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NEW TAUBERIAN THEOREMS FOR CESÀRO SUMMABLE TRIPLE SEQUENCES OF FUZZY NUMBERS

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Abstract. The purpose of this paper is to establish new results on Tauberian theorem for Cesàro summability of triple sequences of fuzzy numbers. Besides, we extend and unify several results in the available literature. Furthermore, a huge number of special cases, theorems and their implications are proved. We show some illustrative examples in support of the results obtained in this paper.

1. Introduction

The notion of the fuzzy set was originally introduced by Zadeh [23]. Later, Matloka [11] established bounded and convergent sequences of fuzzy numbers and proved that every convergent sequence is bounded. Then, Nanda [12] studied the spaces of bounded and convergent sequences of fuzzy numbers and proved that every Cauchy sequence of fuzzy numbers is convergent. Subrahmanyam [14] presented the notion of Cesàro summability of sequences of fuzzy numbers and established Tauberian hypotheses identified with the Cesàro summability method. Talo and Çanak [15] introduced the necessary and sufficient Tauberian conditions, under which convergence follows from Cesàro convergence of sequences of fuzzy numbers. Altın et al. [1] studied the concept of statistical summability by $(C, 1)$-mean for sequences of fuzzy numbers and obtained a Tauberian theorem on that basis. Talo and Başar [16] introduced the concept of slow decreasing sequence for fuzzy numbers and have proved that Cesàro summable sequence (X_n) is convergent, if (X_n) is slowly decreasing. Çanak [3] established the concept of the slow oscillation (that is, both slowly decreasing and

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slowly increasing) sequences for fuzzy numbers and proved that Cesàro summable sequence (X_n) is convergent if (X_n) is slowly oscillating. Later on, many researchers have investigated on sequences and sequences of fuzzy numbers for proving Tauberian theorems. Different classes of sequences and sequences of fuzzy numbers have been presented and studied by Tripathy et al. [22], Dutta [4], Dutta [5], Dutta and Bilgin [6], Tripathy and Debnath [21], Dutta and Başar [7], Jena et al. [9], Jena et al. [10] and many others. Çanak [3] introduced Tauberian theorem for Cesàro summability of sequences of fuzzy numbers. Later on, Jena et al. [8] proved some Tauberian theorems on Cesàro summable double sequences of fuzzy numbers and proved some interesting results. The reader can refer to the monograph [2] and the papers [17–19] and [20] on the classical sequence spaces and related topics. Motivated by the above-mentioned works, in this paper we present the notion of $((C,1,1,1)X)$-summability of a triple sequences of fuzzy numbers defined in Definition 2.11. This paper is organized in two principal parts. In the first one, we provide the necessary definitions which are useful for the development of this paper, and the second one, we show theorems, lemmas and corollaries that we obtained.

2. Notations and Definitions

In this section, we recall some well-know notions which are useful for the developing of this paper. Besides, we define some new notions on Cesàro means $(C,1,1,1)$ of triple sequences (X_{mng}) of fuzzy numbers.

Definition 2.1. Let D denote the set of all closed and bounded intervals $X=[x_1,x_2]$ on the real line $\mathbb{R}$. For $X,Y\in D$, we define

$$d(X,Y)=\max\{|x_1-y_1|,|x_2-y_2|\},$$

where $X=[x_1,x_2]$ and $Y=[y_1,y_2]$

Remark 2.1. It is known that (D,d) is a complete metric space.

Definition 2.2. A fuzzy number X is a fuzzy set on $\mathbb{R}$ and is a mapping $X:\mathbb{R}\to[0,1]$ associating each number t with its grade of membership $X(t)$.

Definition 2.3. A fuzzy number X is said to be convex if,

$$X(t)=\min\{X(s),X(r)\},\quad s<t<r.$$

Definition 2.4. If there exists $t_0\in\mathbb{R}$, such that $X(t_0)=1$, then the fuzzy number X is called normal. Besides, a fuzzy number X is said to be upper semi-continuous if, for each $X^{-1}([0,x+\varepsilon])$ for all $x\in(0,1)$, is open in the usual topology of $\mathbb{R}$. The set of all upper semicontinuous, normal, convex fuzzy numbers is denoted by $\mathbb{R}([0,1])$. For $\alpha\in(0,1]$, α-level set X^α of fuzzy number X is defined by

$$X^\alpha=\{t\in\mathbb{R}:X(t)\geq\alpha\}.$$

Definition 2.5. The set X^0 is defined as the closure of the following set $\{t \in \mathbb{R} : X(t) > 0\}$. We define $\bar{d} : \mathbb{R}([0,1]) \times \mathbb{R}([0,1]) \to \mathbb{R}_+ \cup \{0\}$, by

$$\bar{d}(X,Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

Definition 2.6. A triple sequence (X_{mng}) of fuzzy numbers is a function $X : \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\} \to \mathbb{R}([0,1])$ and is said to be convergent to a fuzzy number X_0 if, for every $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\bar{d}(X_{mng}, X_0) < \varepsilon, \quad \text{as } m, n, g \geq n_0.$$

Remark 2.2. We will denote

$$\begin{aligned}
\Delta_n X_{mng} =& \bar{d}(X_{mng}, X_{m,n-1,g}), \\
\Delta_m X_{mng} =& \bar{d}(X_{mng}, X_{m-1,n,g}), \\
\Delta_g X_{mng} =& \bar{d}(X_{mng}, X_{m,n,g-1})
\end{aligned}$$

and

$$\Delta_{m,n,g} X_{mng} = \bar{d}(X_{mng}, X_{m-1,n,g}) - \bar{d}(X_{m,n-1,g-1}, X_{m-1,n-1,g-1}), \quad X_{-1} = 0.$$

Definition 2.7. A triple sequence (X_{mng}) of fuzzy numbers is said to be bounded, if there exists a positive number $K > 0$ such that

$$\bar{d}(X_{mng}, X_0) \leq K, \quad \text{as } m, n, g \in \mathbb{N} \cup \{0\}.$$

Definition 2.8. The Cesàro transform $(C,1,1,1)X$ of triple sequences (X_{mng}) of fuzzy numbers is defined by

$$\begin{aligned}
((C,1,1,1)X)_{mng} &= \frac{1}{(m+1)(n+1)(g+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \sum_{h=0}^{g} X_{pqh} \\
&= \sum_{p=0}^{m} \sum_{q=0}^{n} \sum_{h=0}^{g} \frac{Y_{pqh}^{(1,1,1)}}{pqh} + X_{000}.
\end{aligned} \tag{2.1}$$

Analogous to (2.1), we can define the $(C,1,0,0)$-, $(C,0,0,1)$- and $(C,0,1,0)$- transforms of a sequence (X_{mng}) as follows

$$\begin{aligned}
((C,1,0,0)X)_{mng} =& \frac{1}{m+1} \sum_{p=0}^{m} X_{png}, \\
((C,0,1,0)X)_{mng} =& \frac{1}{n+1} \sum_{q=0}^{n} X_{mqg}, \\
((C,0,0,1)X)_{mng} =& \frac{1}{g+1} \sum_{h=0}^{g} X_{mnh},
\end{aligned} \tag{2.2}$$

respectively. Additionally, analogues to (2.1) and (2.2), we can define the $(C,1,1,0)$-, $(C,0,1,1)$- and $(C,1,0,1)$-transforms of a sequence (X_{mng}) as follows

$$\begin{aligned}((C,1,1,0)X)_{mng} =& \frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n} X_{pqg},\\ ((C,0,1,1)X)_{mng} =& \frac{1}{(n+1)(q+1)}\sum_{q=0}^{n}\sum_{h=0}^{g} X_{mqh},\\ ((C,1,0,1)X)_{mng} =& \frac{1}{(m+1)(q+1)}\sum_{p=0}^{m}\sum_{h=0}^{g} X_{pnh},\end{aligned}$$

respectively.

Remark 2.3. A triple sequence $X = (X_{mng})$ of fuzzy numbers is $(C,1,1,1)$-summable to a fuzzy number L if for every $\varepsilon > 0$, we have

$$\bar{d}(((C,1,1,1)X)_{mng}, L) < \varepsilon, \quad \text{as } m,n,g \to \infty.$$

Similarly, we say that it is $(C,1,0,0)$-summable to a fuzzy number L if for every $\varepsilon > 0$, we have

$$\bar{d}(((C,1,0,0)X)_{mng}, L) < \varepsilon, \quad \text{as } m,n,g \to \infty,$$

$(C,0,0,1)$-summable to a fuzzy number L if for every $\varepsilon > 0$, we have

$$\bar{d}(((C,0,0,1)X)_{mng}, L) < \varepsilon, \quad \text{as } m,n,g \to \infty,$$

and $(C,0,1,0)$-summable to a fuzzy number L if for every $\varepsilon > 0$, we have

$$\bar{d}(((C,0,1,0)X)_{mng}, L) < \varepsilon, \quad \text{as } m,n,g \to \infty.$$

We say that it is $(C,1,1,0)$-summable, $(C,0,1,1)$-summable and $(C,1,0,1)$- summable to a fuzzy number L if for every $\varepsilon > 0$ we have $\bar{d}(((C,1,1,0)X)_{mng}, L) < \varepsilon$, $\bar{d}(((C,0,1,1)X)_{mng}, L) < \varepsilon$ and $\bar{d}(((C,1,0,1)X)_{mng}, L) < \varepsilon$ as $m,n,g \to \infty$, respectively.

Definition 2.9. For each non-negative integers k, r and j, we define $((C,k,j,r)X)_{mng}$ as follows:

$$((C,k,j,r)X)_{mng} = \begin{cases} \dfrac{1}{(m+1)(n+1)(g+1)}\displaystyle\sum_{p=0}^{m}\sum_{q=0}^{n}\sum_{h=0}^{g}((C,k-1,j-1,r-1)X)_{phq}, & k,r,j \geq 1,\\ X_{mng}, & k,r,j = 0.\end{cases}$$

Definition 2.10. A triple sequence $X = (X_{mng})$ of fuzzy numbers is said to be (C,k,r,j)-summable to a fuzzy number L if for every $\varepsilon > 0$, we have

$$\bar{d}(((C,k,j,r)X)_{mng}, L) < \varepsilon, \quad \text{as } m,n,g \to \infty.$$

Remark 2.4. If $k = 1$, $r = 1$ and $j = 1$, then (C, k, r, j)-summability reduces to $(C, 1, 1, 1)$-summability. Moreover, if $k \neq 0$, $r = 0$ and $j = 0$, then (C, r, k, j)-summability reduces to $(C, k, 0, 0)$-summability. If $k = 0$, $r \neq 0$ and $j = 0$, then (C, r, k, j)-summability reduces to $(C, 0, r, 0)$-summability. Finally, if $k = 0$, $r = 0$ and $j \neq 0$, then (C, r, k, j)-summability reduces to $(C, 0, 0, j)$-summability.

Remark 2.5. Note that, Cesàro summability of $X = (X_{mng})$ refers $(C, 1, 1, 1)$ and (C, k, r, j)-summability of $X = (X_{mng})$.

Remark 2.6. It can also be noted that, the convergence of a triple sequence $X = (X_{mng})$ of fuzzy numbers implies the Cesàro summability of $X = (X_{mng})$, but the converse is not generally true as can be seen in the following example.

Example 2.1. Consider a function $f(a, b, c) = e^{7a}\sin(11b)$. The sequence (X_{mng}) of fuzzy numbers which is the sequence of coefficients in the Taylor's series expansion of the function $f(a, b, c)$ about origin is Cesàro summable but not convergent. For the proof of converse part, certain conditions are presented in terms of oscillatory behavior of triple sequence $X = (X_{mng})$ of fuzzy numbers.

Definition 2.11. Let us define (X_{mng}) as

$$X_{mng} = Y^{(1,1,1)}_{mng} + \sum_{p=1}^{m}\sum_{q=1}^{n}\sum_{h=1}^{g} \frac{Y^{(1,1,1)}_{pqh}}{pqh} + X_{000}, \quad m, n, g \in \mathbb{N},$$

where

$$X_{mng} - ((C, 1, 1, 1)X)_{mng} = Y^{(1,1,1)}_{mng}(\Delta X) = \frac{1}{(m+1)(n+1)(g+1)} \sum_{p=1}^{m}\sum_{q=1}^{n}\sum_{h=1}^{g} pqh(\Delta_{p,q,h}X_{pqh}). \tag{2.3}$$

Further, in analogy to Kronecker identity for a single sequence of fuzzy numbers, we can write

$$Y^{(1,0,0)}_{mng}(\Delta X) = \frac{1}{(m+1)} \sum_{p=1}^{m} p(\Delta_p, X_{png}), \tag{2.4}$$

$$Y^{(0,1,0)}_{mng}(\Delta X) = \frac{1}{(n+1)} \sum_{q=1}^{n} q(\Delta_q, X_{mqg}), \tag{2.5}$$

$$Y^{(0,0,1)}_{mng}(\Delta X) = \frac{1}{(g+1)} \sum_{h=1}^{g} h(\Delta_h X_{mnh}), \tag{2.6}$$

as the $(C, 1, 0, 0)$-mean of the sequence $(m\Delta_m X_{mng})$ of fuzzy numbers, $(C, 0, 1, 0)$-mean of the sequence $(n\Delta_n X_{mng})$ of fuzzy numbers and $(C, 0, 0, 1)$-mean of the sequence $(g\Delta_g X_{mng})$ of fuzzy numbers, respectively.

We define $Y^{(1,1,0)}_{mng}$, $Y^{(1,0,1)}_{mng}$ and $Y^{(0,1,1)}_{mng}$ in the similar manner to (2.4), (2.5) and (2.6)

Remark 2.7. Since the sequence $Y^{(1,1,1)}_{mng}(\Delta_{mng}X_{mng})$ of fuzzy numbers is the $(C,1,1,1)$-mean of the sequence $mng(\Delta_{mng}X_{mng})$ of fuzzy number, the sequence $mng(\Delta_{mng}X_{mng})$ is $(C,1,1,1)$-summable to a fuzzy number L, whenever

$$\bar{d}(Y^{(1,1,1)}_{mng}(\Delta_{mng}X_{mng}),L)<\varepsilon,\quad \text{as } m,n,g\to\infty.$$

Definition 2.12. For each non-negative integers k, r and j, we define $Y^{(k,r,j)}_{mng}$ as follows:

$$Y^{(k,r,j)}_{mng}=\begin{cases}\dfrac{1}{(m+1)(n+1)(g+1)}\displaystyle\sum_{p=0}^{m}\sum_{q=0}^{n}\sum_{h=0}^{g}Y^{(k-1,r-1,j-1)}_{pqh}, & k,r,j\geq 1,\\[2ex] mng(\Delta_{mng}X_{mng}), & k,r,j=0.\end{cases}$$

Definition 2.13. The sequence $mng(\Delta_{mng}X_{mng})$ of fuzzy numbers is said to be (C,k,r,j)-summable to a fuzzy number L if for every $\varepsilon>0$, we have

$$\bar{d}(Y^{krj}_{mng}(\Delta_{mng}X_{mng}),L)<\varepsilon,\quad \text{as } m,n,g\to\infty.$$

Remark 2.8. If $k=1$, $r=1$ and $j=1$, then (C,k,r,j)-summability reduces to $(C,1,1,1)$-summability. Moreover, if $k\neq 0$, $r=0$ and $j=0$, then (C,r,k,j)-summability reduces to $(C,k,0,0)$-summability. Besides, if $k=0$, $r\neq 0$ and $j=0$, then (C,r,k,j)-summability reduces to $(C,0,r,0)$-summability. For $k=0$, $r=0$ and $j\neq 0$, (C,r,k,j)-summability reduces to $(C,0,0,j)$-summability.

Now, we define the De la Vallée Poussin transform of triple sequence (X_{mng}) of fuzzy numbers for sufficiently large non-negative integers m,n,g for $\lambda>1$ and $0<\lambda<1$

$$\tau_{mng}(X)=\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}X_{ipu}$$

and

$$\tau_{mng}(X)=\frac{1}{(m-[\lambda m])(n-[\lambda n])(g-[\lambda g])}\sum_{i=\lambda m+1}^{m}\sum_{p=\lambda n+1}^{n}\sum_{u=\lambda g+1}^{g}X_{ipu},$$

respectively.

Definition 2.14 ([13])**.** A single sequence $X=(X_n)$ of fuzzy numbers is slowly oscillating (in the sense of Stanojevic) if

$$\lim_{\lambda\to 1^+}\limsup_{n}\underline{\max_{n+1\leq k\leq[\lambda n]}}\bar{d}(X_k,X_n)=0.$$

Similar to Definition 2.14, we will define a triple sequence $X=(X_{mng})$ of fuzzy numbers.

Definition 2.15. A triple sequence $X=(X_{mng})$ of fuzzy numbers is slowly oscillating (in the sense of Stanojević) if

$$\lim_{\lambda\to 1^+}\limsup_{m,n,g}\underline{\max_{m+1,n+1,q+1\leq i,p,u\leq[\lambda m],[\lambda n],[\lambda g]}}\bar{d}\left(\sum_{a=m+1}^{i}\sum_{b=m+1}^{p}\sum_{v=+1}^{u}\Delta_{a,b,v}X_{a,b,v},0\right)\leq\varepsilon.$$

3. Main Results

Lemma 3.1. *A triple sequence $X = (X_{mng})$ of fuzzy numbers is slowly oscillating if and only if $(Y_{mng}^{(1,1,1)})$ is slowly oscillating and bounded.*

Proof. Let $X = (X_{mng})$ be a slowly oscillation triple sequence. First of all, let us show that $\bar{d}(V_{mng}^{(1,1,1)}, 0) = O(1)$.

We have by definition of slow oscillation, for $\lambda > 1$,

$$\lim_{\lambda \to 1^+} \limsup_{m,n,g} \max_{\underline{m+1,n+1,g+1 \le i,p,u \le [\lambda m, \lambda n, \lambda g]}} \bar{d}\left(\sum_{a=m+1}^{i} \sum_{b=m+1}^{p} \sum_{v=+1}^{u} \Delta_{a,b,v} X_{a,b,v}, 0 \right) \le \varepsilon$$

and let us rewrite the finite sum $\sum_{i=1}^{m} \sum_{p=1}^{n} \sum_{u=1}^{g} ipu \Delta X_{ipu}$ as the series

$$\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{v=0}^{\infty} \sum_{\underline{\frac{m}{2^a+1}, \frac{n}{2^b+1}, \frac{g}{2^v+1} \le i,p,u \le \frac{m}{2^a}, \frac{n}{2^b}, \frac{g}{2^v}}} ipu \Delta X_{ipu}.$$

Clearly,

$$\begin{aligned}
&\bar{d}\left(\sum_{i=1}^{m} \sum_{p=1}^{n} \sum_{u=1}^{g} ipu \Delta X_{ipu}, 0 \right) \\
\le &\bar{d}\left(\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{v=0}^{\infty} \sum_{\underline{\frac{m}{2^a+1}, \frac{n}{2^b+1}, \frac{g}{2^v+1} \le i,p,u \le \frac{m}{2^a}, \frac{n}{2^b}, \frac{g}{2^v}}} ipu \Delta X_{ipu}, 0 \right) \\
\le &\bar{d}\left(\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{v=0}^{\infty} \frac{mng}{2^{a+n+v}}, 0 \right)
\end{aligned}$$

and

$$\begin{aligned}
&\max_{\underline{\frac{m}{2^a+1}+1, \frac{n}{2^b+1}+1, \frac{g}{2^v+1}+1 \le i,p,u \le \frac{\lambda m}{2^{i+1}}, \frac{\lambda n}{2^{p+1}}, \frac{\lambda g}{2^{u+1}}}} \\
&\bar{d}\left(\sum_{a=\frac{m}{2^{a+1}}+1}^{i} \sum_{b=\frac{n}{2^{b+1}}+1}^{p} \sum_{v=\frac{g}{2^{v+1}}+1}^{u} \Delta_{a,b,v} X_{a,b,v}, 0 \right) \\
\le &mngC \left(\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{v=0}^{\infty} \frac{1}{2^{a+b+v}} \right) = mngC^*, \quad C^* > 0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}\bar{d}(Y^{(1,1,1)}_{mng}(\Delta X),0) =&\bar{d}\left(\frac{1}{(m+1)(n+1)(g+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\sum_{h=0}^{g} pqh(\Delta_{p,q,h}X_{p,q,h}),0\right)\\ =&O(1), \quad \text{as } m,n,g\to\infty.\end{aligned}$$

Since

$$\left\{((C,1,1,1)X)_{mng} = \sum_{p=1}^{m}\sum_{q=1}^{n}\sum_{h=1}^{g}\frac{Y^{(1,1,1)}_{pqh}}{pqh} + X_{000}\right\}$$

is slowly oscillating, hence, $(Y^{(1,1,1)}_{mng})$ is slowly oscillating.

Now, to prove the converse part, consider $(Y^{(1,1,1)}_{mng})$ is bounded and slowly oscillating. Thus, the boundedness of $(Y^{(1,1,1)}_{mng})$ implies that $((C,1,1,1)X)_{mng}$ is slowly oscillating. Moreover, $(Y^{(1,1,1)}_{mng})$ being oscillating slowly, so by Kronecker identity (2.3), (X_{mng}) is oscillating slowly. □

Lemma 3.2. *Let $X = (X_{mng})$ be a triple sequence of fuzzy numbers with m, n sufficiently large, then the following statements hold.*

(a) *For $\lambda > 1$*

(3.1)

$$\begin{aligned}&\bar{d}(X_{mng},((C,1,1,1)X)_{mng})\\ =&\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\{\bar{d}(((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]},((C,1,1,1)X)_{[\lambda m],n,g})\\ &-\bar{d}(((C,1,1,1)X)_{m,[\lambda n],[\lambda g]},((C,1,1,1)X)_{mng})\}\\ &+\frac{[\lambda m]+1}{([\lambda n]-m)([\lambda g]-m)}\bar{d}(((C,1,1,1)X)_{[\lambda m],n,g},((C,1,1,1)X)_{m,n,g})\\ &+\frac{[\lambda n]+1}{([\lambda m]-n)([\lambda g]-n)}\bar{d}(((C,1,1,1)X)_{m,[\lambda n],g},((C,1,1,1)X)_{m,n,g})\\ &+\frac{[\lambda g]+1}{([\lambda m]-g)([\lambda n]-g)}\bar{d}(((C,1,1,1)X)_{m,n,[\lambda g]},((C,1,1,1)X)_{m,n,g}(X))\\ &-\frac{1}{([\lambda m]-m)([\lambda n]-n)(\lambda g]-g)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(X_{ipu},X_{mng})\right).\end{aligned}$$

(b) *For $0<\lambda<1$*

$$\begin{aligned}&\bar{d}(X_{mng},((C,1,1,1)X)_{mng})\\ =&\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{(m-[\lambda m])(n-[\lambda n])(g-[\lambda g])}\{\bar{d}(((C,1,1,1)X)_{mng},((C,1,1,1)X)_{[\lambda m],n,g})\\ &-\bar{d}(((C,1,1,1)X)_{m,[\lambda n],[\lambda g]},((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]})\}\end{aligned} \tag{3.2}$$

$$+\frac{[\lambda m]+1}{(m-[\lambda m])}\bar{d}(((C,1,1,1)X)_{m,n,g},((C,1,1,1)X)_{[\lambda m],n,g})$$
$$+\frac{[\lambda n]+1}{(n-[\lambda n])}\bar{d}(((C,1,1,1)X)_{m,n,g},((C,1,1,1)X)_{m,[\lambda n],g})$$
$$+\frac{[\lambda g]+1}{([g-\lambda g])}\bar{d}(((C,1,1,1)X)_{m,n,g},((C,1,1,1)X)_{m,n,[\lambda g]})$$
$$-\frac{1}{(m-[\lambda m])(n-[\lambda n])(g-\lambda g])}\bar{d}\left(\sum_{i=[\lambda m]+1}^{m}\sum_{p=[\lambda n]+1}^{n}\sum_{u=[\lambda g]+1}^{g}(X_{mng},X_{ipu})\right).$$

Proof. We just prove (3.1), (3.2) by the similar way.

We have by De la Vallée Poussin mean of triple sequence (X_{mng}) of fuzzy numbers

$$\tau_{mng}(X)$$
$$=\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}X_{ipu}$$
$$=\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\left\{\bar{d}\left(\sum_{i=0}^{[\lambda m]},\sum_{i=0}^{[m]}\right)\bar{d}\left(\sum_{p=0}^{[\lambda n]},\sum_{p=0}^{[n]}\right)\bar{d}\left(\sum_{u=0}^{[\lambda g]},\sum_{u=0}^{[g]}\right)\right\}X_{ipu}$$
$$=\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\left\{\bar{d}\left(\sum_{i=0}^{[\lambda m]}\sum_{p=0}^{[\lambda n]}\sum_{u=0}^{[\lambda u]},\sum_{i=0}^{[\lambda m]}\sum_{p=0}^{[n]}\sum_{u=0}^{[u]}\right)X_{ipu}\right\}$$
$$-\left\{\bar{d}\left(\sum_{i=0}^{[m]}\sum_{p=0}^{[\lambda n]}\sum_{u=0}^{[\lambda g]},\sum_{i=0}^{m}\sum_{p=0}^{n}\sum_{u=0}^{g}\right)X_{ipu}\right\}$$
$$=\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}$$
$$\{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}$$
$$-([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)((C,1,1,1)X)_{[\lambda m],n,g}\}$$
$$-\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\{(m+1)([\lambda n]+1)([\lambda g]+1)$$
$$((C,1,1,1)X)_{m,[\lambda n],[\lambda g]}-(m+1)(n+1)(g+1)((C,1,1,1)X)_{mng}\}$$
$$=\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}(C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}$$
$$-\left\{\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}((C,1,1,1)X)_{[\lambda m],n,g}\right.$$
$$\left.-\frac{([\lambda m]+1)}{([\lambda m]-m)}((C,1,1,1)X)_{[\lambda m],n,g}\right\}$$
$$-\left\{\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}((C,1,1,1)X)_{m,[\lambda n],g}\right.$$

$$- \frac{([\lambda n]+1)}{([\lambda n]-n)}((C,1,1,1)X)_{m,[\lambda n],g}\Bigg\}$$

$$- \Bigg\{\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}((C,1,1,1)X)_{m,n,[\lambda g]}$$

$$- \frac{([\lambda g]+1)}{([\lambda g]-g)}((C,1,1,1)X)_{m,n,[\lambda g]}\Bigg\}$$

$$+ \Bigg\{\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}((C,1,1,1)X)_{mng}$$

$$- \frac{([\lambda m]+1)}{([\lambda m]-m)}((C,1,1,1)X)_{mng} - \frac{([\lambda n]+1)}{([\lambda n]-n)}((C,1,1,1)X)_{mng}$$

$$- \frac{([\lambda g]+1)}{([\lambda g]-g)}((C,1,1,1)X)_{mng} + ((C,1,1,1)X)_{mng}\Bigg\},$$

which implies

$$\begin{aligned}
&\tau_{mng} - ((C,1,1,1)X)_{mng} \\
=&\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\{\bar{d}(((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}, \\
&((C,1,1,1)X)_{[\lambda m],n,g}) - \bar{d}(((C,1,1,1)X)_{m,[\lambda n],[\lambda g]}, ((C,1,1,1)X)_{m,n,g})\} \\
&+ \frac{([\lambda m]+1)}{([\lambda m]-m)}\bar{d}(((C,1,1,1)X)_{[\lambda m],n,g}, (C,1,1,1)X)_{mng}) \\
&+ \frac{([\lambda n]+1)}{([\lambda n]-n)}\bar{d}(((C,1,1,1)X)_{m,[\lambda n],g}, ((C,1,1,1),X)_{mng}) \\
&+ \frac{([\lambda g]+1)}{([\lambda g]-g)}\bar{d}(((C,1,1,1)X)_{m,n,[\lambda g]}, ((C,1,1,1)X)_{mng}).
\end{aligned}$$

Besides,

$$X_{mng} = \tau_{mng} - \frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(X_{ipu}, X_{mng})\right).$$

On subtracting $(((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]})$ from the above identity, we have

$$\begin{aligned}
&\bar{d}(X_{mng}, ((C,1,1,1)X)_{mng}) \\
=&\bar{d}(\tau_{mng}(X), ((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]} \\
&- \frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(X_{ipu}, X_{mng})\right) \\
=&\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\{\bar{d}(((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}, ((C,1,1,1)X)_{[\lambda m],n,g}) \\
&- \bar{d}(((C,1,1,1)X)_{m,[\lambda n],[\lambda g]}, ((C,1,1,1)X)_{mng})\}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{[\lambda m]+1}{([\lambda n]-m)([\lambda g]-m)}\bar{d}(((C,1,1,1)X)_{[\lambda m],n,g},((C,1,1,1)X)_{m,n,g})\\
&+\frac{[\lambda n]+1}{([\lambda m]-n)([\lambda g]-n)}\bar{d}(((C,1,1,1)X)_{m,[\lambda n],g},((C,1,1,1)X)_{m,n,g})\\
&+\frac{[\lambda g]+1}{([\lambda m]-g)([\lambda n]-g)}\bar{d}(((C,1,1,1)X)_{m,n,[\lambda g]},((C,1,1,1))_{m,n,g})\\
&-\frac{1}{([\lambda m]-m)([\lambda n]-n)(\lambda g]-g)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(X_{ipu},X_{mng})\right). \qquad \square
\end{aligned}
$$

Theorem 3.1. *If a triple sequence (X_{mng}) of fuzzy number is $(C,1,1,1)$-summable to a fuzzy number L and (X_{mng}) is slowly oscillating (in the sense of Stanojević), then*

$$\bar{d}(X_{mng},L)<\varepsilon, \quad \text{as } m,n,g\to\infty.$$

Proof. Let (X_{mng}) be $(C,1,1,1)$-summable to a fuzzy number L, this implies $\sigma^{(1,1,1)}_{mng}$ is $(C,1,1,1)$-summable to a fuzzy number L. Now, from (2.3), we have $(Y^{(1,1,1)}_{mng})$ is $(C,1,1,1)$-summable to zero. Hence, by Lemma 3.1, $(Y^{(1,1,1)}_{mng})$ is slowly oscillating. Additionally, by Lemma 3.2 part (a), we obtain

$$
\begin{aligned}
(3.3)\quad &\bar{d}(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}(Y^{(1,1,1)}_{mng}))\\
&=\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\{\bar{d}(((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}(Y^{(1,1,1)}_{mng}),\\
&((C,1,1,1)X)_{[\lambda m],n,g}(Y^{(1,1,1)}_{mng}))\\
&-\bar{d}(((C,1,1,1)X)_{m,[\lambda n],[\lambda g]}(Y^{(1,1,1)}_{mng}),((C,1,1,1)X)_{mng}(Y^{(1,1,1)}_{mng}))\}\\
&+\frac{[\lambda m]+1}{([\lambda n]-m)([\lambda g]-m)}\bar{d}(((C,1,1,1)X)_{[\lambda m],n,g}(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}))\\
&+\frac{[\lambda n]+1}{([\lambda m]-n)([\lambda g]-n)}\bar{d}(((C,1,1,1)X)_{m,[\lambda n],g}(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}))\\
&+\frac{[\lambda g]+1}{([\lambda m]-g)([\lambda n]-g)}\bar{d}(((C,1,1,1)X)_{m,n,[\lambda g]}(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}))\\
&-\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y^{(1,1,1)}_{ipu},Y^{(1,1,1)}_{mng})\right).
\end{aligned}
$$

It is easy to verify that for $\lambda>1$ and sufficiently large n and g

$$
\begin{aligned}
\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}&<\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-1-m)([\lambda n]-1-n)([\lambda g]-1-g)}\\
&<\frac{9\lambda^3}{(\lambda-1)^3}.
\end{aligned}
$$

Now, by (3.3),

$$\bar{d}(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}(Y^{(1,1,1)}_{mng}))$$

$$\leq \frac{9\lambda^3}{(\lambda-1)^3}\bar{d}(\tau_{mng}(Y_{mng}^{(1,1,1)}), ((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}(Y_{mng}))$$
$$- \underline{\max_{m+1,n+1,g+1\leq i,p,u\leq [\lambda m],[\lambda n],[\lambda g]}} \bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y_{ipu}^{(1,1,1)}, Y_{mng}^{(1,1,1)})\right).$$

Taking lim sup on both sides in the above inequality, we have

$$\limsup_{m,n,g} \bar{d}(Y_{mng}^{(1,1,1)}, ((C,1,1,1)X)_{mng}(Y_{mng}^{(1,1,1)}))$$
$$\leq \frac{9\lambda^3}{(\lambda-1)^3}\limsup_{m,n,g}\bar{d}(\tau_{mng}(Y_{mng}^{(1,1,1)}), ((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}(Y_{mng}^{(1,1,1)}))$$
$$- \limsup_{m,n,g} \underline{\max_{m+1,n+1,g+1\leq i,p,u\leq [\lambda m],[\lambda n],[\lambda g]}} \bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y_{ipu}^{(1,1,1)}, Y_{mng}^{(1,1,1)})\right).$$

Moreover, $((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}(Y_{mng}^{(1,1,1)}) \to 0$ as $m,n,g\to\infty$, so first term on the right hand side of above inequality, must vanish. This implies,

$$\text{(3.4)}\quad \limsup_{m,n,g} \bar{d}(Y_{mng}^{(1,1,1)}, ((C,1,1,1)X)_{mng}(Y_{mng}^{(1,1,1)}))$$
$$\leq \limsup_{m,n,g} \underline{\max_{m+1,n+1,g+1\leq i,p,u\leq [\lambda m],[\lambda n],[\lambda g]}} \bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y_{ipu}^{(1,1,1)}, Y_{mng}^{(1,1,1)})\right).$$

As $\lambda \to 1^+$ in (3.4), thus we have

$$\limsup_{m,n,g} \bar{d}(Y_{mng}^{(1,1,1)}, ((C,1,1,1)X)_{mng}(Y_{mng}^{(1,1,1)})) \leq 0.$$

This implies $\bar{d}(Y_{mng}^{(1,1,1)}, 0) < \varepsilon$ as $m,n,g\to\infty$. Since (X_{mng}) is summable to a fuzzy number L by $(C,1,1,1)$ mean and $\bar{d}(Y_{mng}^{(1,1,1)}, 0) < \varepsilon$ as $m,n,g\to\infty$, thus $\bar{d}(X_{mng}, L) < \varepsilon$, $m,n,g\to\infty$. □

Corollary 3.1. *If (X_{mng}) is (C,k,r,j)-summable to a fuzzy number L and (X_{mng}) is slowly oscillating (in the sense of Stanojević), then*

$$\bar{d}(X_{mng}, L) < \varepsilon, \quad \textit{as } m,n,g\to\infty.$$

Proof. Let $X=(X_{mng})$ be slowly oscillating. Then, $((C,k,r,j)X)$ is slowly oscillating by Lemma 3.1. Furthermore, since $X=(X_{mng})$ is (C,k,r,j)-summable to a fuzzy number L, we have by Theorem 3.1 that

$$\text{(3.5)}\qquad \bar{d}(((C,k,r,j)X)_{mng}, L) < \varepsilon \quad \text{as} \quad m,n,g\to\infty.$$

Now, from the definition,

$$\text{(3.6)}\qquad ((C,k,r,j)X)_{mng} = ((C,1,1,1)X)_{mng}(((C,k-1,r-1,j-1)X)_{mng}).$$

It is clear that (3.5) and (3.6) imply $X=(X_{mng})$ is $(C,k-1,r-1,j-1)$-summable to a fuzzy number L. Thus, $(((C,k-1,r-1,j-1)X)_{mng})$ is slowly oscillating by Lemma

3.1. Therefore, by Theorem 3.1, we have $\bar{d}(((C,k-1,r-1,j-1)X)_{mng},L)<\varepsilon$ as $m,n,g\to\infty$. Continuing in this way, we get $\bar{d}(X_{mng},L)<\varepsilon$ as $m,n,g\to\infty$. □

Remark 3.1. If $k\neq 0, r=0$ and $j=0$, then (C,r,k,j)-summability reduces to $(C,k,0,0)$-summability. Besides, if $k=0, r\neq 0$ and $j=0$, then (C,r,k,j)-summability reduces to $(C,0,r,0)$-summability. Finally, if $k=0$, $r=0$ and $j\neq 0$, then (C,r,k,j)-summability reduces to $(C,0,0,j)$-summability.

Theorem 3.2. *If a triple sequence (X_{mng}) of fuzzy number is $(C,1,1,1)$-summable to a fuzzy number L and $Y^{(1,1,1)}_{mng}(\Delta_{mng}X_{mng})$ is slowly oscillating, then*

$$\bar{d}(X_{mng},L)<\varepsilon,\quad \text{as } m,n,g\to\infty.$$

Proof. Since (X_{mng}) is $(C,1,1,1)$-summable to a fuzzy number L, so $((C,1,1,1)X)_{mng}$ is $(C,1,1,1)$-summable to a fuzzy number L. Hence, $(Y^{(1,1,1)}_{mng})$ is $(C,1,1,1)$-summable to zero by (2.3). Using identity (2.3) to $(Y^{(1,1,1)}_{mng})$ we have $Y(Y^{(1,1,1)}_{mng})$ is Cesàro summable to zero. This means that $Y(Y^{(1,1,1)}_{mng})$ is slowly oscillating by Lemma 3.1. Now, by Lemma 3.2 part (a), we get

$$\begin{aligned}
(3.7)\quad &\bar{d}(Y(Y^{(1,1,1)}_{mng}),((C,1,1,1)X)_{mng}Y(Y^{(1,1,1)}_{mng}))\\
=&\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\{\bar{d}(((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}Y(Y^{(1,1,1)}_{mng}),\\
&((C,1,1,1)X)_{[\lambda m],n,g}Y(Y^{(1,1,1)}_{mng}))\\
&-\bar{d}(((C,1,1,1)X)_{m,[\lambda n],[\lambda g]}Y(Y^{(1,1,1)}_{mng}),((C,1,1,1)X)_{mng}Y(Y^{(1,1,1)}_{mng}))\}\\
&+\frac{[\lambda m]+1}{([\lambda n]-m)([\lambda g]-m)}\bar{d}(((C,1,1,1)X)_{[\lambda m],n,g}Y(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}))\\
&+\frac{[\lambda n]+1}{([\lambda m]-n)([\lambda g]-n)}\bar{d}(((C,1,1,1)X)_{m,[\lambda n],g}Y(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}))\\
&+\frac{[\lambda g]+1}{([\lambda m]-g)([\lambda n]-g)}\bar{d}(\sigma^{(1,1,1)}_{m,n,[\lambda g]}Y(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}))\\
&-\frac{1}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}Y(Y^{(1,1,1)}_{ipu},Y^{(1,1,1)}_{mng})\right).
\end{aligned}$$

It is easy to verify that for $\lambda>1$ and sufficiently large n and g

$$\begin{aligned}
\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-m)([\lambda n]-n)([\lambda g]-g)}&<\frac{([\lambda m]+1)([\lambda n]+1)([\lambda g]+1)}{([\lambda m]-1-m)([\lambda n]-1-n)([\lambda g]-1-g)}\\
&<\frac{9\lambda^3}{(\lambda-1)^3}.
\end{aligned}$$

Now, by (3.7),

$$\bar{d}(Y(Y^{(1,1,1)}_{mng},((C,1,1,1)X)_{mng}Y(Y^{(1,1,1)}_{mng}))$$

$$\leq \frac{9\lambda^3}{(\lambda-1)^3}\bar{d}(\tau_{mng}Y(Y_{mng}^{(1,1,1)}), ((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}Y(Y_{mng}))$$
$$- \max_{\underline{m+1,n+1,g+1\leq i,p,u\leq [\lambda m],[\lambda n],[\lambda g]}} \bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y(Y_{ipu}^{(1,1,1)}), Y(Y_{mng}^{(1,1,1)}))\right).$$

Taking lim sup on both sides in the above inequality, we have

$$\limsup_{m,n,g} \bar{d}(Y(Y_{mng}^{(1,1,1)}), ((C,1,1,1)X)_{mng}Y(Y_{mng}^{(1,1,1)})))$$
$$\leq \frac{9\lambda^3}{(\lambda-1)^3}\limsup_{m,n,g} \bar{d}(\tau_{mng}Y(Y_{mng}^{(1,1,1)}), ((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}Y(Y_{mng}^{(1,1,1)}))$$
$$- \limsup_{m,n,g} \max_{\underline{m+1,n+1,g+1\leq i,p,u\leq [\lambda m],[\lambda n],[\lambda g]}} \bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y(Y_{ipu}^{(1,1,1)}), Y(Y_{mng}^{(1,1,1)}))\right).$$

Further, $((C,1,1,1)X)_{[\lambda m],[\lambda n],[\lambda g]}Y(Y_{mng}^{(1,1,1)}) \to 0$ as $m,n,g \to \infty$, so first term in the right hand side of above inequality, must vanish. This implies

(3.8)
$$\limsup_{m,n,g} \bar{d}(Y_{mng}^{(1,1,1)}, ((C,1,1,1)X)_{mng}Y(Y_{mng}^{(1,1,1)}))$$
$$\leq \limsup_{m,n,g} \max_{\underline{m+1,n+1,g+1\leq i,p,u\leq [\lambda m],[\lambda n],[\lambda g]}} \bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{p=n+1}^{[\lambda n]}\sum_{u=g+1}^{[\lambda g]}(Y(Y_{ipu}^{(1,1,1)}), Y(Y_{mng}^{(1,1,1)}))\right).$$

Taking $\lambda \to 1^+$ in (3.8), we have

$$\limsup_{m,n,g} \bar{d}(Y(Y_{mng}^{(1,1,1)}), ((C,1,1,1)X)_{mng}Y(Y_{mng}^{(1,1,1)}))) \leq 0,$$

which implies, $\bar{d}(Y(Y_{mng}^{(1,1,1)}),0)) < \varepsilon$ as $m,n,g \to \infty$. Since (X_{mng}) is summable to a fuzzy number L by $(C,1,1,1)$ mean and $\bar{d}(Y(Y_{mng}^{(1,1,1)}),0) < \varepsilon$ as $m,n,g \to \infty$, thus $\bar{d}(X_{mng}, L) < \varepsilon$, as $m,n,g \to \infty$. □

Corollary 3.2. *If (X_{mng}) is (C,k,r,j)-summable to a fuzzy number L and $Y_{mng}^{(1,1,1)}(\Delta X)$ is slowly oscillating, then*

$$\bar{d}(X_{mng}, L) < \varepsilon, \quad \text{as } m,n,g \to \infty.$$

Proof. As $Y_{mng}^{(1,1,1)}(\Delta X)$ is slowly oscillating, setting $X = (X_{mng})$ instead of $Y_{mng}^{(1,1,1)}(\Delta X)$, then $((C,k,r,j)X)_{mng}(Y_{mng}^{(1,1,1)}(\Delta X))$ is slowly oscillating by Lemma 3.1. Moreover, as $Y_{mng}^{(1,1,1)}(\Delta X)$ is (C,k,r,j)-summable to a fuzzy number L, so by Theorem 3.2,

$$\bar{d}(((C,k,r,j)X)_{mng}(Y_{mng}^{(1,1,1)}(\Delta X)), L) < \varepsilon, \quad \text{as } m,n,g \to \infty. \tag{3.9}$$

By definition,

(3.10)
$$((C,k,r,j)X)_{mng}(Y_{mng}^{(1,1,1)}(\Delta X))$$

$$=((C,1,1,1)X)_{mng}(Y^{(1,1,1)}_{mng}(\Delta X))[((C,k-1,r-1,j-1)X)_{mng}(Y^{(1,1,1)}_{mng}(\Delta X))].$$

From (3.9) and (3.10) we have $Y^{(1,1,1)}_{mng}(\Delta X)$ is $(C,k-1,r-1,j-1)$-summable to a fuzzy number L. Thus, $((C,k-1,r-1,j-1)X)_{mng}(Y^{(1,1,1)}_{mng}(\Delta X))$ is slowly oscillating by Lemma 3.1. Therefore, by Theorem 3.1, we have

$$\bar{d}(((C,k-1,r-1,j-1)X)_{mng}(Y^{(1,1,1)}_{mng}(\Delta X)),L)<\varepsilon,\quad \text{as } m,n,g\to\infty.$$

Continuing this way, we get $\bar{d}((Y^{(1,1,1)}_{mng}(\Delta X)),L)<\varepsilon$ as $m,n,g\to\infty$. □

Remark 3.2. If $k\neq 0$, $r=0$ and $j=0$, then (C,r,k,j)-summability reduces to $(C,k,0,0)$-summability. Besides, if $k=0$, $r\neq 0$ and $j=0$, then (C,r,k,j)-summability reduces to $(C,0,r,0)$-summability. Finally, if $k=0$, $r=0$ and $j\neq 0$, then (C,r,k,j)-summability reduces to $(C,0,0,j)$-summability and consequently more corollaries can be generated from the main results of this paper.

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