a_{(2)} = \sum f_0(a_{(1)}) \otimes f_0(a_{(2)}) = \sum (f_0(a))_{(1)} \otimes (f_0(a))_{(2)}.$$

Let us assume that  $\Delta f_k(a) = \sum_{i=0}^k (f_i \otimes f_{k-i}) \Delta(a)$  for  $k \leq n$ . Thus, we have

$$\begin{aligned} (n+1)\Delta f_{n+1}(a) &= \sum_{k=0}^n \Delta f_{n-k} \lambda_{k+1}(a) \\ &= \sum_{k=0}^n \sum_{i=0}^{n-k} (f_i \otimes f_{n-k-i}) \Delta \lambda_{k+1}(a) \\ &= \sum_{k=0}^n \sum_{i=0}^{n-k} (f_i \otimes f_{n-k-i}) (I \otimes \lambda_{k+1} + \lambda_{k+1} \otimes I) \Delta(a) \\ &= \sum_{k=0}^n \sum_{i=0}^{n-k} \sum (f_i \otimes f_{n-k-i}) \left( \sum a_{(1)} \otimes \lambda_{k+1}(a_{(2)}) \otimes \lambda_{k+1}(a_{(1)}) \otimes a_{(2)} \right) \\ &= \sum_{k=0}^n \sum_{i=0}^{n-k} \sum f_i(a_{(1)}) \otimes f_{n-k-i}(\lambda_{k+1}(a_{(2)})) \\ &\quad + f_i(\lambda_{k+1}(a_{(1)}) \otimes f_{n-k-i}(a_{(2)}). \end{aligned}$$

Using our assumption, we can write

$$\begin{aligned} (n+1)\Delta f_{n+1}(a) &= \sum_{i=0}^n \sum f_i(a_{(1)}) \otimes (n-i+1)f_{n-i+1}(a_{(2)}) \\ &\quad + \sum_{i=0}^n \sum (n-i+1) (f_{n-i+1}(a_{(1)}) \otimes f_i(a_{(2)})) \\ &= \sum_{i=0}^n \sum (n+1-i) f_i(a_{(1)}) \otimes f_{n+1-i}(a_{(2)}) \\ &\quad + \sum_{i=1}^{n+1} \sum i (f_i(a_{(1)}) \otimes f_{n+1-i}(a_{(2)})) \\ &= (n+1) \sum_{k=0}^{n+1} \sum f_k(a_{(1)}) \otimes f_{n+1-k}(a_{(2)}) \\ &= (n+1) \sum_{k=0}^{n+1} (f_k \otimes f_{n+1-k}) \Delta(a). \end{aligned}$$

Thus,  $\{f_n\} \in F$ .

Conversely, suppose that  $\{f_n\} \in F$ . Define  $\lambda_n : C \rightarrow C$  by  $\lambda_0 = 0$  and

$$\lambda_n = n f_n - \sum_{k=0}^{n-2} f_{n-1-k} \lambda_{k+1}.$$



Then Proposition 2.2 ensures us that  $\{\lambda_n\} \in \Lambda$ . Now define  $\varphi : \Lambda \rightarrow F$  by  $\varphi(\{\lambda_n\}) = \{f_n\}$ , where

$$f_n = \sum_{i=1}^n \left( \sum_{\sum_{j=1}^i r_j = n} \left( \prod_{j=1}^i \frac{1}{r_i + \dots + r_j} \right) \lambda_{r_i} \cdots \lambda_{r_1} \right).$$

Now  $\varphi$  is clearly a one to one correspondence.  $\square$

Recall that a higher coderivation  $\{f_n\}$  is called ordinary if there is a coderivation  $\lambda$  such that  $f_n = \frac{\lambda^n}{n!}$  for all  $n$ .

**Corollary 2.1.** *A higher coderivation  $\{f_n\} = \varphi(\{\lambda_n\})$  on a coalgebra  $C$  is ordinary if and only if  $\lambda_n = 0$  for  $n \geq 2$ . In this case  $f_n = \frac{f_1^n}{n!}$ .*

### 3. CONCLUSION

In this paper proving an equality for a coderivation on a coalgebra  $C$ , named general coLiebnitz rule for coderivations, we defined higher coderivations on a coalgebra  $C$  and then we characterized them in terms of the coderivations on  $C$ . Indeed, we showed that each higher coderivation is a combination of compositions of coderivations. Finally we proved there is a one to one correspondence between the set of all higher coderivations on  $C$  and all sequences of coderivations on  $C$ . As a corollary we characterize all higher coderivations which are ordinary.

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