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APPROXIMATION BY MODIFIED SZÁSZ OPERATORS WITH A NEW MODIFICATION OF BRENKE TYPE POLYNOMIALS

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Abstract. In the present article we study the approximation properties of modified Szász operators with a new modification of Brenke type polynomials. First, we estimate the rate of convergence, for the newly defined operators, by means of modulus of smoothness, Peetre's K-functional and Lipschitz type functions. Furthermore, we also prove a Voronovskaja type asymptotic theorem.

1. Introduction and Preliminaries

In 1950, Szász [18] extended the theory of well known Bernstein operators for the finite interval $[0,1]$ to infinite interval $\mathbb{R}_0^+ := [0,\infty)$ and established the convergence properties in the infinite interval $\mathbb{R}_0^+$ by defining the operators for $f \in C(\mathbb{R}_0^+)$ as

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0^+, n \in \mathbb{N}. \tag{1.1}$$

A generalization of (1.1) was established by Jakimovski-Leviatan in [12] with the help of the Appell polynomials as

$$P_n(f;x) := \frac{e^{-nx}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0^+, n \in \mathbb{N}, \tag{1.2}$$

where $A(x) = \sum_{n=0}^{\infty} b_n x^n$, $b_n \in \mathbb{R}$, is an analytic function on the disk $|x| < R$, $R > 1$, with $A(1) \neq 0$. The polynomials $p_k(x) = \sum_{i=0}^{k} b_i \frac{x^{k-i}}{(k-i)!}$, $k \in \mathbb{N}$, are the Appell polynomials which are generated by $A(z)e^{zx} = \sum_{k=0}^{\infty} p_k(x) z^k$ under the assumption

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that $p_k(x) \geq 0$ for all $x \in [0, \infty)$. In particular, if $A(z) = 1$, then $p_k(x) = \frac{x^k}{k!}$, and the operators (1.2) reduce to the operators (1.1).

Ismail [11] defined another generalization of (1.1) and (1.2) with the help of Sheffer type polynomials $\{u_k(x)\}_{k\geq 1}$, which are generated by

$$A(s)e^{tB(s)} = \sum_{k=0}^{\infty} u_k(t)s^k, \quad |s| < R,$$

where $A(s) = \sum_{k=0}^{\infty} a_k s^k$, $a_0 \neq 0$ and $B(s) = \sum_{k=1}^{\infty} b_k s^k$, $b_1 \neq 0$, are analytic functions on the disc $|s| < R,\ R > 1$, and a_k and b_k are the real coefficients. Under the following assumptions:

(i) for $t \in \mathbb{R}_0^+$, $u_k(t) \geq 0$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$;
(ii) $A(1) \neq 0$ and $B^{(1)}(1) = 1$,

Ismail introduced and studied some important approximation properties of the following operators

$$Q_n(f;x) = \frac{e^{-nxB(1)}}{A(1)} \sum_{k=0}^{\infty} u_k(nx) f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0^+, n \in \mathbb{N}. \tag{1.3}$$

In particular, when $A(t) = t$ and $B(t) = 1$, the operator (1.3) reduces to the Szász operator (1.1) and for the case $B(t) = t$, the operator $Q_n(f;x)$ yields the operator $P_n(f;x)$ defined in (1.2).

Let $v_k(x) = \sum_{r=0}^{k} a_{k-r} b_r x^r$, $k \in \mathbb{N} \cup \{0\}$, be the Brenke type polynomials on the disk $|x| < R, (R > 1)$ which are generated by

$$A(s)B(xs) = \sum_{k=0}^{\infty} v_k(x)s^k, \tag{1.4}$$

where $A(s) = \sum_{k=0}^{\infty} a_k s^k, a_0 \neq 0$, and $B(s) = \sum_{k=0}^{\infty} b_k s^k$, $b_k \neq 0$, are analytic functions on the disk $|s| < R$, $R > 1$.

Under the following assumptions:

(i) $A(1) \neq 0$, $\frac{a_{r-k} b_r}{A(1)} \geq 0$, $0 \leq r \leq k$, $k \in \mathbb{N} \cup \{0\}$;
(ii) $B : \mathbb{R}_0^+ \to (0, \infty)$;
(iii) (1.4) and the power series $A(t)$ and $B(t)$ converge for $|t| < R$, $R > 1$.

Varma et al. [20] presented a generalization of Szász operators by means of the Brenke type polynomials as

$$R_n(f;x) := \frac{1}{A(1)B(nx)} \sum_{k=0}^{\infty} v_k(nx) f\left(\frac{k}{n}\right), \quad x \geq 0, n \in \mathbb{N}. \tag{1.5}$$

In particular, if $B(t) = e^t$, the operator (1.5) reduces to the operator (1.2) and if $B(t) = e^t$ and $A(t) = 1$ the operator (1.5) reproduces the Szász operator (1.1).

Cheikh and Romdhane [6] defined the d-symmetric d-orthogonal polynomials of Brenke type as

$$\mathcal{A}(t^{\rho+1})\mathcal{B}(xt) = \sum_{k=0}^{\infty} q_k(x)t^k, \tag{1.6}$$

where $\mathcal{A}(t) = \sum_{k=0}^{\infty} a_k t^k$, $\mathcal{B}(t) = \sum_{k=0}^{\infty} b_k t^k$ with $a_0 b_k \neq 0$ for all $k \in \mathbb{N}$, are analytic functions on the disk $|t| < R$, $R > 1$, and ρ is a positive integer. In particular case, $\mathcal{A}(t) = \exp(x)$ and $\mathcal{B}(t) = \exp(x)$, the polynomials (1.6) reduce to the Gould-Hopper polynomials [10] and also when $\rho = 0$, (1.6) reduces to (1.4).

Motivated by the work above, we present a new modification of Szász operators with the generalized form of Brenke type polynomials $q_k(x)$ as

$$\mathcal{D}_n(f;x) := \frac{1}{\mathcal{A}(1)\mathcal{B}(nx)} \sum_{k=0}^{\infty} q_k(nx) f\left(\frac{k}{n}\right), \quad x \geq 0, n \in \mathbb{N}, \tag{1.7}$$

where $q_k(x)$ is defined in (1.6). The purpose of this article is to establish some approximation properties for the operator (1.7), under the following certain conditions

(i) $\mathcal{A}(1) \neq 0$, $\frac{a_{k-m} b_k}{\mathcal{A}(1)} \geq 0$, $0 \leq k \leq m$, $m \in \mathbb{N}_0$;
(ii) $\mathcal{B} : \mathbb{R}_0^+ \to (0, \infty)$;
(iii) (1.6) and the power series for $\mathcal{A}(t)$ and $\mathcal{B}(t)$ converge for $|t| < R$, $R > 1$.

In particular, the operator $\mathcal{D}_n(f;x)$ have the following reductions

(i) if $\rho = 0$, the operator (1.7) reduces to the operator (1.5);
(ii) if $\rho = 0$, and $\mathcal{B}(t) = e^t$, the operator (1.7) reduces to the operator (1.2);
(iii) if $\rho = 0$, $\mathcal{A}(t) = e^t$ and $\mathcal{B}(t) = 1$, the operator (1.7) reproduces the Szász operator (1.1).

For some other recent papers on the topic dealing with the generalization of Szász type operators using different classes of polynomials, see [1–3, 5, 7, 8, 13–15, 17, 19, 21] and the references cited therein.

The rest of the paper is organised as follows. In Section 2, we present some auxiliary results. In Section 3, we estimate the rate of convergence with the help of classical and second-order modulus of smoothness and Peetre's K-functional and also give the order of approximation for the Lipschitz type space. Lastly, we discuss a quantitative Voronovskaja-type theorem.

2. Auxiliary Results

In this section, we present some important auxiliary results which will be used in this later work.

Lemma 2.1. *From the generating function* (1.6) *of the Brenke type polynomials, we have the following equalities:*

$$\sum_{k=0}^{\infty} q_k(nx) = \mathcal{A}(1)\mathcal{B}(nx),$$

$$\begin{aligned}
\sum_{k=0}^{\infty} kq_k(nx) =&(\rho+1)\mathcal{A}^{(1)}(1)\mathcal{B}(nx)+nx\mathcal{B}^{(1)}(nx)\mathcal{A}(1),\\
\sum_{k=0}^{\infty} k^2q_k(nx) =&(\rho+1)^2(\mathcal{A}^{(2)}(1)+\mathcal{A}^{(1)}(1))\mathcal{B}(nx)+2n(\rho+1)x\mathcal{A}^{(1)}(1)\mathcal{B}^{(1)}(nx)\\
&+n^2x^2\mathcal{A}(1)\mathcal{B}^{(2)}(nx)+nx\mathcal{A}(1)\mathcal{B}^{(1)}(nx),\\
\sum_{k=0}^{\infty} k^3q_k(nx) =&(\rho+1)^3(\mathcal{A}^{(3)}(1)+3\mathcal{A}^{(2)}(1))\mathcal{B}(nx)+(\rho^2+1)(\rho+1)\mathcal{A}^{(1)}(1)\mathcal{B}(nx)\\
&+3n(\rho+1)^2x\mathcal{A}^{(2)}(1)\mathcal{B}^{(1)}(nx)+3n(\rho+1)(\rho+2)x\mathcal{A}^{(1)}(1)\mathcal{B}^{(1)}(nx)\\
&+3(\rho+1)n^2x^2\mathcal{A}^{(1)}(1)\mathcal{B}^{(2)}(nx)+n^3x^3\mathcal{A}(1)\mathcal{B}^{(3)}(nx)\\
&+3n^2x^2\mathcal{A}(1)\mathcal{B}^{(2)}(nx)+nx\mathcal{A}^{(1)}(1)\mathcal{B}(nx),\\
\sum_{k=0}^{\infty} k^4q_k(nx) =&(\rho+1)^4(\mathcal{A}^{(4)}(1)+6\mathcal{A}^{(3)}(1)+7\mathcal{A}^{(2)}(1))\mathcal{B}(nx)\\
&+(\rho+1)(\rho^3+3\rho^2-9\rho+1)\mathcal{A}^{(1)}(1)\mathcal{B}(nx)\\
&+4nx(\rho+1)^3\mathcal{A}^{(3)}(1)\mathcal{B}^{(1)}(nx)\\
&+9nx(\rho+2)(\rho+1)^2\mathcal{A}^{(2)}(1)\mathcal{B}^{(1)}(nx)+6n^2x^2(\rho+1)^2\mathcal{A}^{(2)}(1)\mathcal{B}^{(2)}(nx)\\
&+3(\rho+7)(\rho+1)n^2x^2\mathcal{A}^{(1)}(1)\mathcal{B}^{(2)}(nx)+4(\rho+1)n^3x^3\mathcal{A}^{(1)}(1)\mathcal{B}^{(3)}(nx)\\
&+n^4x^4\mathcal{A}(1)\mathcal{B}^{(4)}(nx)+6n^3x^3\mathcal{A}(1)\mathcal{B}^{(3)}(nx)\\
&+7n^2x^2\mathcal{A}(1)\mathcal{B}^{(2)}(nx)+nx\mathcal{A}(1)\mathcal{B}^{(1)}(nx)\\
&+(\rho+1)(\rho^2+20\rho+11)\mathcal{A}^{(1)}(1)\mathcal{B}^{(1)}(nx),
\end{aligned}$$

where $\mathcal{A}^{(r)}(x)=\frac{d^r\mathcal{A}(x)}{dx^r}$ *and* $\mathcal{B}^{(r)}(x)=\frac{d^r\mathcal{B}(x)}{dx^r}$ *for all* $r\in\mathbb{N}$.

Proof. Differentiating (1.6) with respect to t, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} kq_k(x)t^{k-1} =&(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt)+x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt),\\
\sum_{k=0}^{\infty} k^2q_k(x)t^{k-2} =&(\rho+1)^2t^{2\rho}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}(xt)+\rho(\rho+1)t^{\rho-1}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt)\\
&+x^2\mathcal{A}(t^{\rho+1})\mathcal{B}^{(2)}(xt)+x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})(2x\mathcal{B}^{(1)}(xt)+\mathcal{B}(xt)),\\
\sum_{k=0}^{\infty} k^3q_k(x)t^{k-3} =&(\rho+1)^3t^{3\rho}\mathcal{A}^{(3)}(t^{\rho+1})\mathcal{B}(xt)+(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})(3x^2\mathcal{B}^{(2)}(xt)\\
&+\mathcal{B}(xt))+3\rho(\rho+1)^2t^{2\rho-1}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}(xt)\\
&+(\rho+1)t^{\rho-1}\mathcal{A}^{(1)}(t^{\rho+1})(3\rho\mathcal{B}(xt)+3x(\rho+2)\mathcal{B}^{(1)}(xt))\\
&+x^3\mathcal{A}(t^{\rho+1})\mathcal{B}^{(3)}(xt)+3x^2\mathcal{A}(t^{\rho+1})\mathcal{B}^{(2)}(xt)+x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt)
\end{aligned}$$

$$
\begin{aligned}
&+ (\rho+1)^2 t^{2\rho}\mathcal{A}^{(2)}(t^{\rho+1})(3x\mathcal{B}^{(1)}(xt)+3\mathcal{B}(xt))\\
&+ \rho(\rho^2-1)t^{\rho-2}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{\infty} k^4 q_k(x)t^{k-4} =& 4x^3(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(3)}(xt) + 6x^2(\rho+1)^2 t^{2\rho}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}^{(2)}(xt)\\
&+ 6x^2(\rho+1)\rho t^{\rho-1}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(2)}(xt)\\
&+ 4x(\rho+1)^3 t^{3\rho}\mathcal{A}^{(3)}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+ 6x(\rho+1)^2\rho t^{2\rho-1}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+ 6x\rho(\rho+1)^2 t^{2\rho-1}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+ 3x\rho(\rho^2-1)t^{\rho-2}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(1)}(xt) + (\rho+1)^4 t^{4d}\mathcal{A}^{(4)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ 6\rho(\rho+1)^3 t^{3d-1}\mathcal{A}^{(3)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ 3\rho(\rho+1)^2(2\rho-1)t^{2\rho-2}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ x\rho(\rho^2-1)t^{\rho-2}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+ \rho(\rho^2-1)(\rho+1)t^{2\rho-2}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ \rho(\rho^2-1)(d-2)t^{d-3}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt) + x^4\mathcal{A}(t^{\rho+1})\mathcal{B}^{(4)}(xt)\\
&+ 6\Big\{3x^2(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(2)}(xt)\\
&+ 3x(\rho+1)^2 t^{2\rho}\mathcal{A}^{(2)}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+ 3x\rho(\rho+1)t^{\rho-1}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(1)}(xt) + (\rho+1)^3 t^{3\rho}\mathcal{A}^{(3)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ 3\rho(\rho+1)^2 t^{2\rho-1}\mathcal{B}^{(2)}(t^{\rho+1})\mathcal{B}(xt) + \rho(\rho^2-1)t^{\rho-2}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ x^3\mathcal{A}(t^{\rho+1})\mathcal{B}^{(3)}(xt) + 3\Big[2x(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\\
&+ (\rho+1)^2 t^{2\rho}\mathcal{B}^{(2)}(t^{\rho+1})\mathcal{B}(xt) + \rho(\rho+1)t^{\rho-1}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ x^2\mathcal{A}(t^{\rho+1})\mathcal{B}^{(2)}(xt) + (\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\Big]\\
&- 2\Big[(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt) + x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\Big]\Big\}\\
&+ 6\Big\{(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt) + x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\Big\}\\
&- 11\Big\{2x(\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}^{(1)}(xt) + (\rho+1)^2 t^{2\rho}\mathcal{B}^{(2)}(t^{\rho+1})\mathcal{B}(xt)\\
&+ \rho(\rho+1)t^{\rho-1}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt) + x^2\mathcal{A}(t^{\rho+1})\mathcal{B}^{(2)}(xt)
\end{aligned}
$$

$$+ (\rho+1)t^{\rho}\mathcal{A}^{(1)}(t^{\rho+1})\mathcal{B}(xt) + x\mathcal{A}(t^{\rho+1})\mathcal{B}^{(1)}(xt)\Big\}.$$

The desired lemma is obtained by substituting $t = 1$ and $x = nx$, in the above computations. □

Lemma 2.2. *For $x \in \mathbb{R}_0^+$, the r^{th} order moments $\mathcal{D}_n(t^r; x)$, $r = 0, 1, 2, 3, 4$, of the operators $\mathcal{D}_n$ are defined as:*

$$\mathcal{D}_n(1; x) = 1,$$
$$\mathcal{D}_n(t; x) = xb_1 + pa_1,$$
$$\mathcal{D}_n(t^2; x) = x^2 b_2 + \frac{x}{n} b_1\Big(1 + 2npa_1\Big) + p^2\Big(a_2 + a_1\Big),$$
$$\mathcal{D}_n(t^3; x) = 3px^2 a_1 b_2 + 3xp^2 a_2 b_1 + 3xp\Big(p + \frac{1}{n}\Big)a_1 b_1 + p^3 a_3 + 3p^3 a_2 + p\Big(p^2 - \frac{2}{n} + \frac{2}{n^2}\Big)a_1 + x^3 b_3 + \frac{3x^2}{n} b_2 + \frac{x}{n^2} a_1,$$
$$\mathcal{D}_n(t^4; x) = x^4 b_4 + \frac{x^3}{n} b_3\Big(6 + 4npa_1\Big) + \frac{x^2}{n^2} b_2\Big(7 + 6n^2p^2 a_2 + 3np(np + 6)a_1\Big) + \frac{x}{n^3} b_1\Big(1 + 4n^3p^3 a_3 + 9n^2p^2(np + 1)a_2\Big) + \frac{1}{n^3}\Big(n^2p^3 - 18np^2 - 8p\Big)a_1 b_1 + p^4 a_4 + 6p^4 a_3 + 7p^4 a_2 + \Big(p^4 - \frac{12}{n^2}p^2 + \frac{12}{n^3}p\Big)a_1,$$

where $p = \frac{\rho+1}{n}$, $a_r = \frac{\mathcal{A}^{(r)}(1)}{\mathcal{A}(1)}$ and $b_r = \frac{\mathcal{B}^{(r)}(nx)}{\mathcal{B}(nx)}$, $r \in \mathbb{N}$. These notations will be used throughout the paper.

Proof. Using Lemma 2.1 and (1.7), the proof of this lemma can be easily obtained. Hence the details are omitted. □

As a consequence of Lemma 2.2, we have the following result.

Lemma 2.3. *For $x \in \mathbb{R}_0^+$, the central moments $\mathcal{D}_n((t-x)^m; x)$, $m = 1, 2, 4$, are defined by*

$$\mathcal{D}_n(t - x; x) = x(b_1 - 1) + pa_1,$$
$$\mathcal{D}_n((t-x)^2; x) = x^2\Big(b_2 - 2b_1 + 1\Big) + 2xpa_1\Big(b_1 - 1\Big) + \frac{x}{n} b_1 + p^2(a_2 + a_1),$$
$$\mathcal{D}_n((t-x)^4; x) = x^4\Big(1 - 4b_1 + 6b_2 - 4b_3 + b_4\Big) - x^3\Big(-4pa_1 - \frac{12}{n} b_2 + 4pa_1 b_3 + \frac{6}{n} b_1 + 12pa_1 b_1 + \frac{6}{n} b_3 - 12pa_1 b_2\Big) + x^2\Big(6p^2 a_1 - \frac{4}{n^2} a_1 + 6p^2 a_2 - 12p^2 a_2 b_1 - 12p\Big(p + \frac{1}{n}\Big)a_1 b_1 + 6p^2 a_2 b_2 + 3p\Big(p + \frac{6}{n}\Big)a_1 b_2 + \frac{7}{n^2} b_2\Big) - x\Big(4p^3 a_3 - 12p^3 a_2 - 4\Big(p^3 - \frac{2}{n}p^2 - \frac{2}{n^2}p\Big)a_1 + 4p^3 a_3 b_1$$

$$+ 9\Big(p^3 + \frac{p^2}{n}\Big)a_2b_1 + \frac{1}{n^3}b_1\Big) + \Big(\frac{p^3}{n} + \frac{16}{n^2}p^2 - \frac{6}{n^3}p\Big)a_1b_1 + p^4a_4$$
$$+ 6p^4a_3 + 7p^4a_2 + \Big(p^4 - 12\frac{p^2}{n^2} + 12\frac{p}{n^3}\Big)a_1.$$

For the remainder of the work we denote $\xi_n^\rho(x) = \mathcal{D}_n((t-x)^2; x)$ and assume that

$$\lim_{s\to\infty} \frac{\frac{d^r\mathcal{B}(s)}{ds^r}}{\mathcal{B}(s)} = 1, \quad \text{for } 1 \le r \le k, k \in \mathbb{N}. \tag{2.1}$$

Also, let $C_E(\mathbb{R}_0^+)$ be the space of all continuous functions on the interval $\mathbb{R}_0^+$ with $|f(t)| \le \alpha e^{\beta x}$ for all $t \ge 0$ and positive finite numbers α and β.

Theorem 2.1. *Let $f \in C_E(\mathbb{R}_0^+)$. If $\rho \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \mathcal{D}_n(f; x) = f(x),$$

converges uniformly in each compact subset of $\mathbb{R}_0^+$.

Proof. With the help of Lemma 2.2 and condition (2.1), we have

$$\lim_{n\to\infty} \mathcal{D}_n(t^r; x) = x^r, \quad \text{for } r = 0, 1, 2.$$

The above convergence is satisfied uniformly in every compact subset of $\mathbb{R}_0^+$. Hence, by applying Korokin's type theorem (vi) of Theorem 4.1.4 in [4], we get the desired result. □

Next, we present some useful definitions which are needed in the sequel.

Definition 2.1. Let $\delta > 0$ and $f \in C^*(\mathbb{R}_0^+)$. Then the usual modulus of continuity $\omega(f;\delta)$ is defined as

$$\omega(f;\delta) := \sup_{|x-y|\le\delta} |f(x) - f(y)|, \quad \text{for all } x, y \in [0,\infty),$$

where $C^*(\mathbb{R}_0^+)$ be a space of uniformly continuous functions defined on $[0,\infty)$. It is also known that, for any $\delta > 0$,

$$|f(x) - f(y)| \le \omega(f;\delta)\Big(\frac{|x-y|}{\delta} + 1\Big), \quad \text{for all } x, y \in \mathbb{R}_0^+.$$

Definition 2.2. Let $f \in C_B(\mathbb{R}_0^+)$. Then the second order modulus of smoothness is defined by

$$\omega_2(f;\delta) := \sup_{0<t\le\delta} \|f(\cdot + 2t) - 2f(\cdot + t) + f(\cdot)\|_{C_B},$$

where $C_B(\mathbb{R}_0^+)$ is a class of bounded and uniformly continuous real-valued functions with the norm $\|f\|_{C_B} = \sup_{x\in\mathbb{R}_0^+} |f(x)|$.

Definition 2.3 ([9])**.** Let $f \in C_B(\mathbb{R}_0^+)$. The Peetre's K-functional is defined by

$$K(f;\delta) := \inf\left\{\|f-g\|_{C_B} + \delta\|g\|_{C_B^2}\right\}, \quad \text{for all } g \in C_B^2(\mathbb{R}_0^+), \tag{2.2}$$

where $C_B^2(\mathbb{R}_0^+) := \{g \in C_B(\mathbb{R}_0^+) : g' \in AC_{loc}(\mathbb{R}_0^+), g'' \in C_B(\mathbb{R}_0^+)\}$ endowed with the norm $\|g\|_{C_B^2} := \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B}$ and $g' \in AC_{loc}(\mathbb{R}_0^+)$ means that g' is locally absolutely continuous function. It is also known that from [9], there exists an absolute constant $C > 0$, such that

$$K(f;\delta) \le C\omega_2(f;\sqrt{\delta}). \tag{2.3}$$

It is clear that the following inequality

$$K(f,\delta) \le M\left\{\omega_2(f;\sqrt{\delta}) + \min(1,\delta)\|f\|_{C_B}\right\}, \tag{2.4}$$

is valid, for all $\delta > 0$. The constant $M > 0$ is independent of f and δ.

3. The Order of Approximation

In this section, we establish the rate of convergence for the operators $\mathcal{D}_n$ in terms of Peetre's K-functional, classical and second-order modulus of continuity.

Theorem 3.1. *Let $f \in C_E(\mathbb{R}_0^+)$ and $\rho \in \mathbb{N}$. Then the operators $\mathcal{D}_n$ satisfy the following inequality:*

$$|\mathcal{D}_n(f;x) - f(x)| \le 2\omega\left(f;\sqrt{\xi_n^\rho(x)}\right),$$

where $\xi := \xi_n^\rho(x) = \mathcal{D}_n((t-x)^2;x) = x^2\Big(b_2 - 2b_1 + 1\Big) + 2xpa_1\Big(b_1 - 1\Big) + \frac{x}{n}b_1 + p^2(a_2 + a_1)$, see Lemma 2.3.

Proof. In view of the fact that $\mathcal{D}_n(1;x) = 1$ and (1.7), we have

$$\begin{aligned}
|\mathcal{D}_n(f;x) - f(x)| \le& \frac{1}{\mathcal{A}(1)\mathcal{B}(nx)} \sum_{k=0}^{\infty} q_k(nx)\left|f\left(\frac{k}{n}\right) - f(x)\right| \\
\le& \frac{1}{\mathcal{A}(1)\mathcal{B}(nx)} \sum_{k=0}^{\infty} q_k(nx)\left(\frac{1}{\delta}\left|\frac{k}{n} - x\right| + 1\right)\omega(f;\delta) \\
\le& \left\{1 + \frac{1}{\delta\mathcal{A}(1)\mathcal{B}(nx)} \sum_{k=0}^{\infty} q_k(nx)\left|\frac{k}{n} - x\right|\right\}\omega(f;\delta). \qquad (3.1)
\end{aligned}$$

In view of Lemma 2.3 and applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\sum_{k=0}^{\infty} q_k(nx)\left|\frac{k}{n} - x\right| \le& \left\{\sum_{k=0}^{\infty} q_k(nx)\left|\frac{k}{n} - x\right|^2\right\}^{1/2} \\
\le& \left(\sum_{k=0}^{\infty} q_k(nx)\right)^{1/2}\left(\sum_{k=0}^{\infty} q_k(nx)\left|\frac{k}{n} - x\right|^2\right)^{1/2} \\
=& \sqrt{\mathcal{A}(1)\mathcal{B}(nx)}\left(\mathcal{A}(1)\mathcal{B}(nx)\mathcal{D}_n((t-x)^2;x)\right)^{1/2}
\end{aligned}$$

$$=\mathcal{A}(1)\mathcal{B}(nx)\left(\mathcal{D}_n((t-x)^2;x)\right)^{1/2}$$

$$=\mathcal{A}(1)\mathcal{B}(nx)\sqrt{\xi_n^\rho(x)}. \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$|\mathcal{D}_n(f;x)-f(x)| \leq \left\{1+\frac{\sqrt{\xi_n^\rho(x)}}{\delta}\right\}\omega(f;\delta).$$

Choosing $\delta=\sqrt{\xi_n^\rho(x)}$, we obtain the desired result. □

Remark 3.1. For $\rho=0$, Theorem 3.1 represents the Theorem 2 for the operators given by (1.5) (see [20]).

Theorem 3.2. *Let $f \in C_B^2(\mathbb{R}_0^+)$ and $\rho \in \mathbb{N}$. Then we have*

$$|\mathcal{D}_n(f;x)-f(x)| \leq \psi\|f\|_{C_B^2(\mathbb{R}_0^+)},$$

where $\psi := \psi_n^\rho(x) = \Big[\frac{1}{2}\Big(b_2-2b_1+1\Big)x^2+\Big\{n\Big(b_1-1\Big)\Big(pa_1+1\Big)+b_1\Big\}\frac{x}{n}+pa_1+p^2(a_2+a_1)\Big]\|f\|_{C_B^2(\mathbb{R}_0^+)}$.

Proof. Let $x \in \mathbb{R}_0^+$. Applying Taylor's expansion to the function $f \in C_B^2(\mathbb{R}_0^+)$ and using the linearity of $\mathcal{D}_n$, we have

$$\mathcal{D}_n(f;x)-f(x)=f'(x)\mathcal{D}_n(t-x;x)+\frac{1}{2}f^{(2)}(\xi)\mathcal{D}_n((t-x)^2;x), \quad \xi \in (x,t).$$

Using Lemma 2.3, we have

$$\begin{aligned} |\mathcal{D}_n(f;x)-f(x)| \leq & \Big\{x(b_1-1)+pa_1\Big\}\|f'\|_{C_B(\mathbb{R}_0^+)} \\ & +\frac{1}{2}\Big\{x^2\Big(b_2-2b_1+1\Big)+2xpa_1\Big(b_1-1\Big) \\ & +\frac{x}{n}b_1+p^2(a_2+a_1)\Big\}\|f^{(2)}\|_{C_B(\mathbb{R}_0^+)} \\ \leq & \Big[\frac{1}{2}\Big(b_2-2b_1+1\Big)x^2+\Big\{n\Big(b_1-1\Big)\Big(pa_1+1\Big)+b_1\Big\}\frac{x}{n} \\ & +pa_1+p^2(a_2+a_1)\Big]\|f\|_{C_B^2(\mathbb{R}_0^+)}. \end{aligned} \tag{3.3}$$

This completes the proof of the theorem. □

Theorem 3.3. *Let $f \in C_B(\mathbb{R}_0^+)$. Then the following inequality satisfy:*

$$|\mathcal{D}_n(f;x)-f(x)| \leq 2M\left\{\omega_2(f;\sqrt{\delta})+\min(1,\delta)\|f\|_{C_B(\mathbb{R}_0^+)}\right\},$$

where $\delta := \delta_n^\rho(x)=\frac{1}{2}\psi_n^\rho(x)$ and M is a positive constant which is independent of the function f and δ. Also, $\psi_n^\rho(x)$ is defined in Theorem 3.2.

Proof. Let $h \in C_B^2(\mathbb{R}_0^+)$. In view of the Theorem 3.2, we have

$$\begin{aligned}|\mathcal{D}_n(f;x) - f(x)| =& |\mathcal{D}_n(f-h;x)| + |\mathcal{D}_n(h;x) - h(x)| + |f(x) - h(x)|\\ \le& 2\|f-h\|_{C_B} + \psi\|h\|_{C_B^2(\mathbb{R}_0^+)}\\ \le& 2\left[\|f-h\|_{C_B} + \delta\|h\|_{C_B^2(\mathbb{R}_0^+)}\right].\end{aligned}$$

Left-hand side of the above inequality is independent of $h \in C_B^2(\mathbb{R}_0^+)$, so

$$|\mathcal{D}_n(f;x) - f(x)| \le 2K(f;\delta),$$

where $K(f;\delta)$ is defined in (2.2). Taking into account the relation (2.4) in the above inequality, we have

$$|\mathcal{D}_n(f;x) - f(x)| \le 2M\left\{\omega_2(f;\sqrt{\delta}) + \min(1,\delta)\|f\|_{C_B(\mathbb{R}_0^+)}\right\}.$$

This is the required result. □

Theorem 3.4. *Let $x \in \mathbb{R}_0^+$ and $f \in C_B(\mathbb{R}_0^+)$. Then we have the following relation*

$$|\mathcal{D}_n(f;x) - f(x)| \le 4\omega_2\Big(f;\sqrt{\lambda_n^\rho}\Big) + \omega(f;\gamma_n^\rho),$$

where

$$\lambda_n^\rho := \lambda_n^\rho(x) = \frac{1}{8}\left\{\xi_n^\rho(x) + \Big(x(b_1-1) + pa_1 - x\Big)^2\right\} \tag{3.4}$$

and

$$\gamma_n^\rho := \gamma_n^\rho(x) = |x(b_1-1) + pa_1 - x| = |\mathcal{D}_n((t-x);x) - x|. \tag{3.5}$$

Proof. Let us consider a new auxiliary operators $\tilde{\mathcal{D}}_n(f;x)$ on $C_B(\mathbb{R}_0^+)$ defined by

$$\tilde{\mathcal{D}}_n(f;x) := \mathcal{D}_n(f;x) - f\Big(x(b_1-1) + pa_1\Big) + f(x). \tag{3.6}$$

From the above auxiliary operators, it is observe that $\tilde{\mathcal{D}}_n(1;x) = 1$ and $\tilde{\mathcal{D}}_n(t;x) = x$. Let $h \in C_B^2(\mathbb{R}_0^+)$, $C_B^2(\mathbb{R}_0^+) = \{h \in C_B(\mathbb{R}_0^+) : h', h^{(2)} \in C_B(\mathbb{R}_0^+)\}$, then by Taylor series theorem, we have

$$h(t) = h(x) + (t-x)h'(x) + \int_x^t (t-\nu)h^{(2)}(\nu)d\nu.$$

Using Lemma 2.3 and (3.6) and applying the operators $\tilde{\mathcal{D}}_n$ on both sides of the above equation, we have

$$\tilde{\mathcal{D}}_n(h;x) - h(x) = \tilde{\mathcal{D}}_n\bigg(\int_x^t (t-\nu)h^{(2)}(\nu)d\nu; x\bigg).$$

It follows from (3.6) that

$$\begin{aligned}\tilde{\mathcal{D}}_n(h;x) - h(x) =& \mathcal{D}_n\bigg(\int_x^t (t-\nu)h^{(2)}(\nu)d\nu; x\bigg)\\ &+ \int_x^{x(b_1-1)+pa_1} \Big(x(b_1-1) + pa_1 - \nu\Big)h^{(2)}(\nu)d\nu\end{aligned}$$

$$\leq \frac{\|h^{(2)}\|}{2}\mathcal{D}_n((t-x)^2;x) + \frac{\|h^{(2)}\|}{2}\Big(x(b_1-1)+pa_1-x\Big)^2$$
$$= \frac{\|h^{(2)}\|}{2}\Big\{\xi_n^\rho(x) + \Big(x(b_1-1)+pa_1-x\Big)^2\Big\},$$

considering (3.4), we obtain

$$|\tilde{\mathcal{D}}_n(h;x) - h(x)| \leq 4\lambda_n^\rho \|h^{(2)}\|, \tag{3.7}$$

where λ_n^ρ is given in (3.4).

In view of Lemma 2.3 and (3.6), we have

$$|\tilde{\mathcal{D}}_n(f;x)| \leq \|\mathcal{D}_n(f;x)\| + 2\|f\| \leq 3\|f\|, \quad \text{for all } f \in C_B(\mathbb{R}_0^+). \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we obtain

$$\begin{aligned}|\mathcal{D}_n(f;x) - f(x)| \leq & |\tilde{\mathcal{D}}_n(f-h;x) - (f-h)(x)| + |\tilde{\mathcal{D}}_n(h;x) - h(x)| \\ & + \Big|f\Big(x(b_1-1)+pa_1\Big) - f(x)\Big| \\ \leq & 4\Big(\|f-h\| + \lambda_n^\rho\|h^{(2)}\|\Big) + \omega\Big(f;\Big|x(b_1-2)+pa_1\Big|\Big),\end{aligned}$$

taking the infimum on the first term of the above inequality for $h \in C_B^2(\mathbb{R}_0^+)$ and using the inequalities (3.5) and (2.2), we have

$$|\mathcal{D}_n(f;x) - f(x)| \leq 4K(f;\lambda_n^\rho) + \omega(f;\gamma_n^\rho),$$

where γ_n^ρ is given in (3.5) and in view of the relation (2.3), we get our desired result. □

Remark 3.2. It is note that from Theorem 3.1- Theorem 3.4, the operators $\mathcal{D}_n(f;x) \to f(x)$, when λ_n^ρ, γ_n^ρ, ψ_n^α and ξ_n^α tend to zero as $n \to \infty$ with the assumption (2.1).

Now, we estimate the following local approximation result for the function belonging to Lipschitz-type space.

For $\mu \geq 0$, $\nu > 0$ to be fixed, the class of two parameteric Lipschitz type functions [16] is defined as

$$Lip_M^{\zeta,\nu}(\alpha) = \left\{ f \in C_B(\mathbb{R}_0^+) : |f(t) - f(x)| \leq \frac{M|t-x|^\alpha}{(t+\zeta x^2+\nu x)^{\frac{\alpha}{2}}}, t, x \in (0,\infty) \right\},$$

where M is positive constant and $0 < \alpha \leq 1$. In particular, at $\zeta = 0$ and $\nu = 1$, the space $Lip_M^{0,1}(\alpha)$ reduced to the space $L_M^*(\alpha)$ defined in [18].

Theorem 3.5. *Let $f \in L_M^{\zeta,\nu}(\alpha)$ and $\rho \in \mathbb{N}$. Then, for all $x > 0$, we have*

$$|\mathcal{D}_n(f,x) - f(x)| \leq M\left(\frac{\xi_n^\rho(x)}{\zeta x^2 + \nu x}\right)^{\frac{\alpha}{2}},$$

where $\xi_n^\rho(x)$ is defined in Lemma 2.3.

Proof. Let $x \in (0,\infty)$ and $f \in L_M^{\zeta,\nu}(\alpha)$. We have

$$
\begin{aligned}
|\mathcal{D}_n(f,x) - f(x)| &\leq \mathcal{D}_n(|f(t)-f(x)|,x) \\
&\leq M\mathcal{D}_n\Big(\frac{|t-x|^\alpha}{(\zeta x^2+\nu x+t)^{\frac{\alpha}{2}}},x\Big) \\
&\leq \frac{M}{(\zeta x^2+\nu x)^{\frac{\alpha}{2}}}\mathcal{D}_n\Big(|t-x|^\alpha,x\Big). \qquad (3.9)
\end{aligned}
$$

First, we consider the case $\alpha = 1$. Applying Cauchy-Schwarz inequality in (3.9) at $\alpha = 1$, we obtain

$$
|\mathcal{D}_n(f,x) - f(x)| \leq \frac{M}{(\zeta x^2+\nu x)^{\frac{1}{2}}}\Big(\mathcal{D}_n((t-x)^2,x)\Big)^{\frac{1}{2}} \leq M\Big(\frac{\xi_n^\rho(x)}{\zeta x^2+\nu x}\Big)^{\frac{1}{2}}.
$$

Thus, the result holds for $\alpha = 1$.

Now, we prove the result is true for $0<\alpha<1$. Then, for $x \in (0,\infty)$, $f \in L_M^{\zeta,\nu}(\alpha)$ and applying Hölder's inequality in (3.9) by taking $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we obtain

$$
|\mathcal{D}_n(f,x) - f(x)| \leq \frac{M}{(\zeta x^2+\nu x)^{\frac{\alpha}{2}}}\mathcal{D}_n\Big(|t-x|,x\Big)^\alpha.
$$

Finally, applying the Cauchy-Schwartz inequality, we obtain

$$
|\mathcal{D}_n(f;x) - f(x)| \leq \frac{M}{(\zeta x^2+\nu x)^{\frac{\alpha}{2}}}\Big\{\mathcal{D}_n((t-x)^2;x)\Big\}^{\frac{\alpha}{2}} = M\Big(\frac{\xi_n^\alpha(x)}{\zeta x^2+\nu x}\Big)^{\frac{\alpha}{2}}.
$$

This completes the proof of theorem. ☐

4. Voronovskaja-Type Result

The following assumptions are required to discuss a quantitative Voronovskaja-type result for the operators (1.7).

Assumptions:

(i) $\lim_{n\to\infty} n(b_1-1) = \alpha(x)$;
(ii) $\lim_{n\to\infty} n(b_2-2b_1+1) = \beta(x)$;
(iii) $\lim_{n\to\infty} n(b_3-2b_2+b_1) = \lambda(x)$;
(iv) $\lim_{n\to\infty} n(b_3-3b_2+3b_1-1) = \delta(x)$;
(v) $\lim_{n\to\infty} n^2(b_4-4b_3+6b_2-4b_1+1) = \gamma(x)$;

where $\alpha(x)$, $\beta(x)$, $\lambda(x)$, $\delta(x)$ and $\gamma(x)$ are continuous and bounded functions on $\mathbb{R}_0^+$.

Taking into account (2.1), Lemma 2.3 and the above assumptions, we have the following.

Lemma 4.1. *The operators* (1.7) *verify:*

(i) $\lim_{n\to\infty} n\mathcal{D}_n((t-x);x) = x\alpha(x)+pa_1$;
(ii) $\lim_{n\to\infty} n\mathcal{D}_n((t-x)^2;x) = x^2\beta(x)+x$;
(iii) $\lim_{n\to\infty} n^2\mathcal{D}_n((t-x)^4;x) = x^4\gamma(x) - x^3\{4npa_1\delta(x)+6\lambda(x)\} - (3n^2p^2+10np-12)a_1+7$.

Theorem 4.1. *Let $f \in C_B^2(\mathbb{R}_0^+)$. Then we have*

$$\lim_{n\to\infty} n\{\mathcal{D}_n(f;x) - f(x)\} = \Big\{x\alpha(x) + pa_1\Big\} f'(x) + \Big\{x^2\beta(x) + x\Big\}\frac{f^{(2)}(x)}{2}.$$

Proof. Let $x \in \mathbb{R}_0^+$ be an arbitrary but fixed number. Applying the Taylor series theorem to the function $f \in C_B^2(\mathbb{R}_0^+)$, we have

$$f(t) - f(x) = (t-x)f'(x) + \frac{1}{2}(t-x)^2 f^{(2)}(x) + \kappa(t,x)(t-x)^2, \tag{4.1}$$

where $\kappa(t,x) \in C_E(\mathbb{R}_0^+)$ and satisfies $\lim_{t\to x}\kappa(t,x) = 0$. Now, applying the operators $\mathcal{D}_n$ both sides on the equation (4.1), we get

$$\begin{aligned}\lim_{n\to\infty} n\{\mathcal{D}_n(f;x) - f(x)\} &= \lim_{n\to\infty} nf'(x)\mathcal{D}_n(t-x;x) + \lim_{n\to\infty} n\frac{1}{2}\mathcal{D}_n((t-x)^2;x)f^{(2)}(x) \\ &\quad + \lim_{n\to\infty} n\mathcal{D}_n(\kappa(t,x)(t-x)^2;x). \end{aligned} \tag{4.2}$$

In the last term of (4.2), we apply the Cauchy-Schwartz inequality

$$n\mathcal{D}_n(\kappa(t,x)(t-x)^2;x) \leq \sqrt{n^2\mathcal{D}_n((t-x)^4;x)\mathcal{D}_n(\kappa^2(t,x);x)}. \tag{4.3}$$

Since $\kappa(t,x) \to 0$ as $t \to x$, it follows from Theorem 2.1 that

$$\lim_{n\to\infty} \mathcal{D}_n(\kappa^2(t,x);x) = \kappa^2(x,x) = 0, \tag{4.4}$$

uniformly for $x \in [0,b], b > 0$.

Combining the equations from (4.2)–(4.4) and taking into account the Lemma 4.1, we conclude that

$$\lim_{n\to\infty} n\{\mathcal{D}_n(f;x) - f(x)\} = \Big\{x\alpha(x) + pa_1\Big\} f'(x) + \Big\{x^2\beta(x) + x\Big\}\frac{f^{(2)}(x)}{2}.$$

This completes the proof of the theorem. □

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