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NOTE ON THE MULTIFRACTAL FORMALISM OF COVERING NUMBER ON THE GALTON-WATSON TREE

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Abstract. We consider, for t in the boundary of Galton-Watson tree ($\partial\mathsf{T}$), the covering number $\mathsf{N}_n(t)$ by cylinder of generation n. For a suitable set I and a sequence $(s_{n,\gamma})$, we establish almost surely, and uniformly on γ, the Hausdorff and packing dimensions of the set $\{t \in \partial\mathsf{T} : \mathsf{N}_n(t) - nb \sim s_{n,\gamma}\}$ for $b \in I$.

1. Introduction and main results

Let (N, X) be a random vector with independent components taking values in $\mathbb{N}^2$, where $\mathbb{N}$ denotes the set of non-negative integers. Then let $\{(N_u, X_u)\}_{u\in\bigcup_{n\geq 0}\mathbb{N}_+^n}$ be a family of independent copies of the vector (N, X) indexed by the set of finite words over the alphabet $\mathbb{N}_+$: the set of positive integers ($n = 0$ corresponds to the empty sequence denoted $\emptyset$). Let T be the Galton-Watson tree with defining elements $\{N_u\}$: we have $\emptyset \in \mathsf{T}$, if $u \in \mathsf{T}$ and $i \in \mathbb{N}_+$ then ui, the concatenation of u and i, belongs to T if and only if $1 \leq i \leq N_u$ and if $ui \in \mathsf{T}$, then $u \in \mathsf{T}$. Similarly, for each $u \in \bigcup_{n\geq 0}\mathbb{N}_+^n$, denote by $\mathsf{T}(u)$ the Galton-Watson tree rooted at u and defined by the $\{N_{uv}\}$, $v \in \bigcup_{n\geq 0}\mathbb{N}_+^n$.

We assume that $\mathbb{E}(N) > 1$ so that the Galton-Watson tree is supercritical. We also assume that the probability of extinction is equal to 0, so that $\mathbb{P}(N \geq 1) = 1$.

For each infinite word $t = t_1 t_2 \cdots \in \mathbb{N}_+^{\mathbb{N}_+}$ and $n \geq 0$, we set $t_{|n} = t_1 \cdots t_n \in \mathbb{N}_+^n$ ($t_{|0} = \emptyset$). If $u \in \mathbb{N}_+^n$ for some $n \geq 0$, then n is the length of u and it is denoted by $|u|$. We denote by $[u]$ the set of infinite words $t \in \mathbb{N}_+^{\mathbb{N}_+}$ such that $t_{||u|} = u$.

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The set $\mathbb{N}_+^{\mathbb{N}_+}$ is endowed with the standard ultrametric distance

$$d : (u, v) \mapsto e^{-\sup\{|w|: u\in[w], v\in[w]\}},$$

with the convention $\exp(-\infty) = 0$. The boundary of the Galton-Watson tree T is defined as the compact set

$$\partial\mathsf{T} = \bigcap_{n\geq 1} \bigcup_{u\in\mathsf{T}_n} [u],$$

where $\mathsf{T}_n = \mathsf{T} \cap \mathbb{N}_+^n$.

We consider X_u as the covering number of the cylinder $[u]$, that is to say, the cylinder $[u]$ is cut off with probability $p_0 = \mathbb{P}(X = 0)$ and is covered m times with probability $p_m = \mathbb{P}(X = m)$, $m = 1, 2, \ldots$

For $t \in \partial\mathsf{T}$, set

$$\mathsf{N}_n(t) = \sum_{k=1}^{n} X_{t_1\cdots t_k}.$$

Since this quantity depends on $t_1 \cdots t_n$ only, we also denote by $\mathsf{N}_n(u)$ the constant value of $\mathsf{N}_n(\cdot)$ over $[u]$ whenever $u \in \mathsf{T}_n$. The quantity $\mathsf{N}_n(t)$ is called the covered number (or more precisely the n-covered number) of the point t by cylinder of generation k, $k = 1, 2, \ldots, n$.

Consider an individual infinite branch $t_1 \cdots t_n \cdots$ in $\partial\mathsf{T}$. When $\mathbb{E}(X)$ is defined, the strong law of large number yields $\lim_{n\to\infty} n^{-1}\mathsf{N}_n(t) = \mathbb{E}(X)$. It is also well known, in the theory of the birth process, (see [15]) that almost surely (a.s.) $\lim_{n\to\infty} \mathsf{N}_n(t) = +\infty$ for every $t \in \mathcal{D} = \{0, 1\}^{\mathbb{N}}$ if and only if

$$p_0 = \mathbb{P}(X = 0) < \frac{1}{2}.$$

If this condition is satisfied, then a.s. every point is infinitely covered.

We consider, for $b \in \mathbb{R}$, the set

$$E_b = \Big\{t \in \partial\mathsf{T} : \lim_{n\to\infty} \frac{\mathsf{N}_n(t)}{n} = b\Big\}.$$

These level sets can be described geometrically through their Hausdorff dimensions. They have been studied by many authors, see [3,8,11,14,16,21] and [4,7] for a general case. All these papers also deal with the multifractal analysis of associated Mandelbrot measures (see also [1,2,19] for the study of Mandelbrot measures dimension).

We will assume that the free energy of X defined as

$$\tau(q) = \log \mathbb{E}\Big(\sum_{i=1}^{N} e^{qX_i}\Big)$$

is finite over $\mathbb{R}$. We will assume, without loss of generality, that X is not constant so that the function τ is strictly convex. Let τ^* stand for the Legendre transform of the function τ, defined as

$$\tau^*(b) := \inf_{q\in\mathbb{R}} \Big(\tau(q) - qb\Big), \quad b \in \mathbb{R}.$$

We say that the multifractal formalism holds at $b \in \mathbb{R}$ if

$$\dim E_b = \mathrm{Dim}\, E_b = \tau^*(b),$$

where $\dim\, E_b$ is the Hausdorff dimension of E_b and $\mathrm{Dim}\, E_b$ is the packing dimension of E_b (see Section A for the definition). In the following, we define the sets

$$\begin{aligned} J &= \Big\{ q \in \mathbb{R}; \tau(q) - q\tau'(q) > 0 \Big\}, \\ \Omega^1_\alpha &= \mathrm{int}\Big\{ q : \mathbb{E}\Big[\Big| \sum_{i=1}^{N} e^{qX_i} \Big|^\alpha \Big] < \infty \Big\}, \\ \Omega^1 &= \bigcup_{\alpha \in (1,2]} \Omega^1_\alpha, \\ \mathcal{J} &= J \cap \Omega^1 \quad \text{and} \quad I = \Big\{ \tau'(q); q \in \mathcal{J} \Big\}. \end{aligned}$$

Remark 1.1. It is well known, see [6, Proposition 3.1], that $L = \{\alpha \in \mathbb{R},\ \tau^*(\alpha) \geq 0\}$, is a convex, compact and non-empty set. In addition, if we assume that $J = \mathcal{J}$ then $I = \mathrm{int}(L)$, where $\mathrm{int}(L)$ is the interior of L (see also [6, Proposition 3.1.]) In particular, I is an interval.

Next, we define for $b, \gamma \in \mathbb{R}$ and for any positive sequence $s^\gamma = \{s_{n,\gamma}\}_n$ such that $s_{n,\gamma} = \mathrm{o}(n)$ and $\gamma \mapsto s_{n,\gamma}$ is analytic function, the set

$$E_{b,s^\gamma} = \Big\{ t \in \partial\mathsf{T} : \mathsf{N}_n(t) - nb \sim s_{n,\gamma} \text{ as } n \to +\infty \Big\},$$

where $\mathsf{N}_n(t) - nb \sim s_{n,\gamma}$ means that $(\mathsf{N}_n(t) - nb)_n$ and $(s_{n,\gamma})_n$ are two equivalent sequences. It is clear that $E_{b,s^\gamma} \subset E_b$. So, we can get with a simple covering argument, with probability 1, for all $b \in \mathbb{R}$ and $\gamma \in \mathbb{R}$,

$$\dim E_{b,s^\gamma} \leq \dim E_b \leq \mathrm{Dim}\, E_b \leq \tau^*(b), \tag{1.1}$$

(see Proposition 1 in [5] and Proposition 2.7 in [4]). Let us mention that the methods used to compute Hausdorff dimension of the sets E_b in, for example, [4, 7, 17, 18]) do not give results on $\dim\, E_{b,s^\gamma}$. These sets were considered by Kahane and Fan in [15]. The authors considered the space $\{0,1\}^{\mathbb{N}}$ and they compute, for each b, almost surely (a.s.), the Hausdorff dimension of E_{b,s^γ} under the hypothesis :

$$s_{n,\gamma} = \mathrm{o}(n), \quad \eta_n(\gamma) = s_{n,\gamma} - s_{n-1,\gamma} = \mathrm{o}(1) \quad \text{and} \quad \sqrt{n \ln\ln n} = \mathrm{o}(s_{n,\gamma}).$$

A special case of a sequence satisfying the above hypothesis is $s_{n,\gamma} = n^\gamma$ with $\gamma \in (1/2, 1)$. Later, Attia in [5], gives a stronger result in the sense that, a.s. for all $b \in I$, he computed the Hausdorff dimensions of the sets E_{b,s^γ} under the hypothesis

$$s_{n,\gamma} = \mathrm{o}(n), \quad \eta_n(\gamma) = s_{n,\gamma} - s_{n-1,\gamma} = \mathrm{o}(1) \tag{1.2}$$

and there exists $\epsilon_n \to 0$ such that

$$\sum_{n \geq 1} \exp\Big(-\epsilon \sum_{k=1}^{n} \epsilon_k\, \eta_k(\gamma)^2 \Big) < +\infty, \quad \text{for all } \epsilon > 0. \tag{1.3}$$

In particular, we can choose

$$s_{n,\gamma} = \sum_{k=1}^{n} \frac{1}{k^\gamma} \quad \text{with } \gamma \in (0, 1/2).$$

Theorem 1.1 ([5]). *Let s^γ be a positive sequence satisfying* (1.2) *and* (1.3). *Then, a.s. for all $b \in I$*

$$\dim E_{b,s^\gamma} = \dim E_b = \tau^*(b).$$

This requires, for a given sequence s^γ, a simultaneous building of an inhomogeneous Mandelbrot measure and a computing of their dimensions. In particular, for

$$s_{n,\gamma} = \sum_{k=1}^{n} \frac{1}{k^\gamma},$$

we have for all $\gamma \in (0, 1/2)$, a.s. $\dim E_{b,s^\gamma} = \tau^*(b)$. To state our main result, let $s^\gamma = (s_{n,\gamma})_n$ be a positive sequence and we define the set Λ_s to be any set of $\mathbb{R}$ such that

$$\Lambda_s \subseteq \Big\{\gamma \in \mathbb{R}, \text{ such that } (s_{n,\gamma}) \text{ satisfies (1.2) and (1.3)}\Big\} \tag{1.4}$$

and, for $k \geq 1$

$$\widetilde{\eta}_k = \inf_{\gamma \in \Lambda_s} \eta_k(\gamma) > 0. \tag{1.5}$$

We suppose the following hypothesis.

Hypothesis 1.2. There exists a sequence $\epsilon_n \to 0$ such that

$$\sum_{n \geq 1} \exp\Big(- \epsilon \sum_{k=1}^{n} \epsilon_k \widetilde{\eta}_k^2 \Big) < +\infty, \quad \text{for all } \epsilon > 0.$$

Clearly this hypothesis is satisfied, for $s_{n,\gamma} = \sum_{k=1}^{n} \frac{1}{k^\gamma}$, with $\Lambda_s = [\epsilon, 1/2)$, $\epsilon > 0$. Applying the previous theorem we get the conclusion for each $\gamma \in \Lambda_s$ a.s. The goal of this note is to give a uniform result on γ. In addition, we determine the packing dimensions of the sets E_{b,s^γ}. More precisely we have the following result.

Theorem 1.3. *Let $s^\gamma = (s_{n,\gamma})_{n\geq 1}$ be a positive sequence and consider a set Λ_s satisfying* (1.4) *and* (1.5). *Under Hypothesis* 1.2, *we have, a.s.. for all $b \in I$ and for all $\gamma \in \Lambda_s$*

$$\dim E_{b,s^\gamma} = \dim E_b = \operatorname{Dim} E_b = \operatorname{Dim} E_{b,s^\gamma} = \tau^*(b).$$

2. Construction of Inhomogeneous Mandelbrot Measures

We define, for $(q, p) \in \mathcal{J} \times [1, \infty)$, the function

$$\varphi(p, q) = \exp\Big(\tau(pq) - p\tau(q)\Big).$$

From [5], for all nontrivial compact sets $K \subset \mathcal{J}$ there exist $1 < p_K < 2$ and $\widetilde{p}_K > 1$ such that we have

$$\sup_{q \in K} \varphi(p_K, q) < 1, \quad \text{for all } 1 < p \leq p_K, \tag{2.1}$$

and

$$\sup_{q \in K} \mathbb{E}\Big(\Big(\sum_{i=1}^{N} e^{qX_i} \Big)^{\widetilde{p}_K} \Big) < \infty. \tag{2.2}$$

Now, we will construct the inhomogeneous Mandelbrot measure. For $q \in \mathcal{J}$ and $k \geq 1$, we define $\psi_k(q, \gamma)$ as the unique t, such that

$$\tau'(t) = \tau'(q) + \eta_k(\gamma).$$

For $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$ and $q \in \mathcal{J}$ we define, for $1 \leq i \leq N_u$

$$V(ui, q) = \frac{\exp\Big(qX_{ui}\Big)}{\mathbb{E}\Big(\sum_{i=1}^{N} \exp\Big(qX_i\Big) \Big)} = \exp\Big(qX_{ui} - \tau(q)\Big)$$

and, for all $n \geq 0$

$$Y_n(q, \gamma, u) = \sum_{v_1 \cdots v_n \in \mathsf{T}_n(u)} \prod_{k=1}^{n} V\Big(u \cdot v_1 \cdots v_k, \psi_{|u|+k}(q, \gamma)\Big).$$

When $u = \emptyset$, this quantity will be denoted by $Y_n(q, \gamma)$ and when $n = 0$, their values equals 1.

The sequence $\Big(Y_n(q, \gamma, u)\Big)_{n \geq 1}$ is a positive martingale with expectation 1, which converges almost surely and in L^1 norm to a positive random variable $Y(q, \gamma, u)$ (see [9] or [10, Theorem 1]). However, our study will need the almost sure simultaneous convergence of these martingales to positive limits.

Proposition 2.1. (a) *Let $\mathsf{K} = K \times K_\gamma$ be a compact subset of $\mathcal{J} \times \Lambda_s$. There exists $p_{\mathsf{K}} \in (1, 2]$ such that for all $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$ the continuous functions $(q, \gamma) \in \mathsf{K} \mapsto Y_n(q, \gamma, u)$ converge uniformly, almost surely and in $L_{p_{\mathsf{K}}}$ norm, to a limit $(q, \gamma) \in \mathsf{K} \mapsto Y(q, \gamma, u)$. In particular, $\mathbb{E}(\sup_{(q,\gamma) \in \mathsf{K}} Y(q, \gamma, u)^{p_{\mathsf{K}}}) < \infty$. Moreover, $Y(\cdot, \cdot, u)$ is positive almost surely.*

In addition, for all $n \geq 0$, $\sigma\Big(\{(X_{u1}, \ldots, X_{uN_u}), u \in \mathsf{T}_n\}\Big)$ and $\sigma\Big(\{Y(\cdot, \cdot, u), u \in \mathsf{T}_{n+1}\}\Big)$ are independent, and the random functions $Y(\cdot, \cdot, u), u \in \mathsf{T}_{n+1}$, are independent copies of $Y(\cdot, \cdot) := Y(\cdot, \cdot, \emptyset)$.

(b) *With probability 1, for all $q \in \mathcal{J}$ and $\gamma \in \Lambda_s$, the weights*

$$\mu_q^\gamma\Big([u]\Big) = \Big[\prod_{k=1}^{n} \exp\Big(\psi_k(q, \gamma) X_{u_1 \ldots u_k} - \tau(\psi_k(q, \gamma))\Big) \Big] Y(q, \gamma, u)$$

define a measure on $\partial \mathsf{T}$, where $n = |u|$.

The measure μ_q^γ will be used to approximate from below the Hausdorff dimension of the set E_{b,s^γ}.

Proof. (a) Fix a compact $K \subset \mathcal{J}$ and a compact $K_\delta \subset \Lambda_s$. Since $\eta_k(\gamma) = \circ(1)$, we can fix, without loss of generality, a compact neighborhood $K' \subset \mathcal{J}$ of K and suppose that,

$$\forall (q,\gamma) \in \mathsf{K} = K \times K_\gamma, \quad \text{for all k} \geq 1, \psi_k(q,\gamma) \in K'.$$

Fix a compact neighborhood $\mathsf{K}'' = K'' \times K''_\gamma$ of $K' \times K_\gamma$. By (2.2), we can find $\widetilde{p}_{\mathsf{K}''} > 1$, such that

$$\sup_{q \in \mathsf{K}''} \mathbb{E}\Big(\Big(\sum_{i=1}^N e^{qX_i} \Big)^{\widetilde{p}_{\mathsf{K}''}} \Big) < \infty.$$

By (2.1), we can fix $1 < p_{\mathsf{K}} \leq \min(2, \widetilde{p}_{\mathsf{K}''})$ such that $\sup_{q \in K''} \varphi(p_{\mathsf{K}}, q) < 1$. Then for each $(q,\gamma) \in K' \times K$, there exists a neighborhood $V_q \times V_\gamma \subset \mathbb{C}^2$ of (q,γ), whose projection to $\mathbb{R}^2$ is contained in K'', and such that for all $u \in \mathsf{T}$, $(z,z') \in V_q \times V_\gamma$ and $k \geq 1$, the random variable

$$V(u,z) = \frac{\exp(zX_u)}{\mathbb{E}\Big(\sum_{i=1}^N \exp(zX_i) \Big)}, \quad \Gamma(z) = \frac{\mathbb{E}\Big(\sum_{i=1}^N X_i \exp(zX_i) \Big)}{\mathbb{E}\Big(\sum_{i=1}^N \exp(zX_i) \Big)}$$

and the analytic extension of η_k, denoted also by η_k, are well defined. For $(z,z') \in V_q \times V_\gamma$ and $k \geq 1$, we define $\psi_k(z,z')$ as the unique t such that

$$\Gamma(t) = \Gamma(z) + |\eta_k(z')|.$$

Moreover, we have

$$\sup_{z \in V_q} \varphi(p_{\mathsf{K}}, z) < 1, \quad \text{where } \varphi(p_{\mathsf{K}}, z) = \frac{\mathbb{E}\Big(\sum_{i=1}^N |e^{zX_i}|^{p_{\mathsf{K}}} \Big)}{\Big| \mathbb{E}\Big(\sum_{i=1}^N e^{zX_i} \Big) \Big|^{p_{\mathsf{K}}}}.$$

By extracting a finite covering of $K' \times K_\gamma$ from $\bigcup_{q,\gamma} V_q \times V_\gamma$, we find a neighborhood $\mathsf{V} = V_K \times V_{K\gamma} \subset \mathbb{C}^2$ of $K' \times K_\gamma$ such that

$$\sup_{z \in V_K} \varphi(p_{\mathsf{K}}, z) < 1$$

and for all $(z,z') \in \mathsf{V}$, $\psi_k(z,z')$ is defined and belongs to V_K. Since the projection of V_K to $\mathbb{R}$ is included in K'' and the mapping $z \mapsto \mathbb{E}\Big(\sum_{i=1}^N e^{zX_i} \Big)$ is continuous and does not vanish on V_K, by considering a smaller neighborhood of K' included in V_K if necessary, we can assume that

$$C_{V_K} = \sup_{z \in V_K} \mathbb{E}\Big(\Big| \sum_{i=1}^N e^{zX_i} \Big|^{p_{\mathsf{K}}} \Big) \Big| \mathbb{E}\Big(\sum_{i=1}^N e^{zX_i} \Big) \Big|^{-p_{\mathsf{K}}} < \infty.$$

Now, for $u \in \mathsf{T}$, we define the analytic extension to V of $Y_n(q,\gamma,u)$ given by

$$\begin{aligned} Y_n(z,z',u) &= \sum_{v\in\mathsf{T}_n(u)} \prod_{k=1}^n V(u\cdot v_1\cdots v_k, \psi_{|u|+k}(z,z')) \\ &= \Big[\prod_{k=1}^n \mathbb{E}\Big(\sum_{i=1}^N e^{\psi_k(z,z')X_i}\Big)\Big]^{-1} \sum_{v\in\mathsf{T}_n(u)} \prod_{k=1}^n e^{\psi_{|u|+k}(z,z')X(uv_{|k})}. \end{aligned}$$

We denote also $Y_n(z,z',\emptyset)$ by $Y_n(z,z')$. By Lemma 3 in [5], there exists a constant C_{p_K} such that for all $(z,z')\in\mathsf{V}$

$$\begin{aligned} &\mathbb{E}\Big(\,|Y_n(z,z')-Y_{n-1}(z,z')|^{p_\mathsf{K}}\Big) \\ \le & C_{p_\mathsf{K}}\mathbb{E}\left(\Big|\sum_{i=1}^N V(i,\psi_n(z,z'))\Big|^{p_\mathsf{K}}\right)\prod_{k=1}^{n-1}\mathbb{E}\left(\sum_{i=1}^N |V(i,\psi_k(z,z'))|^{p_\mathsf{K}}\right). \end{aligned}$$

Notice that $\mathbb{E}\Big(\sum_{i=1}^N |V(i,\psi_k(z,z'))|^{p_\mathsf{K}}\Big) = \varphi(p_\mathsf{K},\psi_k(z,z'))$. Then

$$\begin{aligned} \mathbb{E}\Big(\,|Y_n(z,z')-Y_{n-1}(z,z')|^{p_\mathsf{K}}\Big) &\le C_{p_\mathsf{K}}\mathbb{E}\left(\Big|\sum_{i=1}^N V(i,\psi_n(z,z'))\Big|^{p_\mathsf{K}}\right)\prod_{k=1}^{n-1}\varphi\Big(p_\mathsf{K},\psi_k(z,z')\Big). \\ &\le C_{p_\mathsf{K}} C_{V_K}\prod_{k=1}^{n-1}\sup_{z\in V_K}\varphi(p_\mathsf{K},z), \end{aligned}$$

where we have used the fact that $\psi_k(z,z')\in V_K$ for all $k\ge 1$. With probability 1, the functions $(z,z')\in\mathsf{V}\mapsto Y_n(z,z')$, $n\ge 0$, are analytic. Fix a closed polydisc $D(z_0,2\rho)\subset\mathsf{V}$ with $z_0=(z_1,z_1')$ and $\rho=(\rho_1,\rho_2)$. Theorem B.1 gives

$$\sup_{(z,z')\in D(z_0,\rho)} |Y_n(z,z')-Y_{n-1}(z,z')| \le 4\int_{[0,1]^2} |Y_n(\zeta(t))-Y_{n-1}(\zeta(t))|\,dt,$$

where, for $t=(t_1,t_2)\in[0,1]^2$

$$\zeta(t)=(\zeta_1(t_1),\zeta_2(t_2))=(z_1+\rho_1 e^{i2\pi t_1}, z_1'+\rho_2 e^{i2\pi t_2}).$$

Furthermore Jensen's inequality and Fubini's Theorem give

$$\begin{aligned} \mathbb{E}\left(\sup_{z\in D(z_0,\rho)}|Y_n(z,z')-Y_{n-1}(z,z')|^{p_\mathsf{K}}\right) &\le \mathbb{E}\left(\left(4\int_{[0,1]^2}|Y_n(\zeta(t))-Y_{n-1}(\zeta(t))|\,dt\right)^{p_\mathsf{K}}\right) \\ &\le 4^{p_\mathsf{K}}\mathbb{E}\left(\int_{[0,1]^2}|Y_n(\zeta(t))-Y_{n-1}(\zeta(t))|^{p_\mathsf{K}}\,dt\right) \\ &= 4^{p_\mathsf{K}}\int_{[0,1]^2}\mathbb{E}\,|Y_n(\zeta(t))-Y_{n-1}(\zeta(t))|^{p_\mathsf{K}}\,dt \\ &\le 4^{p_\mathsf{K}} C_{V_K} C_{p_K}\prod_{k=1}^{n-1}\sup_{z\in V_K}\varphi(p_\mathsf{K},z). \end{aligned}$$

Since $\sup_{z\in V_K} \varphi(p_{\mathsf{K}}, z) < 1$, it follows that

$$\sum_{n\geq 1} \left\| \sup_{(z,z')\in D(z_0,\rho)} |Y_n(z,z') - Y_{n-1}(z,z')| \right\|_{p_{\mathsf{K}}} < \infty.$$

This implies, $(z, z') \mapsto Y_n(z, z')$ converges uniformly, almost surely and in $L^{p_{\mathsf{K}}}$ norm over the compact $D(z_0, \rho)$ to a limit $(z, z') \mapsto Y(z, z')$. This also implies that

$$\left\| \sup_{z\in D(z_0,\rho)} Y(z,z') \right\|_{p_{\mathsf{K}}} < \infty.$$

Since K can be covered by finitely many such discs $D(z_0, \rho)$ we get the uniform convergence, almost surely and in $L^{p_{\mathsf{K}}}$ norm, of the sequence $((q, \gamma) \in \mathsf{K} \mapsto Y_n(q, \gamma))_{n\geq 1}$ to $(q, \gamma) \in \mathsf{K} \mapsto Y(q, \gamma)$. Moreover, since $\mathcal{J} \times \Lambda_s$ can be covered by a countable union of such compact K we get the simultaneous convergence for all $(q, \gamma) \in \mathcal{J} \times \Lambda_s$. The same holds simultaneously for all the functions $(q, \gamma) \in \mathcal{J} \times \Lambda_s \mapsto Y_n(q, \gamma, u)$, $u \in \bigcup_{n\geq 0} \mathbb{N}_+^n$, because $\bigcup_{n\geq 0} \mathbb{N}_+^n$ is countable.

To finish the proof of Proposition 2.1 (1), we must show that with probability 1, $(q, \gamma) \in \mathsf{K} \mapsto Y(q, \gamma)$ does not vanish. Without loss of generality we can suppose that $\mathsf{K} = [0, 1]^2$. If I is a dyadic closed subcube of $[0, 1]^2$, we denote by E_I the event $\{\exists\ (q, \gamma) \in I : Y(q, \gamma) = 0\}$. Let $I_0, I_1,\ I_2, I_3$ stand for the 2^2 dyadic intervals of I in the next generation. The event E_I being a tail event of probability 0 or 1. If we suppose that $\mathbb{P}(E_I) = 1$, then there exists $j \in \{0, 1, 2, 3\}$ such that $\mathbb{P}(E_{I_j}) = 1$. Suppose now that $\mathbb{P}(E_{\mathsf{K}}) = 1$. The previous remark allows to construct a decreasing sequence $(I(n))_{n\geq 0}$ of dyadic subcubes of K such that $\mathbb{P}(E_{I(n)}) = 1$. Let (q_0, γ_0) be the unique element of $\cap_{n\geq 0} I(n)$. Since $(q, \gamma) \mapsto Y(q, \gamma)$ is continuous we have $\mathbb{P}(Y(q_0, \gamma_0) = 0) = 1$, which contradicts the fact that $(Y_n(q_0, \gamma_0))_{n\geq 1}$ converges to $Y(q_0, \gamma_0)$ in L^1.

(b) It is a consequence of the branching property

$$Y_{n+1}(q, \gamma, u) = \sum_{i=1}^{N} \exp\Big(\psi_{n+1}(q, \gamma) X_{ui} - \tau(\psi_{n+1}(q, \gamma))\Big)\, Y_n(q, \gamma, ui). \qquad \square$$

3. Proof of Theorem 1.3

The proof of Theorem 1.3 can be deduced from the two following propositions. Their proof are developed in the next section.

Proposition 3.1. *Suppose Hypothesis* 1.2*, with probability* 1*, for all* $q \in \mathcal{J}$ *and* $\gamma \in \Lambda_s$*,*

$$\mathsf{N}_n(t) - nb \sim s_{n,\gamma}, \qquad \textit{for } \mu_q^\gamma\textit{-almost every } t \in \partial\mathsf{T},$$

where $b = \tau'(q)$*.*

Proposition 3.2. *With probability* 1*, for all* $(q, \gamma) \in \mathcal{J} \times \Lambda_s$*, for* μ_q^γ*-almost every* $t \in \partial\mathsf{T}$

$$\lim_{n\to\infty} \frac{\log Y(q, \gamma, t_{|n})}{n} = 0.$$

From Proposition 3.1, we have with probability 1, for all $q \in \mathcal{J}$ and $\gamma \in \Lambda_s$, that $\mu_q^\gamma\Big(E_{b,s^\gamma}\Big) = 1, \quad (b = \tau'(q))$. In addition, with probability 1, for all $(q,\gamma) \in \mathcal{J} \times \Lambda_s$, for μ_q^γ-almost every $t \in E_{b,s^\gamma}$, from the same Proposition and proposition 3.2, we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\log(\mu_q^\gamma[t_{|n}])}{\log(\operatorname{diam}([t_{|n}]))} \\
&= \lim_{n\to\infty} -\frac{1}{n} \log\left(\prod_{k=1}^{n} \exp\Big(\psi_k(q,\gamma) X_{t_1\dots t_k} - \tau(\psi_k(q,\gamma))\Big) Y(q,\gamma,t_{|n}) \right) \\
&= \lim_{n\to\infty} -\frac{1}{n} \sum_{k=1}^{n} \psi_k(q,\gamma) X_{t_1\dots t_k} + \frac{1}{n}\sum_{k=1}^{n} \tau(\psi_k(q,\gamma)) - \frac{\log Y(q,\gamma,t_{|n})}{n} \\
&= \lim_{n\to\infty} -\frac{1}{n} \sum_{k=1}^{n} \psi_k(q,\gamma) X_{t_1\dots t_k} + \frac{1}{n}\sum_{k=1}^{n} \tau(\psi_k(q,\gamma)).
\end{aligned}$$

Since $\eta_k(\gamma) = \circ(1)$ and then $\psi_k(q,\gamma) \to q$, we get

$$\lim_{n\to\infty} \frac{\log(\mu_q^\gamma[t_{|n}])}{\log(\operatorname{diam}([t_{|n}]))} = -q\tau'(q) + \tau(q) = \tau^*(\tau'(q)).$$

We deduce the result from the mass distribution principle (Theorem A.1) and (1.1).

4. Proof of Propositions 3.1 and 3.2

4.1. **Proof of Proposition** 3.1. Let $\mathsf{K} = K \times K_\gamma$ be a compact subset of $\mathcal{J} \times \Lambda_s$. For $b = \tau'(q)$, $q \in \mathcal{J}$, $\gamma \in \Lambda_s$, $n \geq 1$, $\epsilon > 0$ and $s^\gamma = (s_{n,\gamma})_{n\geq 1}$ we set

$$\begin{aligned}
E^1_{b,n,\gamma,\epsilon} &= \Big\{ t \in \partial\mathsf{T} : \sum_{k=1}^{n} \Big(X_{t_1\cdots t_k}(t) - b - \eta_k(\gamma) \Big) \geq \epsilon \sum_{k=1}^{n} \eta_k(\gamma) \Big\}, \\
E^{-1}_{b,n,\gamma,\epsilon} &= \Big\{ t \in \partial\mathsf{T} : \sum_{k=1}^{n} \Big(X_{t_1\cdots t_k}(t) - b - \eta_k(\gamma) \Big) \leq -\epsilon \sum_{k=1}^{n} \eta_k(\gamma) \Big\}.
\end{aligned}$$

Suppose that we have shown that for, $\lambda \in \{-1, 1\}$, we have:

$$\mathbb{E}\Big(\sup_{(q,\gamma)\in\mathsf{K}} \sum_{n\geq 1} \mu_q^\gamma(E^\lambda_{b,n,\gamma,\epsilon}) \Big) < \infty. \tag{4.1}$$

Then, with probability 1, for all $(q,\gamma) \in \mathcal{J} \times \Lambda_s$, $\lambda \in \{-1,1\}$, and $\epsilon \in \mathbb{Q}_+^*$,

$$\sum_{n\geq 1} \mu_q^\gamma(E^\lambda_{b,n,\gamma,\epsilon}) < \infty,$$

consequently, by the Borel-Cantelli lemma, for μ_q^γ-almost every t, we have

$$\sum_{k=1}^{n} X_{t_1\cdots t_k}(t) - b - \eta_k(\gamma) = \circ\Big(\sum_{k=1}^{n} \eta_k(\gamma) \Big), \quad \text{so } \mathsf{N}_n(t) - nb \sim s_{n,\gamma},$$

which yields the desired result.

Let us prove (4.1) when $\lambda = 1$ (the case $\lambda = -1$ is similar). Let $\theta = (\theta_n)$ be a positive sequence and $(q, \gamma) \in \mathsf{K}$. One has

$$\sup_{(q,\gamma)\in\mathsf{K}} \mu_q^\gamma\Big(E^1_{b,n,\gamma,\epsilon}\Big) \le \sup_{(q,\gamma)\in\mathsf{K}} \sum_{u\in\mathsf{T}_n} \mu_q^\gamma([u])\, \mathbf{1}_{\Big\{E^1_{b,n,\gamma,\epsilon}\Big\}}(t_u),$$

where t_u is any point in $[u]$. Denote t_u simply by t, then

$$\begin{aligned}
&\sup_{(q,\gamma)\in\mathsf{K}} \mu_q^\gamma\Big(E^1_{b,n,\gamma,\epsilon}\Big)\\
&\le \sup_{(q,\gamma)\in\mathsf{K}} \sum_{u\in\mathsf{T}_n} \mu_q^\gamma[u] \prod_{k=1}^n \exp\Big(\theta_k X_{t_1\cdots t_k} - \theta_k b - \theta_k \eta_k(\gamma)(1+\epsilon)\Big)\\
&\le \sup_{(q,\gamma)\in\mathsf{K}} \sum_{u\in\mathsf{T}_n} \prod_{k=1}^n \exp\Big((\psi_k(q,\gamma)+\theta_k)X_{t_1\cdots t_k} - \tau(\psi_k(q,\gamma)) - \theta_k b - \theta_k\eta_k(\gamma)(1+\epsilon)\Big)\\
&\quad \times \quad Y(q,\gamma,u).
\end{aligned}$$

For $(q, \gamma) \in \mathsf{K}$, $\theta = (\theta_n)$ and $n \ge 1$, we set

$$\begin{aligned}
&H_n(q,\gamma,\theta)\\
&= \sum_{u\in\mathsf{T}_n} \prod_{k=1}^n \exp\Big((\psi_k(q,\gamma)+\theta_k)X_{t_1\cdots t_k} - \tau(\psi_k(q,\gamma)) - \theta_k b - \theta_k\eta_k(\gamma)(1+\epsilon)\Big) M(u),
\end{aligned}$$

where

$$M(u) = \sup_{(q,\gamma)\in\mathsf{K}} Y(q,\gamma,u).$$

Recall the proof of Proposition 2.1, there exists a neighborhood $\mathsf{V} = V_K \times V_{K_\gamma} \subset \mathbb{C}^2$ of $\mathsf{K} = K \times K_\gamma$ such that

$$\Gamma(z) = \frac{\mathbb{E}\Big(\sum_{i=1}^N X_i \exp(zX_i)\Big)}{\mathbb{E}\Big(\sum_{i=1}^N \exp(zX_i)\Big)}$$

is well defined for $z \in V_K$, for $k \ge 1$, $\eta_k(z')$ is defined for $z' \in V_{K_\gamma}$ and $\forall (z, z') \in \mathsf{V}$, $\psi_k(z, z')$ is defined and belongs to V_K.

For $\epsilon > 0$, $(z, z') \in \mathsf{V}$ and $n \ge 1$, we define

$$\begin{aligned}
H_n(z,z',\theta) = \sum_{u\in\mathsf{T}_n} \prod_{k=1}^n &\exp\Big((\psi_k(z,z')+\theta_k)X_{u_{|k}} - \theta_k\Gamma(z) - \theta_k\eta_k(z')(1+\epsilon)\Big)\\
&\times \mathbb{E}\Big(\sum_{i=1}^N \exp\Big(\psi_k(z,z')X_i\Big)\Big)^{-1} M(u).
\end{aligned}$$

Proposition 4.1. *There exist a neighborhood* $\mathsf{V}' \subset \mathsf{V}$ *of* K*, a positive constant* $\mathcal{C}_\mathsf{K}$ *and a positive sequence* θ *such that for all* $(z, z') \in \mathsf{V}'$*, for all* $n \in \mathbb{N}^*$

$$\mathbb{E}(|H_n(z,z',\theta)|) \le \mathcal{C}_\mathsf{K} \exp\Big(-\frac{\epsilon}{4}\sum_{k=1}^n \epsilon_k \widetilde{\eta}_k^2\Big),$$

where the sequences $(\epsilon_n)_n$ *and* $(\widetilde{\eta}_n)_n$ *are the sequences used in Hypothesis* 1.2.

Lemma 4.1. *There exist a positive sequence* $\theta = (\theta_n)$ *and a positive constant* $\mathcal{C}_{\mathsf{K}}$ *such that for all* $(q,\gamma) \in \mathsf{K}$ *we have*

$$\mathbb{E}\Big(H_n(q,\gamma,\theta)\Big) \le \mathcal{C}_{\mathsf{K}} \exp\Big(-\frac{\epsilon}{2}\sum_{k=1}^n \epsilon_k \widetilde{\eta}_k^2\Big).$$

Proof of Lemma 4.1. Let $\theta = (\theta_n)$ be a positive sequence, clearly we have

$$\begin{aligned}\mathbb{E}\Big(H_n(q,\gamma,\theta)\Big) =& \prod_{k=1}^n \mathbb{E}\Big(\sum_{i=1}^N \exp\Big((\psi_k(q,\gamma)+\theta_k)X_i\Big)\\ &\times \exp\Big(-\tau(\psi_k(q,\gamma)) - \theta_k b - \theta_k\eta_k(\gamma)(1+\epsilon)\Big)\mathbb{E}(M(u))\\ \le& \mathcal{C}'_{\mathsf{K}} \prod_{k=1}^n \exp\Big(\tau(\psi_k(q,\gamma)+\theta_k) - \tau(\psi_k(q,\gamma)) - \theta_k b - \theta_k\eta_k(\gamma)(1+\epsilon)\Big),\end{aligned}$$

where, by Proposition 2.1, $\mathcal{C}'_{\mathsf{K}} = \mathbb{E}\Big(M(u)\Big) = \mathbb{E}\Big(M(\emptyset)\Big) < \infty$ for all $u \in \bigcup_{n\ge 0}\mathbb{N}_+^n$.

Since $\eta_k(\gamma) = \mathrm{o}(1)$, we can fix a compact neighborhood K' of K and suppose that for all $k \ge 1$ and $(q,\gamma) \in \mathsf{K}$, we have $\psi_k(q,\gamma) \in K'$. For $(q,\gamma)\in\mathsf{K}$ and $k \ge 1$, writing the Taylor expansion with integral rest of order 2 of the function $g : \theta \mapsto \tau(\psi_k(q,\gamma)+\theta)$ at 0, we get

$$g(\theta) = g(0) + \theta g'(0) + \theta^2 \int_0^1 (1-t)g''(t\theta)dt,$$

with $g''(t\theta) \le m_{\mathsf{K}} = \sup_{t\in[0,1]}\sup_{q\in K'}\sup_{\gamma\in K_\gamma} g''(t\theta)$. It follows that for all $k \ge 1$

$$\tau(\psi_k(q,\gamma)+\theta_k) - \tau((\psi_k(q,\gamma)) - \theta_k\tau'((\psi_k(q,\gamma)) \le \theta_k^2 m_{\mathsf{K}}.$$

Recall that $\tau'(\psi_k(q,\gamma)) = \tau'(q) + \eta_k(\gamma)$. Then

$$\begin{aligned}\mathbb{E}\Big(H_n(q,\gamma,\theta)\Big) &\le \mathcal{C}'_{\mathsf{K}} \prod_{k=1}^n \exp\Big(\tau(\psi_k(q,\gamma)+\theta_k) - \tau(\psi_k(q,\gamma)) - \theta_k b - \theta_k\eta_k(\gamma)(1+\epsilon)\Big),\\ &\le \mathcal{C}'_{\mathsf{K}} \prod_{k=1}^n \exp\Big(-\theta_k\eta_k(\gamma)\epsilon + \theta_k^2 m_{\mathsf{K}}\Big).\end{aligned}$$

Choose the sequence θ such that $\theta_k = \epsilon_k\widetilde{\eta}_k$. Then

$$\mathbb{E}\Big(H_n(q,\gamma,\theta)\Big) \le \mathcal{C}'_{\mathsf{K}} \prod_{k=1}^n \exp\Big(-\epsilon_k\widetilde{\eta}_k^2(\epsilon - \epsilon_k m_{\mathsf{K}})\Big).$$

Since $\epsilon_k \to 0$ then for k large enough we have $\epsilon - \epsilon_k m_{\mathsf{K}} > \dfrac{\epsilon}{2}$. Then, there exists a constant $\mathcal{C}_{\mathsf{K}}$ such that

$$\mathbb{E}\Big(H_n(q,\gamma,\theta)\Big) \le \mathcal{C}_{\mathsf{K}} \exp\Big(-\frac{\epsilon}{2}\sum_{k=1}^n \epsilon_k\widetilde{\eta}_k^2\Big). \qquad \square$$

Proof of Proposition 4.1. Since $\mathbb{E}(|H_n(q,\gamma,\theta)|) \leq \mathcal{C}_{\mathsf{K}} \exp\Big(-\frac{\epsilon}{2}\sum_{k=1}^n \epsilon_k \widetilde{\eta}_k^2\Big)$ for $q \in K$, there exists a neighborhood $\mathsf{V}_{q,\gamma} \subset \mathsf{V}$ of (q,γ) such that for all $(z,z') \in \mathsf{V}_{q,\gamma}$ we have

$$\mathbb{E}(|H_n(z,z',\theta)|) \leq \mathcal{C}_{\mathsf{K}} \exp\Big(-\frac{\epsilon}{4}\sum_{k=1}^n \epsilon_k \widetilde{\eta}_k^2\Big).$$

By extracting a finite covering of K from $\bigcup_{(q,\gamma)\in\mathsf{K}} V_{q,\gamma}$, we find a neighborhood $\mathsf{V}' \subset \mathsf{V}$ of K such that

$$\mathbb{E}(|H_n(z,z',\theta)|) \leq \mathcal{C}_{\mathsf{K}} \exp\Big(-\frac{\epsilon}{4}\sum_{k=1}^n \epsilon_k \widetilde{\eta}_k^2\Big). \qquad \square$$

With probability 1, the functions $(z,z') \in \mathsf{V}' \mapsto H_n(z,z',\theta)$ are analytic. Fix a closed polydisc $D(z_0, 2\rho) \subset V$, with $z_0 = (z_1, z_1')$ and $\rho = (\rho_1, \rho_2)$. Theorem B.1 gives

$$\sup_{(z,z')\in D(z_0,\rho)} \Big|H_n(z,z',\theta)\Big| \leq 2\int_{[0,1]^2} \Big|H_n(\zeta(t),\theta)\Big| dt,$$

where for $t = (t_1,t_2) \in [0,1]^2$

$$\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2)) = (z_1 + \rho_1 e^{i2\pi t_1}, z_1' + \rho_2 e^{i2\pi t_2}).$$

Furthermore Fubini's Theorem gives

$$\begin{aligned}\mathbb{E}\left(\sup_{z\in D(z_0,\rho)} |H_n^s(z,z',\theta)|\right) &\leq \mathbb{E}\Big(2\int_{[0,1]^2} |H_n(\zeta(t),\theta)|\, dt\Big)\\ &\leq 4\int_{[0,1]^2} \mathbb{E}\,|H_n(\zeta(t),\theta)|\, dt\\ &\leq 4\exp\left(-\frac{\epsilon}{4}\sum_{k=1}^n \epsilon_k \widetilde{\eta}_k^2\right).\end{aligned}$$

Finally, we get

$$\mathbb{E}\left(\sup_{(q,\gamma)\in\mathsf{K}} \mu_q^\gamma\left(E^1_{b,n,\gamma,\epsilon}\right)\right) \leq 4\exp\Big(-\frac{\epsilon}{4}\sum_{k=1}^n \epsilon_k \widetilde{\eta}_k^2\Big)$$

and, then, under Hypothesis 1.2, we get (4.1), which finish the proof of Proposition 3.1.

4.2. **Proof of Propostion** 3.2. Let $\mathsf{K} = K \times K_\gamma$ be a compact subset of $\mathcal{J} \times \Lambda_s$. For $a > 1$, $(q,\gamma) \in \mathsf{K}$ and $n \geq 1$, we set

$$E^+_{n,a} = \Big\{t \in \partial\mathsf{T} : Y(q,\gamma,t_{|n}) > a^n\Big\}$$

and

$$E^-_{n,a} = \Big\{t \in \partial\mathsf{T} : Y(q,\gamma,t_{|n}) < a^{-n}\Big\}.$$

It is sufficient to show that for $E \in \{E^+_{n,a}, E^-_{n,a}\}$

$$\mathbb{E}\Big(\sup_{(q,\gamma)\in\mathsf{K}} \sum_{n\geq 1} \mu_q^\gamma(E)\Big) < \infty. \tag{4.2}$$

Indeed, if this holds, then with probability 1, for each $(q,\gamma)\in\mathsf{K}$ and $E\in\{E^+_{n,a},E^-_{n,a}\}$, $\sum_{n\ge 1}\mu_q^\gamma(E)<\infty$, hence by the Borel-Cantelli lemma, for μ_q^γ-almost every $t\in\partial\mathsf{T}$, if n is big enough we have

$$-\log a\le\liminf_{n\to\infty}\frac{1}{n}\log Y(q,\gamma,t_{|n})\le\limsup_{n\to\infty}\frac{1}{n}\log Y(q,\gamma,t_{|n})\le\log a.$$

Letting a tend to 1 along a countable sequence yields the result.

Let us prove (4.2) for $E=E^+_{n,a}$ (the case $E=E^-_{n,a}$ is similar). At first we have,

$$\begin{aligned}\sup_{(q,\gamma)\in\mathsf{K}}\mu_q^\gamma(E^+_{n,a})&=\sup_{(q,\gamma)\in\mathsf{K}}\sum_{u\in\mathsf{T}_n}\mu_q^\gamma([u])\mathbf{1}_{\left\{Y(q,\gamma,u)>a^n\right\}}\\
&=\sup_{(q,\gamma)\in\mathsf{K}}\sum_{u\in\mathsf{T}_n}Y(q,\gamma,u)\prod_{k=1}^n\exp\Big(\psi_k(q,\gamma)X(u)-\tau\big(\psi_k(q,\gamma)\big)\Big)\mathbf{1}_{\left\{Y(q,\gamma,u)>a^n\right\}}\\
&\le\sup_{(q,\gamma)\in\mathsf{K}}\sum_{u\in\mathsf{T}_n}(Y(q,\gamma,u))^{1+\nu}\prod_{k=1}^n\exp\Big(\psi_k(q,\gamma)X_u-\tau\big((\psi_k(q,\gamma)\big)\Big)a^{-\nu},\\
&\le\sup_{(q,\gamma)\in\mathsf{K}}\sum_{u\in\mathsf{T}_n}M(u)^{1+\nu}\prod_{k=1}^n\exp\Big(\psi_k(q,\gamma)X_u-\tau\big(\psi_k(q,\gamma)\big)\Big)a^{-\nu},\end{aligned}$$

where $M(u)=\sup_{(q,\gamma)\in\mathsf{K}}Y(q,\gamma,u)$ and $\nu>0$ is an arbitrary parameter. For $q\in K$, $\gamma\in K_\gamma$ and $\nu>0$ we set

$$L_n(q,\gamma,\nu)=\sum_{u\in\mathsf{T}_n}M(u)^{1+\nu}\prod_{k=1}^n\exp\Big(\psi_k(q,\gamma)X_u-\tau\big(\psi_k(q,\gamma)\big)\Big)a^{-\nu}.$$

Recall the proof of Proposition 2.1, there exists a neighborhood $\mathsf{V}\subset\mathbb{C}^2$ of K such that for all $(z,z')\in\mathsf{V}$ and $k\ge 1$ $\psi_k(z,z')$ is well defined and $\mathbb{E}\left(\sum_{i=1}^N e^{\psi_k(z,z')X_i}\right)\ne 0$.

Lemma 4.2. *Fix $a>1$. For $(z,z')\in\mathsf{V}$ and $\nu>0$, let*

$$\begin{aligned}L_n(z,z',\nu)=&\left[\prod_{k=1}^n\mathbb{E}\bigg(\sum_{i=1}^N\exp\big(\psi_k(z,z')X_i\big)\bigg)^{-1}\right]\\
&\times\sum_{u\in\mathsf{T}_n}M(u)^{1+\nu}\prod_{k=1}^n\exp\Big(\psi_k(z,z')X_{u_{|k}}\Big)a^{-\nu}.\end{aligned}$$

There exist a neighborhood $\mathsf{V}'\subset\mathbb{C}^2$ of K and a positive constant C_{K} such that, for all $(z,z')\in\mathsf{V}'$, for all integer $n\ge 1$

$$\mathbb{E}\Big(\Big|L_n(z,z',p_{\mathsf{K}}-1)\Big|\Big)\le C_{\mathsf{K}}a^{-n(p_{\mathsf{K}}-1)/4},\tag{4.3}$$

where p_{K} provided by Proposition 2.1.

Proof. Write $V=V_K\times V_{K_\gamma}$. For $z\in V_K$ and $\nu>0$, let

$$\widetilde{L}_1(z,\nu)=\left|\mathbb{E}\bigg(\sum_{i=1}^N\exp\big(zX_i\big)\bigg)\right|^{-1}\mathbb{E}\bigg(\sum_{i=1}^N\Big|\exp\big(zX_i\big)\Big|\bigg)\ a^{-\nu}.$$

Let $q \in K$. Since $\mathbb{E}(\widetilde{L}_1(q,\nu)) = a^{-\nu}$, there exists a neighborhood $V_q \subset V_K$ of q such that for all $z \in V_q$ we have $\mathbb{E}\Big(\Big|\widetilde{L}_1(z,\nu)\Big|\Big) \leq a^{-\nu/2}$. Let $\gamma \in K_\gamma$. Recall the proof of Proposition 2.1 and since $\eta_k(\gamma) = \circ(1)$, we can find a neighborhood $V_\gamma \subset V_{K_\gamma}$ of K_γ such that, for all $k \geq 1$, $(z,z') \in V_q \times V_\gamma$, we have

$$\mathbb{E}\Big(\Big|\widetilde{L}_1(\psi_k(z,z'),\nu)\Big|\Big) \leq a^{-\nu/3}.$$

By extracting a finite covering of K from $\bigcup_{(q,\gamma)} V_q \times V_\gamma$, we find a neighborhood $\mathsf{V}' \subset \mathsf{V}$ of K such that for all $(z,z') \in \mathsf{V}'$ and $k \geq 1$

$$\mathbb{E}\Big(\Big|\widetilde{L}_1(\psi_k(z,z'),\nu)\Big|\Big) \leq a^{-\nu/4}.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\Big(\Big|L_n(z,z',\nu)\Big|\Big)\\
&= \left[\prod_{k=1}^n \left|\mathbb{E}\Big(\sum_{i=1}^N \exp\Big(\psi_k(z,z')X_i\Big)\Big)\right|^{-1}\right] \mathbb{E}\left(\left|\sum_{u\in\mathsf{T}_n} M(u)^{1+\nu}\prod_{k=1}^n \exp\Big(\psi_k(z,z')X_u\Big)\right|\right) a^{-n\nu}\\
&\leq \left[\prod_{k=1}^n \left|\mathbb{E}\Big(\sum_{i=1}^N \exp\Big(\psi_k(z,z')X_i\Big)\Big)\right|^{-1}\right] \mathbb{E}\left(\sum_{u\in\mathsf{T}_n} M(u)^{1+\nu}\prod_{k=1}^n \left|\exp\Big(\psi_k(z,z')X_u\Big)\right|\right) a^{-n\nu}.
\end{aligned}$$

By Proposition 2.1, there exists $p_{\mathsf{K}} \in (1,2]$ such that for all $u \in \bigcup_{n\geq 0}\mathbb{N}_+^n$,

$$\mathbb{E}\Big(M(u)^{p_{\mathsf{K}}}\Big) = \mathbb{E}\Big(M(\emptyset)^{p_{\mathsf{K}}}\Big) = C_{\mathsf{K}} < \infty.$$

Now take $\nu = p_{\mathsf{K}} - 1$ in the last calculation, it follows, from the independence of $\sigma\Big(\{Y(\cdot,\cdot,u),\, u \in \mathsf{T}_n\}\Big)$ and $\sigma\Big(\{(X_{u1},\dots,X_{uN_u}),\, u \in \mathsf{T}_{n-1}\}\Big)$ for all $n \geq 1$, that

$$\begin{aligned}
&\mathbb{E}\Big(\Big|L_n(z,z',p_{\mathsf{K}}-1)\Big|\Big)\\
&\leq \left[\prod_{k=1}^n \left|\mathbb{E}\Big(\sum_{i=1}^N \exp\Big(\psi_k(z,z')X_i\Big)\Big)\right|^{-1}\right] \prod_{k=1}^n \mathbb{E}\Big(\sum_{i=1}^N \Big|\exp\Big(\psi_k(z,z')X_i\Big)\Big|\Big)^n C_{\mathsf{K}} a^{-n(p_{\mathsf{K}}-1)}\\
&= C_{\mathsf{K}} \prod_{k=1}^n \mathbb{E}\Big(\Big|\widetilde{L}_1(\psi_k(z,z'),p_{\mathsf{K}}-1)\Big|\Big)\\
&\leq C_{\mathsf{K}} a^{-n(p_K-1)/4},
\end{aligned}$$

then lemma is now proved. □

With probability 1, the functions $(z,z') \in \mathsf{V}' \mapsto L_n(z,z',\nu)$ are analytic. Fix a closed polydisc $D(z_0,2\rho) \subset \mathsf{V}'$, with $z_0 = (z_1,z_1')$ and $\rho = (\rho_1,\rho_2)$. Theorem B.1 gives

$$\sup_{z\in D(z_0,\rho)} \Big|L_n(z,p_{\mathsf{K}}-1)\Big| \leq 4\int_{[0,1]^2} \Big|L_n(\zeta(t),p_{\mathsf{K}}-1)\Big| dt,$$

where, for $t = (t_1, t_2) \in [0, 1]^2$

$$\zeta(t) = (\zeta_1(t_1), \zeta_2(t_2)) = (z_1 + \rho_1 e^{i2\pi t_1}, z_1' + \rho_2 e^{i2\pi t_2}).$$

Furthermore Fubini's Theorem gives

$$\begin{aligned}\mathbb{E}\left(\sup_{z\in D(z_0,\rho)} |L_n(z, p_{\mathsf{K}} - 1)|\right) &\le \mathbb{E}\left(4\int_{[0,1]^2} |L_n(\zeta(t), p_{\mathsf{K}} - 1)|\, dt\right)\\ &\le 4\int_{[0,1]^2} \mathbb{E}\,|L_n(\zeta(t), p_{\mathsf{K}} - 1)|\, dt\\ &\le 4C_{\mathsf{K}} a^{-n(p_{\mathsf{K}}-1)/4}.\end{aligned}$$

Since $a > 1$ and $p_{\mathsf{K}} - 1 > 0$, we get (4.2).

Appendix A. Hausdorff and Packing Dimensions

Given a subset K of $\mathbb{N}_+^{\mathbb{N}_+}$ endowed with a metric d making it σ-compact, $s > 0$ and E a subset of K, the s-dimensional Hausdorff measure of E is defined as

$$\mathcal{H}^s(E) = \lim_{\delta\to 0^+} \inf\left\{\sum_{i\in\mathbb{N}} (\mathrm{diam}(U_i)^s\right\},$$

the infimum being taken over all the countable coverings $(U_i)_{i\in\mathbb{N}}$ of E by subsets of K of diameters less than or equal to δ. Then, the Hausdorff dimension of E is defined as

$$\dim E = \sup\{s > 0 : \mathcal{H}^s(E) = \infty\} = \inf\{s > 0 : \mathcal{H}^s(E) = 0\},$$

with the convention $\sup\emptyset = 0$ and $\inf\emptyset = \infty$.

Packing measures and dimensions are defined as follows. Given $s > 0$ and $E \subset K$ as above, one first defines

$$\overline{P}^s(E) = \lim_{\delta\to 0^+} \sup\left\{\sum_{i\in\mathbb{N}} (\mathrm{diam}(B_i)^s\right\},$$

the supremum being taken over all the packings $\{B_i\}_{i\in\mathbb{N}}$ of E by balls centered on E and with diameter smaller than or equal to δ. Then, the s-dimensional packing measure of E is defined as

$$P^s(E) = \lim_{\delta\to 0^+} \inf\left\{\sum_{i\in\mathbb{N}} \overline{P}^s(E_i)\right\},$$

the infimum being taken over all the countable coverings $(E_i)_{i\in\mathbb{N}}$ of E by subsets of K of diameters less than or equal to δ. Then, the packing dimension of E is defined as

$$\mathrm{Dim}\, E = \sup\{s > 0 : P^s(E) = \infty\} = \inf\{s > 0 : P^s(E) = 0\},$$

with the convention $\sup\emptyset = 0$ and $\inf\emptyset = \infty$. For more details the reader is referred to [13, 20].

If μ is a positive and finite Borel measure supported on K, then its lower Hausdorff and packing dimensions is defined as

$$\underline{\dim}(\mu) = \inf\Big\{ \dim F : F \text{ Borel}, \ \mu(F) > 0\Big\}$$
$$\underline{\operatorname{Dim}}(\mu) = \inf\Big\{\operatorname{Dim} F : F \text{ Borel}, \ \mu(F) > 0\Big\}$$

and its upper Hausdorff and packing dimensions are defined as

$$\overline{\dim}(\mu) = \inf\Big\{ \dim F : F \text{ Borel}, \ \mu(F) = \|\mu\|\Big\}$$
$$\overline{\operatorname{Dim}}(\mu) = \inf\Big\{\operatorname{Dim} F : F \text{ Borel}, \ \mu(F) = \|\mu\|\Big\}.$$

We have (see [12])

$$\underline{\dim}(\mu) = \operatorname{ess\,inf}_\mu \liminf_{r\to 0^+} \frac{\log \mu(B(t,r))}{\log(r)},$$
$$\underline{\operatorname{Dim}}(\mu) = \operatorname{ess\,inf}_\mu \limsup_{r\to 0^+} \frac{\log \mu(B(t,r))}{\log(r)}$$

and

$$\overline{\dim}(\mu) = \operatorname{ess\,sup}_\mu \liminf_{r\to 0^+} \frac{\log \mu(B(t,r))}{\log(r)},$$
$$\overline{\operatorname{Dim}}(\mu) = \operatorname{ess\,sup}_\mu \limsup_{r\to 0^+} \frac{\log \mu(B(t,r))}{\log(r)},$$

where $B(t,r)$ stands for the closed ball of radius r centered at t. If $\underline{\dim}(\mu) = \overline{\dim}(\mu)$ (resp. $\underline{\operatorname{Dim}}(\mu) = \overline{\operatorname{Dim}}(\mu)$), this common value is denoted $\dim\mu$ (resp. $\operatorname{Dim}(\mu)$), and if $\dim\mu = \operatorname{Dim}\mu$, one says that μ is exact dimensional.

Recall the mass distribution principle.

Theorem A.1. ([13, Theorem 4.2]). *Let ν be a positive and finite Borel probability measure on a compact metric space (X,d). Assume that $M \subseteq X$ is a Borel set such that $\nu(M) > 0$ and*

$$M \subseteq \left\{ t \in X : \liminf_{r\to 0^+} \frac{\log \nu(B(t,r))}{\log r} \geq \delta \right\}.$$

Then the Hausdorff dimension of M is bounded from below by δ.

Appendix B. Cauchy Formula in Several Variables

Let us recall the Cauchy formula for holomorphic functions in several variables.

Definition B.1. Let $d \geq 1$, a subset D of $\mathbb{C}^d$ is an open polydisc if there exist open discs $D_1, \dots, D_d$ of $\mathbb{C}$ such that $D = D_1 \times \cdots \times D_d$. If we denote by ζ_j the centre of D_j, then $\zeta = (\zeta_1, \dots, \zeta_d)$ is the centre of D and if r_j is the radius of D_j then $r = (r_1, \dots, r_d)$ is the multiradius of D. The set $\partial D = \partial D_1 \times \cdots \times \partial D_d$ is the distinguished boundary of D. We denote by $D(\zeta, r)$ the polydisc with center ζ and radius r.

Let $D = D(\zeta, r)$ be a polydisc of $\mathbb{C}^d$ and $g \in C(\partial D)$ a continuous function on ∂D. We define the integral of g on ∂D as

$$\int_{\partial D} g(\zeta) d\zeta_1 \cdots d\zeta_d = (2i\pi)^d r_1 \cdots r_d \int_{[0,1]^d} g(\zeta(\theta)) e^{i2\pi\theta_1} \cdots e^{i2\pi\theta_d} d\theta_1 \cdots d\theta_d,$$

where $\zeta(\theta) = (\zeta_1(\theta), \ldots, \zeta_d(\theta))$ and $\zeta_j(\theta) = \zeta_j + r_j e^{i2\pi\theta_j}$ for $j = 1, \ldots, d$.

Theorem B.1. *Let $D = D(a, r)$ be polydisc in $\mathbb{C}^d$ with a multiradius whose components are positive, and f be a holomorphic function in a neiborhood of D. Then, for all $z \in D$*

$$f(z) = \frac{1}{(2i\pi)^d} \int_{\partial D} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_d}{(\zeta_1 - z_1) \cdots (\zeta_d - z_d)}.$$

It follows that

$$\sup_{z \in D(a, r/2)} |f(z)| \leq 2^d \int_{[0,1]^d} |f(\zeta(\theta))| \, d\theta_1 \cdots d\theta_d.$$

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