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EXISTENCE OF CLASSICAL SOLUTIONS FOR BROER-KAUP EQUATIONS

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Abstract. In this paper we investigate the Cauchy problem for one dimensional Broer-Kaup equations for existence of global classical solutions. We give conditions under which the considered equations have at least one and at least two classical solutions. To prove our main results we propose a new approach based upon recent theoretical results.

1. Introduction

Study of existence of global classical solutions of nonlinear models is one of the important works in nonlinear science. In this paper, we investigate the Cauchy problem for a model describing the bi-directional propagation of long waves in shallow water which was proposed by Broer and Kaup [2, 9] and called Broer-Kaup (BK) equations. Namely, we are concerned with the following system:

$$\begin{aligned} u_t + uu_x + v_x &= 0, \quad t \in (0,\infty), x \in \mathbb{R},\\ v_t + u_x + 2(uv)_x + u_{xxx} &= 0, \quad t \in (0,\infty), x \in \mathbb{R},\\ u(0,x) &= u_0(x), \quad x \in \mathbb{R},\\ v(0,x) &= v_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where

(H1): $u_0, v_0 \in \mathcal{C}^1(\mathbb{R})$, $0 \le u_0, v_0 \le B$ on $\mathbb{R}$ for some positive constant B.

Here the unknowns $u = u(t,x)$ and $v = v(t,x)$ denote respectively, the horizontal velocity and the elevation of the water wave. The Broer-Kaup equations of system

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(1.1) can be obtained from the symmetry constraints of the Kadomtsev-Petviashvili (KP) equation and are a mathematical model of many nonlinear waves, see [12]. More precisely, they describe the evolution of the horizontal velocity component $u(t,x)$ of water waves of height $v(t,x)$ propagating in both directions in an infinite narrow channel of finite constant depth. Several methods have been used to capture different nature of solutions contained in Broer-Kaup equations like traveling wave solutions, periodic wave solutions, dromion solutions, solitary wave solutions and soliton-like solutions, see [21] and [29]. By qualitative analysis method, a sufficient condition for the existence of peaked periodic wave solutions to the Broer-Kaup equations was given in [8] and some exact explicit expressions of peaked periodic wave solutions were also presented. In [16], fission and fusion phenomena were revealed and soliton solutions were obtained. A family of traveling wave solutions is given in [18,19] and [7]. Solitary wave solutions to the Broer-Kaup equations are considered in [14] by using the first integral method. By application of the sub-ode method [25], new and more general form solutions are obtained for the Broer-Kaup equations. Using a consistent tanh expansion method, Chen et al. [1] gave the interaction solutions between the solitons and other different types of nonlinear waves. In [13], some smooth and peaked solitary wave solutions have been constructed by the bifurcation method of dynamical system. By using a Darboux transformation, Zhou et al. [27] obtained new exact solutions for Broer-Kaup system. In [6], new type of solitary wave solutions for the Broer-Kaup equations were presented by using the He's variational principle.

Various algebraic aspects of BK equations solutions have been studied. Kupershmidt [11] showed that BK equations are integrable and possess infinite number of conservation laws and tri-Hamiltonian structure. In [4], The geometric properties of non-Noether symmetries as well as their applications were discussed.

The analysis by many methods of the (2+1)-dimensional BK system can be found in [26] and [28] and the references therein. The (1+1)-dimensional and the (2 + 1)-dimensional higher order Broer-Kaup equation was considered for example in [23] and [20], respectively. Concerning generalized and variable coefficient Broer-Kaup equations, see for example [22] and [10]. Recently, fractional and stochastic Broer-Kaup system, were studied in [3] and [24].

The aim of this paper is to investigate the initial value problem (1.1) for existence and nonuniqueness of global classical solutions. For goal, a new topological approach which uses the abstract theory of the sum of two operators is used for investigations of existence of at least one and at least two classical solutions. This basic and new idea can be used for investigations for existence of global classical solutions for many of the interesting equations of mathematical physics. Here, by a classical solution to the Broer-Kaup equations we mean a solution at least three times continuously differentiable in x and once in t for any t ≥ 0. In other words, (u,v) belongs to the space $\mathcal{C}^1([0,\infty),\mathcal{C}^3(\mathbb{R}))\times\mathcal{C}^1([0,\infty),\mathcal{C}^3(\mathbb{R}))$ of continuously differentiable functions on $[0,\infty)$ with values in the Banach space $\mathcal{C}^3(\mathbb{R})$.

The paper is organized as follows. In the next section, we give some properties of solutions of problem (1.1). First, we give an integral representation of these solutions, then we prove some *a priori* estimates in a sense that will be defined later on. In Section 3 we prove our main results about existence and multiplicity of solutions for the Broer-Kaup system (1.1). Finally, in Section 4 we give an example to illustrate our main results.

2. Some Properties of Solutions of Problem (1.1)

Let $X = X^1 \times X^1$, where $X^1 = \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R}))$. For $(u,v) \in X$, define the operators S_1^1, S_1^2 and S_1 as follows.

$$S_1^1(u,v)(t,x) = u(t,x) - u_0(x) + \int_0^t (u(t_1,x)u_x(t_1,x) + v_x(t_1,x))dt_1,$$

$$S_1^2(u,v)(t,x) = v(t,x) - v(0,x) + \int_0^t \Big(u_x(t_1,x) + 2u_x(t_1,x)v(t_1,x) + 2u(t_1,x)v_x(t_1,x) + u_{xxx}(t_1,x)\Big)dt_1,$$

$$S_1(u,v)(t,x) = \left(S_1^1(u,v)(t,x), S_1^2(u,v)(t,x)\right), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

2.1. **Integral representation of the solutions.**

Lemma 2.1. *Suppose that* **(H1)** *is satisfied. If* $(u,v) \in X$ *satisfies the equation*

$$S_1(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \tag{2.1}$$

then (u,v) *is a solution of the IVP* (1.1).

Proof. Let $(u,v) \in X$ be a solution of the equation (2.1). Then

$$S_1^1(u,v)(t,x) = 0, \quad S_1^2(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{2.2}$$

We differentiate both equations of (2.2) with respect to t and x and we find

$$u_t(t,x) + u(t,x)u_x(t,x) + v_x(t,x) = 0,$$

$$v_t(t,x) + u_x(t,x) + 2u_x(t,x)v(t,x) + 2u(t,x)v_x(t,x) + u_{xxx}(t,x) = 0,$$

$(t,x) \in [0,\infty) \times \mathbb{R}$. We put $t = 0$ in both equations of (2.2) and we arrive at

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \mathbb{R}.$$

This completes the proof. □

Lemma 2.2. *Suppose* **(H1)** *and let* $h \in \mathcal{C}([0,\infty) \times \mathbb{R})$ *be a positive function almost everywhere on* $[0,\infty) \times \mathbb{R}$. *If* $(u,v) \in X$ *satisfies the following integral equations:*

$$\int_0^t \int_0^x (t-t_1)(x-x_1)^3 h(t_1,x_1) S_1^1(u,v)(t_1,x_1)dx_1dt_1 = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

and

$$\int_0^t \int_0^x (t-t_1)(x-x_1)^3 h(t_1,x_1) S_1^2(u,v)(t_1,x_1)dx_1dt_1 = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

then (u,v) *is a solution to the IVP* (1.1).

Proof. We differentiate three times with respect to t and three times with respect to x the integral equations of Lemma 2.2 and we find

$$h(t,x)S_1(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

whereupon

$$S_1(u,v)(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Hence and Lemma 2.1, we conclude that (u,v) is a solution to the IVP (1.1). This completes the proof. $\square$

2.2. **A priori estimates.** In the sequel, $X = X^1 \times X^1$ where $X^1 = \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R}))$ will be endowed with the norm

$$\|(u,v)\| = \max\left\{\|u\|_{X^1}, \|v\|_{X^1}\right\}, \quad (u,v) \in X,$$

with

$$\|u\|_{X^1} = \max\Bigg\{ \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u(t,x)|, \quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_t(t,x)|, \quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_x(t,x)|,$$
$$\sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_{xx}(t,x)|, \quad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_{xxx}(t,x)| \Bigg\},$$

provided it exists. Let

$$B_1 = 4(B + B^2).$$

Lemma 2.3. *Under hypothesis* **(H1)** *and for* $(u,v) \in X$ *with* $\|(u,v)\| \le B$, *the following estimates hold:*

$$|S_1^1(u,v)(t,x)| \le B_1(1+t), \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

and

$$|S_1^2(u,v)(t,x)| \le B_1(1+t), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Proof. Suppose that **(H1)** is satisfied and let $(u,v) \in X$ with $\|(u,v)\| \le B$.

(i) Estimation of $|S_1^1(u,v)(t,x)|$, $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned}
|S_1^1(u,v)(t,x)| &= \left| u(t,x) - u_0(x) + \int_0^t \left(u(t_1,x)u_x(t_1,x) + v_x(t_1,x)\right) dt_1 \right| \\
&\le |u(t,x)| + |u_0(x)| + \int_0^t \left(|u(t_1,x)||u_x(t_1,x)| + |v_x(t_1,x)|\right) dt_1 \\
&\le 2B + (B + B^2)t \\
&\le (3B + B^2)(1+t) \\
&\le B_1(1+t).
\end{aligned}$$

(ii) Estimation of $|S_1^2(u,v)(t,x)|$, $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned}\left|S_1^2(u,v)(t,x)\right| &= \left|v(t,x) - v(0,x) + \int_0^x \Big(u_x(t_1,x) + 2u_x(t_1,x)v(t_1,x)\right. \\ &\qquad \left. + 2u(t_1,x)v_x(t_1,x) + u_{xxx}(t_1,x)\Big) dx_1 dt_1\right| \\ &\le |v(t,x)| + |v(0,x)| + \int_0^x \Big(|u_x(t_1,x)| + 2|u_x(t_1,x)||v(t_1,x)| \\ &\qquad + 2|u(t_1,x)||v_x(t_1,x)| + |u_{xxx}(t_1,x)|\Big) dx_1 dt_1 \\ &\le 2B + (2B + 4B^2)t \\ &\le 4(B + B^2)(1+t) \\ &= B_1(1+t).\end{aligned}$$

This completes the proof. □

Suppose

(H2): $g \in \mathcal{C}([0,\infty) \times \mathbb{R})$ is a positive function almost everywhere on $[0,\infty) \times \mathbb{R}$ such that

$$16(1+t)^2 \left(1 + |x| + x^2 + |x|^3\right) \int_0^t \left| \int_0^x g(t_1,x_1) dx_1 \right| dt_1 \le A,$$

$(t,x) \in [0,\infty) \times \mathbb{R}$, for some constant $A > 0$.

In the last section, we will give an example for a function g that satisfies **(H2)**. For $(u,v) \in X$, define the operators

$$\begin{aligned}S_2^1(u,v)(t,x) &= \int_0^t \int_0^x (t-t_1)(x-x_1)^3 g(t_1,x_1) S_1^1(u,v)(t_1,x_1) dx_1 dt_1, \\ S_2^2(u,v)(t,x) &= \int_0^t \int_0^x (t-t_1)(x-x_1)^3 g(t_1,x_1) S_1^2(u,v)(t_1,x_1) dx_1 dt_1\end{aligned}$$

and

$$S_2(u,v)(t,x) = \left(S_2^1(u,v)(t,x), S_2^2(u,v)(t,x)\right), \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{2.3}$$

Lemma 2.4. *Under hypothesis* **(H1)** *and* **(H2)** *and for* $(u,v) \in X$, *with* $\|(u,v)\| \le B$, *the following estimate holds:*

$$\|S_2(u,v)\| \le AB_1.$$

Proof. Suppose **(H1)** and **(H2)** and let $(u,v) \in X$, with $\|(u,v)\| \le B$.

(i) Estimation of $|S_2^1(u,v)(t,x)|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}
|S_2^1(u,v)(t,x)| &= \left|\int_0^t\int_0^x (t-t_1)(x-x_1)^3 g(t_1,x_1)S_1^1(u,v)(t_1,x_1)dx_1dt_1\right|\\
&\le \int_0^t\left|\int_0^x (t-t_1)|x-x_1|^3 g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)|dx_1\right|dt_1\\
&\le B_1(1+t)\int_0^t\left|\int_0^x (t-t_1)|x-x_1|^3 g(t_1,x_1)dx_1\right|dt_1\\
&\le 8B_1(1+t)^2|x|^3\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\
&\le 8B_1(1+t)^2\left(1+|x|+x^2+|x|^3\right)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\
&\le AB_1.
\end{aligned}$$

(ii) Estimation of $\left|\frac{\partial}{\partial t}S_2^1(u,v)(t,x)\right|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}
\left|\frac{\partial}{\partial t}S_2^1(u,v)(t,x)\right| &= \left|\int_0^t\int_0^x (x-x_1)^3 g(t_1,x_1)S_1^1(u,v)(t_1,x_1)dx_1dt_1\right|\\
&\le \int_0^t\left|\int_0^x |x-x_1|^3 g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)|dx_1\right|dt_1\\
&\le B_1(1+t)\int_0^t\left|\int_0^x |x-x_1|^3 g(t_1,x_1)dx_1\right|dt_1\\
&\le 8B_1(1+t)^2|x|^3\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\
&\le 8B_1(1+t)^2\left(1+|x|+x^2+|x|^3\right)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\
&\le AB_1.
\end{aligned}$$

(iii) Estimation of $\left|\frac{\partial}{\partial x}S_2^1(u,v)(t,x)\right|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}
\left|\frac{\partial}{\partial x}S_2^1(u,v)(t,x)\right| &= 3\left|\int_0^t\int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)S_1^1(u,v)(t_1,x_1)dx_1dt_1\right|\\
&\le 3B_1(1+t)\int_0^t\left|\int_0^x (t-t_1)(x-x_1)^2 g(t_1,x_1)dx_1\right|dt_1\\
&\le 12B_1(1+t)^2x^2\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\
&\le 12B_1(1+t)^2\left(1+|x|+x^2+|x|^3\right)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\
&\le AB_1.
\end{aligned}$$

(iv) Estimation of $\left|\frac{\partial^2}{\partial x^2}S_2^1(u,v)(t,x)\right|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}\left|\frac{\partial^2}{\partial x^2}S_2^1(u,v)(t,x)\right| &= 6\left|\int_0^t\int_0^x(t-t_1)(x-x_1)g(t_1,x_1)S_1^1(u,v)(t_1,x_1)dx_1dt_1\right|\\ &\le 6\int_0^t\left|\int_0^x(t-t_1)|x-x_1|g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)|dx_1\right|dt_1\\ &\le 6B_1(1+t)\int_0^t\left|\int_0^x(t-t_1)|x-x_1|g(t_1,x_1)dx_1\right|dt_1\\ &\le 12B_1(1+t)^2|x|\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\le 12B_1(1+t)^2\left(1+|x|+x^2+|x|^3\right)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\le AB_1.\end{aligned}$$

(v) Estimation of $\left|\frac{\partial^3}{\partial x^3}S_2^1(u,v)(t,x)\right|$, $(t,x)\in[0,\infty)\times\mathbb{R}$:

$$\begin{aligned}\left|\frac{\partial^3}{\partial x^3}S_2^1(u,v)(t,x)\right| &= 6\left|\int_0^t\int_0^x(t-t_1)g(t_1,x_1)S_1^1(u,v)(t_1,x_1)dx_1dt_1\right|\\ &\le 6\int_0^t\left|\int_0^x(t-t_1)g(t_1,x_1)|S_1^1(u,v)(t_1,x_1)|dx_1\right|dt_1\\ &\le 6B_1(1+t)\int_0^t\left|\int_0^x(t-t_1)g(t_1,x_1)dx_1\right|dt_1\\ &\le 12B_1(1+t)^2\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\le 12B_1(1+t)^2\left(1+|x|+x^2+|x|^3\right)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1\\ &\le AB_1.\end{aligned}$$

Similarly, the same estimates (i)-(v) can be proved for the operator S_2^2. Finally,

$$\|S_2(u,v)\|\le AB_1.$$

This completes the proof. □

3. Main Results

3.1. **Existence of nonnegative solutions.** The following theorem (see its proof in [17]) will be used to prove Theorem 3.2.

Theorem 3.1. *Let E be a Banach space and*

$$E_1=\{x\in E:\|x\|\le R\},$$

with $R > 0$. Consider two operators T and S, where

$$Tx = -\epsilon x, \quad x \in E_1,$$

with $\epsilon > 0$ and $S : E_1 \to E$ be continuous and such that

(i) $(I - S)(E_1)$ *resides in a compact subset of E and*

(ii) $\{x \in E : x = \lambda(I - S)x, \|x\| = R\} = \emptyset$ *for any* $\lambda \in \left(0, \frac{1}{\epsilon}\right)$.

Then there exists $x^* \in E_1$ such that

$$Tx^* + Sx^* = x^*.$$

In the sequel, suppose that the constants B and A which appear in the conditions **(H1)** and **(H2)**, respectively, satisfy the following inequality:

(H3): $AB_1 < B$, where $B_1 = 4(B + B^2)$.

Our first main result for existence of classical solutions of the IVP (1.1) is as follows.

Theorem 3.2. *Assume that the hypotheses* **(H1)***,* **(H2)** *and* **(H3)** *are satisfied. Then the IVP* (1.1) *has at least one nonnegative solution* $(u, v) \in \mathcal{C}^1([0, \infty), \mathcal{C}^3(\mathbb{R})) \times \mathcal{C}^1([0, \infty), \mathcal{C}^3(\mathbb{R}))$.

Proof. Choose $\epsilon \in (0, 1)$, such that $\epsilon B_1(1 + A) < B$.

For $(u, v) \in X = \mathcal{C}^1([0, \infty), \mathcal{C}^3(\mathbb{R})) \times \mathcal{C}^1([0, \infty), \mathcal{C}^3(\mathbb{R}))$, we will write

$$(u, v) \geq 0 \text{ if } u(t, x) \geq 0 \text{ and } v(t, x) \geq 0, \quad \text{for any } (t, x) \in [0, \infty) \times \mathbb{R}.$$

Let $\tilde{\tilde{\tilde{Y}}}$ denotes the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$, $\tilde{\tilde{Y}} = \overline{\tilde{\tilde{\tilde{Y}}}}$ be the closure of $\tilde{\tilde{\tilde{Y}}}$, $\tilde{Y} = \tilde{\tilde{Y}} \cup \{(u_0, v_0)\}$ and

$$Y = \{(u, v) \in \tilde{Y} : (u, v) \geq 0, \|(u, v)\| \leq B\}.$$

Note that Y is a compact set in X. For $(u, v) \in X$, define the operators

$$T(u, v)(t, x) = -\epsilon(u, v)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R},$$
$$S(u, v)(t, x) = (u, v)(t, x) + \epsilon(u, v)(t, x) + \epsilon S_2(u, v)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}.$$

For $(u, v) \in Y$ and by using Lemma 2.4, it follows that

$$\begin{aligned}\|(I - S)(u, v)\| &= \|\epsilon(u, v) - \epsilon S_2(u, v)\| \\ &\leq \epsilon\|(u, v)\| + \epsilon\|S_2(u, v)\| \\ &\leq \epsilon B_1 + \epsilon AB_1 \\ &= \epsilon B_1(1 + A) \\ &< B.\end{aligned}$$

Thus, $S : Y \to X$ is continuous and $(I - S)(Y)$ resides in a compact subset of X. Now, suppose that there is a $(u, v) \in X$ so that $\|(u, v)\| = B$ and

$$(u, v) = \lambda(I - S)(u, v)$$

or

$$\frac{1}{\lambda}(u,v) = (I - S)(u,v) = -\epsilon(u,v) - \epsilon S_2(u,v)$$

or

$$\left(\frac{1}{\lambda} + \epsilon\right)(u,v) = -\epsilon S_2(u,v),$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Hence, $\|S_2(u,v)\| \leq AB_1 < B$,

$$\epsilon B < \left(\frac{1}{\lambda} + \epsilon\right) B = \left(\frac{1}{\lambda} + \epsilon\right) \|(u,v)\| = \epsilon \|S_2(u,v)\| < \epsilon B,$$

which is a contradiction. In virtue of Theorem 3.1, the operator $T + S$ has a fixed point $(u^*, v^*) \in Y$. Therefore,

$$\begin{aligned}(u^*,v^*)(t,x) =& T(u^*,v^*)(t,x) + S(u^*,v^*)(t,x)\\ =& -\epsilon(u^*,v^*)(t,x) + (u^*,v^*)(t,x) + \epsilon(u^*,v^*)(t,x) + \epsilon S_2(u^*,v^*)(t,x),\end{aligned}$$

$(t,x) \in [0,\infty) \times \mathbb{R}$, whereupon

$$0 = S_2(u^*,v^*)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Lemma 2.2 yields that (u^*, v^*) is a solution to the IVP (1.1). This completes the proof. □

3.2. **Multiplicity of nonnegative solutions.** Let E be a real Banach space.

Definition 3.1. A closed, convex set $\mathcal{P}$ in E is said to be cone if

(a) $\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$;
(b) $x, -x \in \mathcal{P}$ implies $x = 0$.

Definition 3.2. A mapping $K : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 3.3. Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X,$$

for any $x, y \in X$.

The following result (see details of its proof in [5] and [17]) will be used to prove Theorem 3.4.

Theorem 3.3. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $w_0 \in \mathcal{P}\setminus\{0\}$ such that the following conditions hold:*

(i) *$Sx \neq (I - T)(x - \lambda w_0)$ for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda w_0)$;*
(ii) *there exists $\varepsilon > 0$ such that $Sx \neq (I - T)(\lambda x)$ for all $\lambda \geq 1 + \varepsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$;*

(iii) $Sx \neq (I-T)(x-\lambda w_0)$ *for all* $\lambda > 0$ *and* $x \in \partial U_3 \cap (\Omega + \lambda w_0)$.

Then $T+S$ *has at least two non-zero fixed points* $x_1, x_2 \in \mathcal{P}$ *such that*

$$x_1 \in \partial U_2 \cap \Omega \quad \text{and} \quad x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \quad \text{and} \quad x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

In the sequel, suppose that the constants B and A which appear in the conditions **(H1)** and **(H2)**, respectively, satisfy the following inequality:

(H4): $AB_1 < \frac{L}{5}$, where $B_1 = 4(B+B^2)$ and L is a positive constant that satisfies the following conditions:

$$r < L < R_1 \leq B, \quad R_1 > \left(\frac{2}{5m}+1\right)L,$$

with r and R_1 are positive constants and $m > 0$ is large enough.

Our second main result for existence and multiplicity of classical solutions of the IVP (1.1) is as follows.

Theorem 3.4. *Assume that the hypotheses* **(H1)**, **(H2)** *and* **(H4)** *are satisfied. Then the IVP* (1.1) *has at least two nonnegative solutions*

$$(u_1, v_1), (u_2, v_2) \in \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R})) \times \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R})).$$

Proof. Set $X = \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R})) \times \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R}))$ and let

$$\widetilde{P} = \{(u,v) \in X : (u,v) \geq 0 \text{ on } [0,\infty) \times \mathbb{R}\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $(u,v) \in X$, define the operators

$$T_1(u,v)(t,x) = (1+m\epsilon)(u,v)(t,x) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right), \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

$$S_3(u,v)(t,x) = -\epsilon S_2(u,v)(t,x) - m\epsilon(u,v)(t,x) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right), \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

where ϵ is a positive constant, $m > 0$ is large enough and the operator S_2 is given by formula (2.3). Note that any fixed point $(u,v) \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (1.1). Now, let us define

$$U_1 = \mathcal{P}_r = \{(u,v) \in \mathcal{P} : \|(u,v)\| < r\},$$
$$U_2 = \mathcal{P}_L = \{(u,v) \in \mathcal{P} : \|(u,v)\| < L\},$$
$$U_3 = \mathcal{P}_{R_1} = \{(u,v) \in \mathcal{P} : \|(u,v)\| < R_1\},$$
$$\Omega = \overline{\mathcal{P}_{R_2}} = \{(u,v) \in \mathcal{P} : \|(u,v)\| \leq R_2\}, \quad \text{with } R_2 = R_1 + \frac{A}{m}B_1 + \frac{L}{5m}.$$

(a) Let $(u_1,v_1), (u_2,v_2) \in \Omega$, then

$$\|T_1(u_1,v_1) - T_1(u_2,v_2)\| = (1+m\epsilon)\|(u_1,v_1) - (u_2,v_2)\|,$$

whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.

(b) Let $(u,v) \in \overline{\mathcal{P}_{R_1}}$, then Lemma 2.4 yields

$$\|S_3(u,v)\| \le \epsilon\|S_2(u,v)\| + m\epsilon\|(u,v)\| + \epsilon\frac{L}{10} \le \epsilon\bigg(AB_1 + mR_1 + \frac{L}{10}\bigg).$$

Therefore, $S_3(\overline{\mathcal{P}_{R_1}})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}_{R_1}})$ is equi-continuous. Consequently, $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is completely continuous.

(c) Let $(u_1,v_1) \in \overline{\mathcal{P}_{R_1}}$ and set

$$(u_2,v_2) = (u_1,v_1) + \frac{1}{m}S_2(u_1,v_1) + \left(\frac{L}{5m}, \frac{L}{5m}\right).$$

Note that $S_2^1(u_1,v_1) + \frac{L}{5} \ge 0$, $S_2^2(u_1,v_1) + \frac{L}{5} \ge 0$ on $[0,\infty) \times \mathbb{R}$. We have $u_2, v_2 \ge 0$ on $[0,\infty) \times \mathbb{R}$ and

$$\|(u_2,v_2)\| \le \|(u_1,v_1)\| + \frac{1}{m}\|S_2(u_1,v_1)\| + \frac{L}{5m} \le R_1 + \frac{A}{m}B_1 + \frac{L}{5m} = R_2.$$

Therefore, $(u_2,v_2) \in \Omega$ and

$$-\epsilon m(u_2,v_2) = -\epsilon m(u_1,v_1) - \epsilon S_2(u_1,v_1) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$(I - T_1)(u_2,v_2) = -\epsilon m(u_2,v_2) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = S_3(u_1,v_1).$$

Consequently, $S_3(\overline{\mathcal{P}_{R_1}}) \subset (I - T_1)(\Omega)$.

(d) Assume that for any $(w_0,z_0) \in \mathcal{P}^* = \mathcal{P} \setminus \{0\}$ there exist $\lambda \ge 0$ and $(u,v) \in \partial\mathcal{P}_r \cap (\Omega + \lambda(w_0,z_0))$ or $(u,v) \in \overline{\mathcal{P}_{R_1}} \cap (\Omega + \lambda(w_0,z_0))$ such that

$$S_3(u,v) = (I - T_1)((u,v) - \lambda(w_0,z_0)).$$

Then

$$-\epsilon S_2(u,v) - m\epsilon(u,v) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = -m\epsilon((u,v) - \lambda(w_0,z_0)) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$-S_2(u,v) = \lambda m(w_0,z_0) + \left(\frac{L}{5}, \frac{L}{5}\right).$$

Hence,

$$\|S_2 v\| = \left\|\lambda m(w_0,z_0) + \left(\frac{L}{5}, \frac{L}{5}\right)\right\| > \frac{L}{5}.$$

This is a contradiction.

(e) Let $\varepsilon_1 = \frac{2}{5m}$. Assume that there exist $(u_1,v_1) \in \partial\mathcal{P}_L$ and $\lambda_1 \ge 1 + \varepsilon_1$ such that $\lambda_1(u_1,v_1) \in \overline{\mathcal{P}_{R_1}}$ and

$$S_3(u_1,v_1) = (I - T_1)(\lambda_1(u_1,v_1)). \tag{3.1}$$

Since $(u_1,v_1) \in \partial\mathcal{P}_L$ and $\lambda_1(u_1,v_1) \in \overline{\mathcal{P}_{R_1}}$, it follows that

$$\left(\frac{2}{5m} + 1\right)L < \lambda_1 L = \lambda_1\|(u_1,v_1)\| \le R_1.$$

Moreover,

$$-\epsilon S_2(u_1, v_1) - m\epsilon(u_1, v_1) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = -\lambda_1 m\epsilon(u_1, v_1) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$S_2(u_1, v_1) + \left(\frac{L}{5}, \frac{L}{5}\right) = (\lambda_1 - 1)m(u_1, v_1).$$

From here,

$$2\frac{L}{5} \geq \left\| S_2(u_1, v_1) + \left(\frac{L}{5}, \frac{L}{5}\right) \right\| = (\lambda_1 - 1)m\|(u_1, v_1)\| = (\lambda_1 - 1)mL$$

and

$$\frac{2}{5m} + 1 \geq \lambda_1,$$

which is a contradiction.

Therefore, all conditions of Theorem 3.4 hold. Hence, the IVP (1.1) has at least two solutions (u_1, v_1) and (u_2, v_2) so that

$$\|(u_1, v_1)\| = L < \|(u_2, v_2)\| \leq R_1$$

or

$$r \leq \|(u_1, v_1)\| < L < \|(u_2, v_2)\| \leq R_1.$$

□

4. An Example

Below, we will illustrate our main results. Let

$$h(s) = \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Then

$$h'(s) = \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})},$$

$$l'(s) = \frac{11\sqrt{2}s^{10}(1 + s^{20})}{1 + s^{40}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Therefore,

$$-\infty < \lim_{s \to \pm\infty} (1 + s + s^2)h(s) < \infty,$$

$$-\infty < \lim_{s \to \pm\infty} (1 + s + s^2)l(s) < \infty.$$

Hence, there exists a positive constant C_1 so that

$$(1+s+s^2+s^3+s^4+s^5+s^6)\left(\frac{1}{44\sqrt{2}} \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}} \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}\right) \leq C_1,$$

$$(1+s+s^2+s^3+s^4+s^5+s^6)\left(\frac{1}{44\sqrt{2}} \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}} \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}\right) \leq C_1,$$

$s \in \mathbb{R}$. Note that $\lim\limits_{s\to\pm 1} l(s) = \frac{\pi}{2}$ and by [15, pp. 707, Integral 79], we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}} \log \frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^{44})(1+s+s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(t,x) = Q(t)Q(x), \quad t \in [0,\infty), \quad x \in \mathbb{R}.$$

Then there exists a constant $C_2 > 0$ such that

$$12(1+t)^2 \left(1+|x|+x^2+|x|^3\right) \int_0^t \left| \int_0^x g_1(t_1,x_1)dx_1 \right| dt_1 \le C_2, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Let

$$g(t,x) = \frac{A}{C_2} g_1(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Then

$$12(1+t)^2 \left(1+|x|+x^2+|x|^3\right) \int_0^t \left| \int_0^x g(t_1,x_1)dx_1 \right| dt_1 \le A, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

i.e., **(H2)** holds. Now, consider the initial value problem

$$\begin{aligned} u_t + uu_x + v_x &= 0, \quad t \in (0,\infty), \quad x \in \mathbb{R}, \\ v_t + u_x + 2(uv)_x + u_{xxx} &= 0, \quad t \in (0,\infty), \quad x \in \mathbb{R}, \\ u(0,x) &= \frac{1}{1+x^2+x^4}, \quad x \in \mathbb{R}, \\ v(0,x) &= \frac{1}{1+3x^2+x^8}, \quad x \in \mathbb{R}, \end{aligned} \tag{4.1}$$

so that **(H1)** holds, with $B = 10$, for example. Take

$$B = 10 \quad \text{and} \quad A = \frac{1}{10^4}.$$

Then

$$AB_1 = A.4(B+B^2) = \frac{1}{10^4} \cdot 4(10+10^2) < B.$$

So, condition **(H3)** is fulfilled. Thus, the conditions **(H1)**, **(H2)** and **(H3)** are satisfied. Hence, by Theorem 3.2, it follows that problem (4.1) has at least one solution $(u,v) \in \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R})) \times \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R}))$.

In the sequel, take

$$R_1 = B = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \epsilon = \frac{1}{10^4}.$$

Clearly,

$$r < L < R_1 \leq B, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right) L, \quad AB_1 < \frac{L}{5},$$

i.e., **(H4)** holds. Hence, by Theorem 3.4, it follows that the initial value problem (4.1) has at least two nonnegative solutions $(u_1, v_1), (u_2, v_2) \in \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R})) \times \mathcal{C}^1([0,\infty), \mathcal{C}^3(\mathbb{R}))$.

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