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FOLDING THEORY APPLIED TO FUZZY (POSITIVE) IMPLICATIVE (PRE)FILTERS IN EQ-ALGEBRAS

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Abstract. In this paper, the concepts of fuzzy n-fold positive implicative and fuzzy n-fold implicative (pre)filters in EQ-algebras are introduced and several properties of them are provided. Moreover, the relationship between fuzzy n-fold positive implicative (pre)filter and fuzzy n-fold implicative (pre)filter is considered. Using the level subset of a fuzzy set in EQ-algebras, some characterizations of fuzzy n-fold (positive) implicative (pre)filters in EQ-algebras are given. Also, we investigate under what conditions the fuzzy n-fold positive implicative (pre)filter is a fuzzy n-fold implicative (pre)filter in EQ-algebras.

1. Introduction

EQ-algebras were proposed by Novák and De Baets [6,8]. One of the motivations was to introduce a special algebra as the correspondence of truth values for high-order fuzzy type theory (FTT) [7] that generalizes the system of classical type theory [1] in which the sole basic connective is equality. Analogously, the basic connective in (FTT) should be fuzzy equality. Another motivation is from the equational style of proof in logic. It has three connectives: meet $\wedge$, product $\odot$ and fuzzy equality $\sim$. The implication operation $\to$ is the derived of the fuzzy equality $\sim$ and it together with $\odot$ no longer strictly form the adjoint pair in general. EQ-algebras are interesting and important for studying and researching and residuated lattices are particular cases of EQ-algebras. In fact, EQ-algebras generalize non-commutative residuated lattices [3]. From the point of view of logic, the main diference between residuated lattices and EQ-algebras lies in the way the implication operation is obtained. While in residuated

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lattices it is obtained from (strong) conjunction, in EQ-algebras it is obtained from equivalence. Consequently, the two kinds of algebras differ in several essential points despite their many similar or identical properties. Since residuated lattices (BL-algebras, MV-algebras, MTL-algebras, and R_0-algebras) are particular types of EQ-algebras, it is natural and meaningful to extend some notions of residuated lattices to EQ-algebras. Filter theory plays an important role in studying these algebras because the properties of filters have a strong influence on the structure properties of algebras. From a logical point of view, various filters correspond to various sets of provable formulas. Up to now, some types of (fuzzy) filters on ordered algebras based logical algebras have been widely studied [2,4,5,10] and some important results have been obtained. Fuzzy algebra is an important branch of fuzzy mathematics and Rosenfeld [12] started the study of fuzzy algebraic structures with the introduction of the concept of fuzzy sub-groups in 1971. Since then these ideas have been applied to other algebraic structures such as semigroups, rings, ideals, modules and vector spaces. So generalization existing results in BL-algebras and residuated lattices, to EQ-algebras is important tool for studying various algebraic and logical systems in special case EQ-algebras.

In BL-algebras, residuated lattices, MTL-algebra fuzzy (n-fold) implicative filters and fuzzy (n-fold) positive implicative filters were provided. In EQ-algebras, the notions of implicative filters and positive implicative filters were introduced by Liu and Zhang. Moreover, Paad and et al. [11] extended this filters to n-fold implicative filters and n-fold positive implicative filters in EQ-algebras and fuzzy implicative filters and fuzzy positive implicative filters were provided by Xin and et al. [13]. This motivates us to extend different types of fuzzy (implicative, positive implicative) (pre)filters of EQ-algebras. Hence, in this paper, we introduce the notions fuzzy n-fold implicative and fuzzy n-fold positive implicative (pre)filters in EQ-algebras and investigate the properties and characterized them as it have done in residuated lattices. Moreover, we study the relationship between fuzzy n-fold positive implicative (pre)filters and fuzzy n-fold implicative (pre)filters. In the follow, by using the level subset of a fuzzy set in EQ-algebras, we give some characterizations of fuzzy n-fold (positive) implicative (pre)filters in EQ-algebras. Also, we investigate under what conditions the fuzzy n-fold positive implicative (pre)filter is fuzzy n-fold implicative (pre)filter in EQ-algebras.

2. Preliminaries

Definition 2.1 ([3,6])**.** An EQ-algebra is an algebra $(L, \wedge, \odot, \sim, 1)$ of type $(2, 2, 2, 0)$ satisfying the following axioms.

($E1$) $(L, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. We set $x \leq y$ if and only if $x \wedge y = x$.

($E2$) $(L, \odot, 1)$ is a commutative monoid and $\odot$ is isotone with respect to $\leq$.

($E3$) $x \sim x = 1$ (reflexivity axiom).

($E4$) $((x \wedge y) \sim z) \odot (s \sim x) \leq z \sim (s \wedge y)$ (substitution axiom).

$(E5)$ $(x \sim y) \odot (s \sim t) \leq (x \sim s) \sim (y \sim t)$ (congruence axiom).
$(E6)$ $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$ (monotonicity axiom).
$(E7)$ $x \odot y \leq x \sim y$ (boundedness axiom),
for all $s, t, x, y, z \in L$.

Let L be an EQ-algebra. Then for all $x, y \in L$, we put

$$x \to y = (x \wedge y) \sim x, \quad \tilde{x} = x \sim 1.$$

The derived operation $\to$ is called implication and if L contains an element 0 such that $0 \leq x$, for any $x \in L$, then 0 is called bottom element and we may define the unary operation $\neg$ on L by $\neg x = x \sim 0$.

Definition 2.2 ([6])**.** Let L be an EQ-algebra. Then L is called:
(i) separated, if $x \sim y = 1$ implies $x = y$ for all $x, y \in L$;
(ii) good, if $\tilde{x} = x$ for all $x \in L$;
(iii) residuated, if $(x \odot y) \wedge z = x \odot y$ if and only if $x \wedge ((y \wedge z) \sim y) = x$ for all $x, y, z \in L$.

Lemma 2.1 ([3,6])**.** *Let L be an EQ-algebra. Then the following properties hold for any $x, y, z \in L$:*
(i) *$x \sim y = y \sim x$, $x \sim y \leq x \to y$, $x \odot y \leq x \wedge y \leq x, y$;*
(ii) *$x \leq 1 \sim x = 1 \to x \leq y \to x$;*
(iii) *$x \to y \leq (y \to z) \to (x \to z)$;*
(iv) *$x \to y \leq (z \to x) \to (z \to y)$;*
(v) *if $x \leq y$, then $x \to y = 1$;*
(vi) *if $x \leq y$, then $z \to x \leq z \to x$, $y \to z \leq x \to z$;*
(vii) *if L contains a bottom element 0, then $\neg 0 = 1$, $\neg x = x \to 0$.*

In general, the identity $x \to (y \to z) = y \to (x \to z)$ may not be true in EQ-algebras. We call that EQ-algebra L has exchange principle if $x \to (y \to z) = y \to (x \to z)$ for any $x, y, z \in L$.

Theorem 2.1 ([6])**.** *Let L be an EQ-algebra. Then the following are equivalent:*
(i) *L is good;*
(ii) *L is separated and satisfies exchange principle;*
(iii) *L is separated and satisfies $x \leq (x \to y) \to y$ for any $x, y \in L$.*

Theorem 2.2 ([3])**.** *Let L be a residuated EQ-algebra. Then for any $x, y, z \in L$:*
(i) *$x \odot y \leq z$ if and only if $x \leq y \to z$;*
(ii) *$(x \odot y) \to z = x \to (y \to z)$.*

Definition 2.3 ([3])**.** Let L be an EQ-algebra and $\emptyset \neq F \subseteq L$. Then F is called a prefilter of L if it satisfies for any $x, y \in L$
$(F1)$ $1 \in F$;
$(F2)$ if $x \in F$, $x \to y \in F$, then $y \in F$.
A prefilter F is said to be a filter if it satisfies:
$(F3)$ if $x \to y \in F$, then $(x \odot z) \to (y \odot z) \in F$ for any $x, y, z \in L$.

Note that prefilter and filters are coincide in residuated EQ-algebras.

Definition 2.4 ([5]). Let F be a prefilter of EQ-algebra L. Then we say that F has weak exchange principle, if it satisfies for any $x, y, z \in L$

$$x \to (y \to z) \in F \text{ implies } y \to (x \to z) \in F.$$

Definition 2.5 ([5, 11]). Let F be a prefilter of EQ-algebra L. Then F is called an n-fold positive implicative prefilter if it satisfies:

$(F5)$ $x^n \to (y \to z) \in F$, $x^n \to y \in F$ imply $x^n \to z \in F$ for all $x, y, z \in L$.

If F is a filter and satisfies $(F5)$, then F is called an n-fold positive implicative filter and 1-fold positive implicative (pre)filter is called positive implicative (pre)filter.

Theorem 2.3 ([5]). *Let F be a prefilter of EQ-algebra L. Then the following are equivalent.*

(i) *F is a positive implicative prefilter.*

(ii) $(x \wedge (x \to y)) \to y \in F$ *for any* $x, y \in L$.

Definition 2.6 ([5, 11]). Let L be an EQ-algebra and $\emptyset \neq F \subseteq L$. Then F is called an n-fold implicative prefilter if

(i) $1 \in F$;

(ii) $z \to ((x^n \to y) \to x) \in F$ and $z \in F$ imply $x \in F$ for any $x, y, z \in L$.

1-fold implicative (pre)filter is called implicative (pre)filter.

Theorem 2.4 ([11]). *Let $F \subseteq Q$ be two prefilters of EQ-algebra L and L has exchange principle. If F is an n-fold positive implicative prefilter, then so is Q.*

Theorem 2.5 ([11]). *Let F and G be two prefilters of EQ-algebra L such that $F \subseteq G$. If F is an n-fold implicative prefilter with the weak exchange principle, then G is an n-fold implictive prefilter.*

Theorem 2.6 ([11]). *Let F be an n-fold implicative filter of residuated EQ-algebra L. Then F is an n-fold positive implicative filter of L.*

A fuzzy set of L is a mapping $\mu : L \to [0, 1]$ and for all $t \in [0, 1]$, the set $\mu_t = \{x \in L \mid \mu(x) \geq t\}$ is called a level subset of μ.

Definition 2.7 ([13]). Let μ be a fuzzy set of EQ-algebra L. Then μ is called a fuzzy prefilter of L if it satisfies for all $x, y \in L$:

$(FF1)$ $\mu(x) \leq \mu(1)$;

$(FF2)$ $\mu(x) \wedge \mu(x \to y) \leq \mu(y)$.

A fuzzy prefilter μ is called a fuzzy filter if it satisfies:

$(FF3)$ $\mu(x \to y) \leq \mu((x \odot z) \to (y \odot z))$ for all $x, y, z \in L$.

Definition 2.8 ([13]). Let μ be a fuzzy prefilter of EQ-algebra L. Then μ is called a fuzzy positive implicative prefilter of L if it satisfies for all $x, y, z \in L$:

$(FF4)$ $\mu(x \to (y \to z)) \wedge \mu(x \to y) \leq \mu(x \to z)$.

A fuzzy filter μ of L is called a fuzzy positive implicative filter if it satisfies $(FF4)$.

Definition 2.9 ([13])**.** Let μ be a fuzzy set of EQ-algebra L. Then μ is called a fuzzy implicative prefilter of L if it satisfies for all $x, y, z \in L$:
$(FF1)$ $\mu(x) \leq \mu(1)$;
$(FF5)$ $\mu(z \to ((x \to y) \to x)) \wedge \mu(z) \leq \mu(x)$.

Proposition 2.1 ([13])**.** *Let μ be a fuzzy prefilter of EQ-algebra L. Then for any $x, y, z \in L$:*
(i) if $x \leq y$, then $\mu(x) \leq \mu(y)$;
(ii) if μ is a fuzzy filter, then $\mu(x \to y) \wedge \mu(y \to z) \leq \mu(x \to z)$.

Theorem 2.7 ([13])**.** *Let μ be a fuzzy filter of EQ-algebra L. Then μ is a fuzzy positive implicative filter of L if and only if $\mu((x \wedge (x \to y)) \to y) = \mu(1)$ for all $x, y \in L$.*

Note. From now on, in this paper, L will denote a EQ-algebra, unless otherwise stated.

3. Fuzzy n-Fold Positive Implicative (Pre)filters in EQ-Algebras

In this section, we introduce the concept of fuzzy n-fold positive implicative (pre)filters in EQ-algebras and we give some relate results.

Definition 3.1. Let μ be a fuzzy prefilter of L. Then μ is called a *fuzzy n-fold positive implicative prefilter* of L if for all $x, y, z \in L$, it satisfies
$(FF6)$ $\mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) \leq \mu(x^n \to z)$.

A fuzzy n-fold positive implicative prefilter μ is called a *fuzzy n-fold positive implicative filter* of L if it satisfies $(FF3)$.

Example 3.1 ([13]). Let $L = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

,

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	1
1	0	a	1	1

,

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	1	1

.

Routine calculation shows that $(L, \wedge, \odot, \sim, 1)$ is an EQ-algebra. Define fuzzy set μ in L as follows: $\mu(1) = 0.8$, $\mu(b) = 0.6$ and $\mu(0) = \mu(a) = 0.4$. One can check that μ is a fuzzy n-fold positive implicative prefilter of L for any natural number n.

Theorem 3.1. *Let μ be a fuzzy prefilter of L. Then μ is a fuzzy n-fold positive implicative prefilter if and only if for any $t \in [0, 1]$, $\emptyset \neq \mu_t$ is an n-fold positive implicative prefilter of L.*

Proof. The proof is straightforward. □

Theorem 3.2. *Let μ be a fuzzy prefilter of L. Then μ is a fuzzy n-fold positive implicative prefilter of L if and only if for any $a \in L$, $\mu^{a,n} : L \to [0, 1]$ is a fuzzy prefilter of L, where $\mu^{a,n}(x) = \mu(a^n \to x)$ for any $x \in L$.*

Proof. Suppose that μ is a fuzzy n-fold positive implicative prefilter of L. Since $a^n \to 1 = 1$, we get that $\mu(a^n \to 1) = \mu(1)$ and so $\mu^{a,n}(1) = \mu(a^n \to 1) = \mu(1)$. From $a^n \to x \leq 1$, we have $\mu(a^n \to x) \leq \mu(1)$, that is $\mu^{a,n}(x) \leq \mu(1)$. Therefore, $\mu^{a,n}(x) \leq \mu^{a,n}(1)$, for any $x \in L$. On the other hand, since μ is a fuzzy n-fold positive implicative prefilter of L, we conclude that $\mu(a^n \to (x \to y)) \wedge \mu(a^n \to x) \leq \mu(a^n \to y)$, for any $a, x, y \in L$, that is $\mu^{a,n}(x \to y) \wedge \mu^{a,n}(x) \leq \mu^{a,n}(y)$. Therefore, $\mu^{a,n}$ is a fuzzy prefilter in L.

Conversely, let for any $a \in L$, $\mu^{a,n}$ is a fuzzy prefilter of L. Then $\mu^{x,n}$ is a fuzzy prefilter of L and so it follows that $\mu^{x,n}(y \to z) \wedge \mu^{x,n}(y) \leq \mu^{x,n}(z)$, for any $y, z \in L$. Hence, $\mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) \leq \mu(x^n \to z)$ for any $x, y, z \in L$. Therefore, μ is a fuzzy n-fold positive implicative prefilter of L. ☐

Proposition 3.1. *Let μ be a fuzzy n-fold positive implicative prefilter of L. Then for any $a \in L$, $\mu^{a,n}$ is the fuzzy prefilter containing μ.*

Proof. Assume that μ is a fuzzy n-fold positive implicative prefilter of L, then by Theorem 3.2, $\mu^{a,n}$ is a fuzzy prefilter of L. Since by Lemma 2.1 (ii), $x \leq a^n \to x$, by Proposition 2.1 (i), we get that $\mu(x) \leq \mu(a^n \to x)$ and so $\mu(x) \leq \mu^{a,n}(x)$. Therefore, $\mu^{a,n}$ is the fuzzy prefilter containing μ. ☐

Proposition 3.2. *Let μ and ν be two fuzzy prefilters of L. Then for any $a, b \in L$ and natural number n, the following statements hold:*

(i) *$\mu^{a,n} = \mu$ if and only if $\mu(a^n) = \mu(1)$;*

(ii) *$a \leq b$ implies that $\mu^{b,n} \subseteq \mu^{a,n}$;*

(iii) *$\mu \subseteq \nu$ implies that $\mu^{a,n} \subseteq \nu^{a,n}$;*

(iv) *$(\mu \cap \nu)^{a,n} = \mu^{a,n} \cap \nu^{a,n}$, $(\mu \cup \nu)^{a,n} = \mu^{a,n} \cup \nu^{a,n}$.*

Proof. (i) Let $\mu^{a,n} = \mu$, for $a \in L$ and natural number n. Then $\mu(a^n) = \mu^{a,n}(a^n) = \mu(a^n \to a^n) = \mu(1)$. Conversely, assume that $\mu(a^n) = \mu(1)$, since by Lemma 2.1 (ii), $x \leq a^n \to x$, for any $x \in L$ and since μ is a fuzzy prefilter, we get that $\mu(x) \leq \mu(a^n \to x) = \mu^{a,n}(x)$. Hence, $\mu \subseteq \mu^{a,n}$. On the other hand, since μ is a fuzzy prefilter, we have $\mu(a^n \to x) = \mu(a^n \to x) \wedge \mu(1) = \mu(a^n \to x) \wedge \mu(a^n) \leq \mu(x)$, for all $x \in L$. Hence, $\mu^{a,n}(a) \leq \mu(x)$ and so $\mu^{a,n} \subseteq \mu$. Therefore, $\mu^{a,n} = \mu$.

(ii) Suppose that $a, b, x \in L$ and $a \leq b$. Then by $(EQ2)$, $a^n \leq b^n$ and so by Lemma 2.1 (iv), $b^n \to x \leq a^n \to x$ and since μ is a fuzzy prefilter, we get that $\mu(b^n \to x) \leq \mu(a^n \to x)$. Hence, $\mu^{b,n}(x) \leq \mu^{a,n}(x)$ and so $\mu^{b,n} \subseteq \mu^{a,n}$.

(iii) Suppose that $\mu \subseteq \nu$ and $x \in L$, then $\mu(a^n \to x) \leq \nu(a^n \to x)$ and so $\mu^{a,n} \subseteq \nu^{a,n}$.

(iv) For any $x \in L$, we have

$$(\mu \cup \nu)^{a,n}(x) = (\mu \cup \nu)(a^n \to x) = \mu(a^n \to x) \vee \nu(a^n \to x) = \mu^{a,n}(x) \vee \nu^{a,n}(x).$$

Thus, $(\mu \cup \nu)^{a,n} = \mu^{a,n} \cup \nu^{a,n}$. Similarly, $(\mu \cap \nu)^{a,n} = \mu^{a,n} \cap \nu^{a,n}$. ☐

Theorem 3.3. *Let μ be a fuzzy n-fold positive implicative prefilter of L. Then for any $x, y \in L$*

$$\mu(x^n \odot (x \to y)^n \to y) = \mu(1).$$

Proof. Let μ be a fuzzy n-fold positive implicative prefilter of L and $x, y \in L$. Then $\mu(x^n \odot (x \to y)^n \to (x \to y)) \wedge \mu(x^n \odot (x \to y)^n \to x) \leq \mu(x^n \odot (x \to y)^n \to y)$. Since by Lemma 2.1 (i), $x^n \odot (x \to y)^n \leq (x \to y)^n \leq x \to y$ and $x^n \odot (x \to y)^n \leq x^n \leq x$, we conclude that $x^n \odot (x \to y)^n \to (x \to y) = 1$ and $x^n \odot (x \to y)^n \to x = 1$. Hence, $\mu(x^n \odot (x \to y)^n \to (x \to y)) = \mu(1)$ and $\mu(x^n \odot (x \to y)^n \to x) = \mu(1)$ and so $\mu(x^n \odot (x \to y)^n \to y) = \mu(1)$. □

Theorem 3.4. *Let μ be a fuzzy prefilter of L. Then the following statements are equivalent:*

(i) *μ is a fuzzy n-fold positive implicative prefilter of L;*

(ii) *$\mu(x^n \to (x^n \to y)) \leq \mu(x^n \to y)$ for all $x, y \in L$;*

(ii) *$\mu(x^n \to (x^n \to y)) = \mu(x^n \to y)$ for all $x, y \in L$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that μ is a fuzzy n-fold positive implicative prefilter of L. Then by Definition 3.1, we have

$$\begin{aligned}\mu(x^n \to (x^n \to y)) &= \mu(x^n \to (x^n \to y)) \wedge \mu(1)\\ &= \mu(x^n \to (x^n \to y)) \wedge \mu(x^n \to x^n)\\ &\leq \mu(x^n \to y).\end{aligned}$$

Therefore, $\mu(x^n \to (x^n \to y)) \leq \mu(x^n \to y)$.

$(ii) \Rightarrow (iii)$ Since by Lemma 2.1 (i), $x^n \to y \leq x^n \to (x^n \to y)$, by Proposition 2.1 (i), we get that $\mu(x^n \to y) \leq \mu(x^n \to (x^n \to y))$ and so by (ii), we that conclude that $\mu(x^n \to (x^n \to y)) = \mu(x^n \to y)$.

$(iii) \Rightarrow (i)$ Since by Lemma 2.1 (iii), we have $x^n \to (y \to z) \leq ((y \to z) \to (x^n \to z)) \to (x^n \to (x^n \to z))$ and $x^n \to y \leq (y \to z) \to (x^n \to z)$, we get that $\mu(x^n \to (y \to z)) \leq \mu(((y \to z) \to (x^n \to z)) \to (x^n \to (x^n \to z)))$ and $\mu(x^n \to y) \leq \mu((y \to z) \to (x^n \to z))$, by Proposition 2.1 (i). Hence, by (iii) we conclude that

$$\begin{aligned}\mu(x^n \to y) \wedge \mu(x^n \to (y \to z)) \leq & \mu((y \to z) \to (x^n \to z))\\ & \wedge \mu(((y \to z) \to (x^n \to z)) \to (x^n \to (x^n \to z)))\\ \leq & \mu(x^n \to (x^n \to z))\\ = & \mu(x^n \to z).\end{aligned}$$

Therefore, μ is a fuzzy n-fold positive implicative prefilter of L. □

Proposition 3.3 ([9]). *The following properties are equivalent:*

(i) *an EQ-algebra L is residuated;*

(ii) *the EQ-algebra L is good, and*

$$(x \odot y) \to z \leq x \to (y \to z),$$

holds for all $x, y, z \in L$.

Definition 3.2. Let L be an EQ-algebra. Then we say that L has condition $(*)$, if for any $x, y, z \in L$

$$(x \odot y) \to z \leq x \to (y \to z). \quad (*)$$

Note that by Proposition 3.3, every residuated EQ-algebra satisfying in the condition $(*)$ and the following example shows that the EQ-algebra satisfying the condition $(*)$ may not be a residuated EQ-algebra.

Example 3.2 ([5]). Let $L = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

,

$\sim$	0	a	b	1
0	1	a	a	a
a	a	1	b	b
b	a	b	1	1
1	a	b	1	1

,

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	a	b	1	1
1	a	b	1	1

.

Routine calculation shows that $(L, \wedge, \odot, \sim, 1)$ is an EQ-algebra. It is easily checked that L satisfies $(*)$ for any $x, y, z \in L$ and L is not residuated because $1 \leq 1 \to b$, but $1 \odot 1 \nleq b$.

Theorem 3.5. *Let L be an EQ-algebra by condition $(*)$ and μ be a fuzzy n-fold positive implicative prefilter of L. Then for any $x \in L$ $\mu(x^n \to x^{2n}) = \mu(1)$.*

Proof. Assume that μ is a fuzzy n-fold positive implicative prefilter of L and $x \in L$. Since $x^{2n} \to x^{2n} = x^n \odot x^n \to x^n \odot x^n = 1$, we get that $\mu(x^n \odot x^n \to x^n \odot x^n) = \mu(1)$ and since by $(*)$, $x^n \odot x^n \to x^n \odot x^n \leq x^n \to (x^n \to (x^n \odot x^n))$, we conclude that $\mu(x^n \odot x^n \to x^n \odot x^n) \leq \mu(x^n \to (x^n \to (x^n \odot x^n)))$. Hence, $\mu(x^n \to (x^n \to (x^n \odot x^n))) = \mu(1)$ and so by Theorem 3.4, we conclude that $\mu(x^n \to x^n \odot x^n) = \mu(x^n \to x^{2n}) = \mu(1)$. $\square$

Theorem 3.6. *Let μ be a fuzzy prefilter of good EQ-algebra L. Then the following statements are equivalent:*

(i) μ is a fuzzy n-fold positive implicative prefilter;

(ii) $\mu(x^n \to (x^n \to y)) \leq \mu(x^n \to y)$ for any $x, y \in L$;

(iii) $\mu(x^n \to (y \to z)) \leq \mu((x^n \to y) \to (x^n \to z))$ for any $x, y, z \in L$.

Proof. $(i) \Rightarrow (ii)$ Let μ be a fuzzy n-fold positive implicative prefilter of L and $x, y \in L$. Then by Lemma 2.1 (i) we have:

$$\begin{aligned}\mu(x^n \to (x^n \to y)) &= \mu(x^n \to (x^n \to y)) \wedge \mu(1)\\ &= \mu(x^n \to (x \to y)) \wedge \mu(x^n \to x^n)\\ &\leq \mu(x^n \to y).\end{aligned}$$

$(ii) \Rightarrow (iii)$ Let $x, y, z \in L$. Then by Lemma 2.1 (iv) $y \to z \leq (x^n \to y) \to (x^n \to z)$ and so by Lemma 2.1 (vi), we conclude that $x^n \to (y \to z) \leq x^n \to ((x^n \to y) \to (x^n \to z))$. Hence, by Theorem 2.1, we have

$$x^n \to ((x^n \to y) \to (x^n \to z)) = x^n \to (x^n \to ((x^n \to y) \to z)),$$

and so by Proposition 2.1 (i), we obtain

$$\mu(x^n \to (y \to z)) \leq \mu(x^n \to (x^n \to ((x^n \to y) \to z)).$$

Now, by (ii) we have

$$\mu(x^n \to (x^n \to ((x^n \to y) \to z)) \leq \mu(x^n \to ((x^n \to y) \to z)),$$

and since $\mu(x^n \to ((x^n \to y) \to z)) = \mu((x^n \to y) \to (x^n \to z))$, we conclude that

$$\mu(x^n \to (y \to z)) \leq \mu((x^n \to y) \to (x^n \to z)).$$

$(iii) \Rightarrow (i)$ Let $x, y, z \in L$. Then, by (iii), we have

$$\begin{aligned}\mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) &\leq \mu((x^n \to y) \to (x^n \to z)) \wedge \mu(x^n \to y)\\ &\leq \mu(x^n \to z).\end{aligned}$$

Therefore, μ is a fuzzy n-fold positive implicative prefilter of L. □

Theorem 3.7. *Let μ be a fuzzy prefilter of residuated EQ-algebra L. Then the following statements are equivalent:*
(i) *μ is a fuzzy n-fold positive implicative filter of L;*
(ii) *$\mu(x^{n+1} \to y) \leq \mu(x^n \to y)$ for any $x, y \in L$;*
(ii) *$\mu(x^n \to (y \to z)) \leq \mu((x^n \to y) \to (x^n \to z))$ for any $x, y, z \in L$.*

Proof. The proof is similar to the proof of Theorem 3.6. □

Definition 3.3 ([13])**.** Let μ be a fuzzy prefilter of L. Then we say that μ has weak exchange principle, if it satisfies $\mu(x \to (y \to z)) = \mu(y \to (x \to z))$ for all $x, y, z \in L$.

Theorem 3.8. *Let μ, ν be two fuzzy prefilters of L and satisfy weak exchange principle such that $\mu \subseteq \nu$ and $\mu(1) = \nu(1)$. If μ is a fuzzy n-fold positive implicative prefilter, then so is ν.*

Proof. Let μ be a fuzzy n-fold positive implicative prefilter of L. Then, by Theorem 3.1, for each $t \in [0, 1]$, $\emptyset \neq \mu_t$ is an n-fold positive implicative prefilter of L and satisfies weak exchange principle because if $x \to (y \to z) \in \mu_t$, then $\mu(y \to (x \to z)) = \mu(x \to (y \to z)) \geq t$ and so $y \to (x \to z) \in \mu_t$. Hence, μ_t satisfies weak exchange principle and by similar way ν_t satisfies weak exchange principle. Now, since $\mu \subseteq \nu$, we get that $\mu_t \subseteq \nu_t$, for each $t \in [0, 1]$ and so, by Theorem 2.4, for each $t \in [0, 1]$, $\emptyset \neq \nu_t$ is an n-fold positive implicative prefilter of L. Thus, by Theorem 3.1, ν is a fuzzy n-fold positive implicative prefilters of L. □

Theorem 3.9. *Let μ be a fuzzy filter of L and satisfy weak exchange principle such that $\mu(x^n \to x^n \odot x^n) = \mu(1)$ and $\mu((x^n \odot (x^n \to y)) \to y) = \mu(1)$ for any $x, y \in L$. Then μ is a fuzzy n-fold positive implicative filter of L.*

Proof. Let μ be a fuzzy filter of L such that satisfy weak exchange principle and $x, y, z \in L$. Then by Proposition 2.1 (ii)

$$\begin{aligned}
\mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) &= \mu(y \to (x^n \to z)) \wedge \mu(x^n \to y) \\
&\leq \mu((x^n \odot y) \to (x^n \odot (x^n \to z))) \\
&\wedge \mu((x^n \odot x^n) \to (x^n \odot y)) \\
&\leq \mu((x^n \odot x^n) \to (x^n \odot (x^n \to z))) \\
&= \mu((x^n \odot x^n) \to (x^n \odot (x^n \to z))) \wedge \mu(1) \\
&= \mu((x^n \odot x^n) \to (x^n \odot (x^n \to z))) \\
&\wedge \mu(x^n \to (x^n \odot x^n)) \\
&\leq \mu(x^n \to (x^n \odot (x^n \to z)) \\
&= \mu(x^n \to (x^n \odot (x^n \to z)) \wedge \mu(1) \\
&= \mu(x^n \to (x^n \odot (x^n \to z))) \\
&\wedge \mu((x^n \odot (x^n \to z)) \to z) \\
&\leq \mu(x^n \to z).
\end{aligned}$$

Therefore, μ is a fuzzy n-fold positive implicative filter of L. □

Proposition 3.4. *If μ be a fuzzy positive implicative filter of L, then $\mu((x \odot (x \to y)) \to y) = \mu(1)$ for any $x, y \in L$.*

Proof. If μ be a fuzzy positive implicative filter of L, then by Theorem 2.7, $\mu((x \wedge (x \to y)) \to y) = \mu(1)$, for any $x, y \in L$. From $x \odot (x \to y) \leq x \wedge (x \to y)$, we have $(x \wedge (x \to y)) \to y \leq (x \odot (x \to y)) \to y$. Hence, $\mu((x \wedge (x \to y)) \to y) \leq \mu((x \odot (x \to y)) \to y)$. Therefore, $\mu((x \odot (x \to y)) \to y) = \mu(1)$. □

Theorem 3.10. *Let L be an EQ-algebra with condition $(*)$ and μ be a fuzzy filter of L such that satisfy weak exchange principle. Then the following statements are equivalent:*
(i) *μ is a fuzzy positive implicative filter;*
(ii) *$\mu(x \to x^2) = \mu(1)$ and $\mu((x \odot (x \to y)) \to y) = \mu(1)$ for any $x, y \in L$.*

Proof. It follows from Theorem 3.5, Theorem 3.9 and Proposition 3.4, whenever $n = 1$. □

Theorem 3.11. *Let μ be a fuzzy filter of residuated EQ-algebra L. Then μ is a fuzzy n-fold positive implicative filter of L if and only if $\mu(x^n \to x^{2n}) = \mu(1)$ for any $x \in L$.*

Proof. Let μ be a fuzzy n-fold positive implicative filter of residuated EQ-algebra L. Then L satisfies in condition $(*)$ and so, by Theorem 3.5, $\mu(x^n \to x^{2n}) = \mu(1)$, for any $x \in L$. Conversely, let $\mu(x^n \to x^{2n}) = \mu(1)$ for any $x \in L$. Then, by Theorem

2.2 (ii) and Proposition 2.1 (ii) for $x, y \in L$ we have

$$\begin{aligned}\mu(x^n \to (x^n \to y)) &= \mu(x^n \odot x^n \to y)\\ &= \mu(x^{2n} \to y)\\ &= \mu(x^{2n} \to y) \wedge \mu(1)\\ &= \mu(x^{2n} \to y) \wedge \mu(x^n \to x^{2n})\\ &\leq \mu(x^n \to y).\end{aligned}$$

Therefore, by Theorem 2.2, μ is a fuzzy n-fold positive implicative filter of L. □

Theorem 3.12. *Let L be an EQ-algebra with condition $(*)$ and μ be a fuzzy n-fold positive implicative filter of L. Then μ is a fuzzy $(n+1)$-fold positive implicative filter of L.*

Proof. Let μ be a fuzzy n-fold positive implicative filter of L and $x, y, z \in L$. Then by Theorem 3.5, $\mu(x^n \to x^{2n}) = \mu(1)$. By Lemma 2.1 (i) and (vi), we have $x^{2n} = x^{n+n} \leq x^{n+1}$ and so $x^{n+1} \to (y \to z) \leq x^{n+n} \to (y \to z)$ and $x^{n+1} \to y \leq x^{n+n} \to y$ and since $x^{n+n} \to (y \to z) = (x^2)^n \to (y \to z)$ and $x^{n+n} \to y = (x^2)^n \to y$, by Proposition 2.1 (i), we get that $\mu(x^{n+1} \to (y \to z)) \leq \mu((x^2)^n \to (y \to z))$ and $\mu(x^{n+1} \to y) \leq \mu((x^2)^n \to y)$. Now, by Proposition 2.1 (ii), we have

$$\begin{aligned}\mu(x^{n+1} \to (y \to z)) \wedge \mu(x^{n+1} \to y) &\leq \mu((x^2)^n \to (y \to z)) \wedge \mu((x^2)^n \to y)\\ &\leq \mu((x^2)^n \to z)\\ &= \mu((x^2)^n \to z) \wedge \mu(1)\\ &\leq \mu((x^2)^n \to z) \wedge \mu(x^n \to x^{2n})\\ &\leq \mu(x^n \to z)\\ &\leq \mu(x^{n+1} \to z).\end{aligned}$$

Therefore, μ is a fuzzy $(n+1)$-fold positive implicative filter of L. □

Theorem 3.13. *Let L be a residuated EQ-algebra and μ be a fuzzy filter of L. Then the following statements are equivalent:*

(i) *μ is a fuzzy n-fold positive implicative filter;*
(ii) *$\mu(x^{n+1} \to y) \leq \mu(x^n \to y)$ for any $x, y \in L$;*
(iii) *$\mu(x^n \to (x^n \to y)) \leq \mu(x^n \to y)$ for any $x, y \in L$;*
(iv) *$\mu(x^n \to (y \to z)) \leq \mu((x^n \to y) \to (x^n \to z))$ for any $x, y, z \in L$;*
(v) *$\mu(x^n \to x^{2n}) = \mu(1)$ for any $x \in L$;*
(vi) *$\mu((x^n \odot y) \to z) \leq \mu((x \wedge y)^n \to z)$ for any $x, y, z \in L$.*

Proof. Let L be a residuated EQ-algebra and μ be a fuzzy filter of L. Then by Theorem 3.7, Theorem 3.6 and Theorem 3.11, the parts (i), (ii), (iii), (iv) and (v) are equivalent.

$(i) \Rightarrow (vi)$ Let μ be a fuzzy n-fold positive implicative filter of L and $x, y, z \in L$. Then by Lemma 2.1 (vi) and Proposition 2.1 (i), we have

$$\begin{aligned}
\mu((x^n \odot y) \to z) &= \mu(x^n \to (y \to z)) \\
&= \mu(y \to (x^n \to z)) \\
&\leq \mu((x \wedge y) \to (x^n \to z)) \\
&= \mu(x^n \to ((x \wedge y) \to z) \\
&\leq \mu((x \wedge y)^n \to ((x \wedge y) \to z)) \\
&= \mu((x \wedge y)^n \to ((x \wedge y) \to z)) \wedge \mu(1) \\
&= \mu((x \wedge y)^n \to ((x \wedge y) \to z) \wedge \mu((x \wedge y)^n \to (x \wedge y)) \\
&\leq \mu((x \wedge y)^n \to z).
\end{aligned}$$

$(vi) \Rightarrow (v)$ Let $x, y \in L$. Then by (vi), we have

$$\begin{aligned}
\mu(x^{n+1} \to y) &= \mu((x^n \odot x) \to y) \\
&\leq \mu((x \wedge x)^n \to y) \\
&= \mu(x^n \to y),
\end{aligned}$$

and since (v) and (i) are equivalent, we conclude that (vi) and (i) are equivalent and the proof is complete. $\square$

4. Fuzzy n-Fold Implicative (Pre)filters in EQ-Algebras

In this section we introduce the concept of fuzzy n-fold implicative (pre)filters in EQ-algebras and we give some related results.

Definition 4.1. Let μ be a fuzzy set of L. Then μ is called a *fuzzy n-fold implicative prefilter* of L if it satisfies for all $x, y, z \in L$

$(FF1)$ $\mu(x) \leq \mu(1)$;

$(FF7)$ $\mu(z \to ((x^n \to y) \to x)) \wedge \mu(z) \leq \mu(x)$.

A fuzzy n-fold implicative prefilter μ is called a *fuzzy n-fold implicative filter* of L if it satisfies $(FF3)$.

Example 4.1 ([13]). Let $L = \{0, a, b, c, 1\}$ be a chain with Cayley tables as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	0	0	c
1	0	a	b	c	1

,

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	b	b	b
b	0	b	1	c	c
c	0	b	c	1	1
1	0	b	c	1	1

,

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	1	1
c	0	b	c	1	1
1	0	b	c	1	1

.

Routine calculation shows that $(L, \wedge, \odot, \sim, 1)$ is an EQ-algebra. Define a fuzzy set μ in L as follows: $\mu(1) = \mu(a) = \mu(b) = \mu(c) = t_1$ and $\mu(0) = t_2$, where $0 \leq t_2 < t_1 \leq 1$. We can see that μ is a fuzzy 2-fold implicative prefilter of L.

Theorem 4.1. *Let μ be a fuzzy set of L. Then μ is a fuzzy n-fold implicative prefilter if and only if for any $t \in [0,1]$, μ_t is an n-fold implicative prefilter of L.*

Proof. The proof is straightforward. □

Theorem 4.2. *Every fuzzy n-fold implicative prefilter of L is a fuzzy prefilter.*

Proof. Suppose that μ is a fuzzy n-fold implicative prefilter of L. Then $\mu(x) \leq \mu(1)$, for all $x \in L$. Firstly, we prove if $x \leq y$, then $\mu(x) \leq \mu(y)$. Let $x, y \in L$ such that $x \leq y$. Then by Lemma 2.1 (i) and (vi), we have $y \leq (y^n \to y) \to y$ and so $x \to y \leq x \to ((y^n \to y) \to y)$ and since by Lemma 2.1 (v), $x \to y = 1$, we get that $x \to ((y^n \to y) \to y) = 1$. Hence, $\mu(x \to ((y^n \to y) \to y)) = \mu(1)$ and since μ is a fuzzy n-fold implicative prefilter, we have

$$\mu(x) = \mu(x) \wedge \mu(1) = \mu(x) \wedge \mu(x \to ((y^n \to y) \to y)) \leq \mu(y).$$

Now, by $y \leq 1 \to y$, we get that $x \to y \leq x \to (1 \to y)$ and so $\mu(x \to y) \leq \mu(x \to (1 \to y))$. Hence, by Definition 4.1, we have

$$\begin{aligned}\mu(x \to y) \wedge \mu(x) &\leq \mu(x \to (1 \to y)) \wedge \mu(x)\\ &= \mu(x \to ((y^n \to 1) \to y)) \wedge \mu(x)\\ &\leq \mu(y).\end{aligned}$$

Therefore, μ is a fuzzy prefilter of L. □

Theorem 4.3. *Let μ be a fuzzy (pre)filter of L. The following statements are equivalent:*

(i) *μ is a fuzzy n-fold implicative (pre)filter of L;*
(ii) *$\mu((x^n \to y) \to x) \leq \mu(x)$ for all $x, y \in L$;*
(iii) *$\mu((x^n \to y) \to x) = \mu(x)$ for all $x, y \in L$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that μ is a fuzzy n-fold implicative (pre)filter of L. Then

$$\mu(1 \to ((x^n \to y) \to x)) = \mu(1 \to ((x^n \to y) \to x) \wedge \mu(1) \leq \mu(x).$$

Since by Lemma 2.1 (ii), $(x^n \to y) \to x \leq 1 \to ((x^n \to y) \to x)$, by Proposition 2.1 (i), we conclude that $\mu((x^n \to y) \to x) \leq \mu(1 \to ((x^n \to y) \to x))$. Consequently, we have $\mu((x^n \to y) \to x) \leq \mu(x)$.

$(ii) \Rightarrow (iii)$ Since by Lemma 2.1 (ii), $x \leq (x^n \to y) \to x$, it follows that by Proposition 2.1 (i), $\mu(x) \leq \mu((x^n \to y) \to x)$. Combining (ii), we get $\mu((x^n \to y) \to x) = \mu(x)$.

$(iii) \Rightarrow (i)$ Let μ be a fuzzy prefilter of L. Then for $x, y, z \in L$, $\mu(x) \leq \mu(1)$ and by (iii) we have

$$\mu(z \to ((x^n \to y) \to x)) \wedge \mu(z) \leq \mu((x^n \to y) \to x) = \mu(x).$$

Therefore, μ is a fuzzy n-fold implicative prefilter of L. □

Theorem 4.4. *Let μ be a fuzzy n-fold implicative prefilter of residuated EQ-algebra L. Then μ is a fuzzy n-fold positive implicative prefilter.*

Proof. Let μ be a fuzzy n-fold implicative prefilter of residuated EQ-algebra L. Then by Theorem 4.1, for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is an n-fold implicative prefilter of L and so by Theorem 2.6, for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is a n-fold positive implicative prefilter of L. Therefore, by Theorem 3.1, μ is a fuzzy n-fold positive implicative prefilter of L. □

Theorem 4.5. *Let L be an EQ-algebra with bottom element 0 and μ be a fuzzy (pre)filter of L. The following statements are equivalent:*
(i) *μ is a fuzzy n-fold implicative (pre)filter of L;*
(ii) *$\mu(\neg x^n \to x) \leq \mu(x)$ for all $x \in L$;*
(iii) *$\mu(\neg x^n \to x) = \mu(x)$ for all $x \in L$.*

Proof. (i) $\Rightarrow$ (ii) Assume that μ is a fuzzy n-fold implicative (pre)filter of L. Then by Theorem 4.3, for all $x \in L$,

$$\mu(\neg x^n \to x) = \mu((x^n \to 0) \to x) \leq \mu(x).$$

(ii) $\Rightarrow$ (iii) Since by Lemma 2.1 (ii), $x \leq \neg x^n \to x$, we get that $\mu(x) \leq \mu(\neg x^n \to x)$ as μ is a fuzzy prefilter of L. Combining (ii), we get $\mu(\neg x^n \to x) = \mu(x)$.

(iii) $\Rightarrow$ (i) Let μ be a fuzzy prefilter of L. Then by Lemma 2.1 (vi) and by $0 \leq y$, we have $\neg x^n = x^n \to 0 \leq x^n \to y$ and so $(x^n \to y) \to x \leq \neg x^n \to x$. Hence, by Proposition 2.1 (i), $\mu((x^n \to y) \to x) \leq \mu(\neg x^n \to x)$. Combining ($iii$), we get that $\mu((x^n \to y) \to x) \leq \mu(x)$. Therefore, by Theorem 4.3, μ is a fuzzy n-fold implicative (pre)filter of L. □

Theorem 4.6. *Let μ and ν be two fuzzy (pre)filters of L such that $\mu \subseteq \nu$. If μ is a fuzzy n-fold implicative (pre)filter with weak exchange principle of L, then ν is a fuzzy n-fold implicative (pre)filter of L.*

Proof. Let μ be a fuzzy n-fold implicative (pre)filter of L. Then by Theorem 4.1, for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is an n-fold implicative (pre)filter of L and since $\mu \subseteq \nu$, we get that $\mu_t \subseteq \nu_t$, for each $t \in [0,1]$. Now, since for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is an n-fold implicative (pre)filter of L, by Theorem 2.5, we conclude that for each $t \in [0,1]$, $\emptyset \neq \nu_t$ is an n-fold implicative (pre)filter of L. Therefore, by Theorem 3.1, ν is a fuzzy n-fold implicative (pre)filters of L. □

Theorem 4.7. *Let μ be a fuzzy (pre)filter of EQ-algebra L. If μ is a fuzzy n-fold implicative (pre)filter of L, then μ is a fuzzy $(n+1)$-fold implicative (pre)filter of L.*

Proof. Let μ be a fuzzy n-fold positive implicative (pre)filter of L. Since $x^{n+1} \leq x^n$, we get that by Lemma 2.1 (vi), $x^n \to y \leq x^{n+1} \to y$ and so $(x^{n+1} \to y) \to x \leq (x^n \to y) \to x$. Hence, by Proposition 2.1 (i), we have $\mu((x^{n+1} \to y) \to x) \leq \mu((x^n \to y) \to x)$ and since by Theorem 4.5, $\mu((x^n \to y) \to x) \leq \mu(x)$, we conclude that $\mu((x^{n+1} \to y) \to x) \leq \mu(x)$. Therefore, by Theorem 4.5, μ is a fuzzy $(n+1)$-fold implicative (pre)filter of L. □

Theorem 4.8. *Let L be an EQ-algebra with a bottom element 0 and μ be a fuzzy prefilter of L with weak exchange principle. Then the following statements are equivalent:*

(i) *μ is a fuzzy n-fold implicative prefilter of L;*

(ii) *$\mu(x \to (\neg z^n \to y)) \wedge \mu(y \to z) \leq \mu(x \to z)$ for all $x, y, z \in L$;*

(iii) *$\mu(x \to (\neg z^n \to z)) \leq \mu(x \to z)$ for all $x, z \in L$;*

(iv) *$\mu(x \to (\neg z^n \to z)) = \mu(x \to z)$ for all $x, z \in L$.*

Proof. $(i) \Rightarrow (ii)$ Let μ be a fuzzy n-fold implicative prefilter of L and $x, y, z \in L$. Then by Lemma 2.1 (iii) and (iv), $y \to z \leq (x \to y) \to (x \to z)$ and $\neg z^n \to (x \to y) \leq ((x \to y) \to (x \to z)) \to (\neg z^n \to (x \to z))$, and so by Proposition 2.1 (i), $\mu(y \to z) \leq \mu((x \to y) \to (x \to z))$ and $\mu(\neg z^n \to (x \to y)) \leq \mu(((x \to y) \to (x \to z)) \to (\neg z^n \to (x \to z)))$. Now, by weak exchange principle we have

$$\begin{aligned}
\mu(y \to z) \wedge \mu(x \to (\neg z^n \to y)) &= \mu(y \to z) \wedge \mu(\neg z^n \to (x \to y)) \\
&\leq \mu((x \to y) \to (x \to z)) \\
&\wedge \mu(((x \to y) \to (x \to z)) \to (\neg z^n \to (x \to z))) \\
&\leq \mu(\neg z^n \to (x \to z)),
\end{aligned}$$

and since by Lemma 2.1 (ii), $z \leq x \to z$, by $(EQ2)$ we get that $z^n \leq (x \to z)^n$ and so $\neg(x \to z)^n \leq \neg z^n$. Hence, $\neg z^n \to (x \to z) \leq \neg(x \to z)^n \to (x \to z)$ and so by Proposition 2.1 (i), $\mu(\neg z^n \to (x \to z)) \leq \mu(\neg(x \to z)^n \to (x \to z))$ and since μ is a fuzzy n-fold implicative prefilter of L, by Theorem 4.5 we conclude that $\mu(\neg(x \to z)^n \to (x \to z)) \leq \mu(x \to z)$. Consequently, we obtain

$$\mu(x \to (\neg z^n \to y)) \wedge \mu(y \to z) \leq \mu(x \to z).$$

$(ii) \Rightarrow (i)$ Suppose that μ satisfies $\mu(x \to (\neg z^n \to y)) \wedge \mu(y \to z) \leq \mu(x \to z)$, for all $x, y, z \in L$. Then

$$\begin{aligned}
\mu(\neg x^n \to x) &= \mu(\neg x^n \to x) \wedge \mu(1) \\
&\leq \mu(1 \to (\neg x^n \to x)) \wedge \mu(x \to x) \\
&\leq \mu(1 \to x) \\
&= \mu(1 \to x) \wedge \mu(1) \\
&\leq \mu(x).
\end{aligned}$$

Therefore, by Theorem 4.5, μ is a fuzzy n-fold implicative prefilter of L.

$(ii) \Rightarrow (iii)$ Let $x, z \in L$. Then by (ii), we have:

$$\begin{aligned}
\mu(x \to (\neg z^n \to z)) &= \mu(x \to (\neg z^n \to z)) \wedge \mu(1) \\
&= \mu(x \to (\neg z^n \to z)) \wedge \mu(z \to z) \\
&\leq \mu(x \to z).
\end{aligned}$$

$(iii) \Rightarrow (iv)$ From $z \leq \neg z^n \to z$, it follows that $x \to z \leq x \to (\neg z^n \to z)$. Then $\mu(x \to z) \leq \mu(x \to (\neg z^n \to z))$ as μ is a fuzzy prefilter. Combining (iii), we get $\mu(x \to (\neg z^n \to z)) = \mu(x \to z)$.

$(iv) \Rightarrow (i)$ Let $\mu(x \to (\neg z^n \to z)) = \mu(x \to z)$, for all $x, z \in L$. Then $\mu(1 \to (\neg x^n \to x)) = \mu(1 \to x)$ and since μ is a fuzzy prefilter, we get that $\mu(1 \to x) = \mu(1 \to x) \wedge \mu(1) \leq \mu(x)$ and so $\mu(1 \to (\neg x^n \to x)) \leq \mu(x)$. Moreover, from $\neg x^n \to x \leq 1 \to (\neg x^n \to x)$, it follows that by Proposition 2.1 (i), $\mu(\neg x^n \to x) \leq \mu(1 \to (\neg x^n \to x))$. Consequently, we obtain $\mu(\neg x^n \to x) \leq \mu(x)$. Therefore, by Theorem 4.5 μ is a fuzzy n-fold implicative prefilter of L. □

Theorem 4.9. *Let μ be a fuzzy n-fold implicative prefilter with the weak exchange principle. Then for any $x, y \in L$*

$$\mu((x^n \to y) \to y) \leq \mu((y \to x) \to x).$$

Proof. Let μ be a fuzzy n-fold implicative prefilter of L and put $u = (y \to x) \to x$. Then, by Lemma 2.1 (iii), $(x^n \to y) \to y \leq (y \to x) \to ((x^n \to y) \to x)$ and so, by Proposition 2.1 (i),

$$\begin{aligned}\mu((x^n \to y) \to y) &\leq \mu((y \to x) \to ((x^n \to y) \to x))\\ &= \mu((x^n \to y) \to ((y \to x) \to x))\\ &= \mu((x^n \to y) \to u),\end{aligned}$$

by Lemma 2.1 (ii), we have $x \leq (y \to x) \to x = u$ and so by $(EQ2)$ we get that $x^n \leq u^n$. Hence, by Lemma 2.1 (vi), we have $u^n \to y \leq x^n \to y$ and so $(x^n \to y) \to u \leq (u^n \to y) \to u$. Hence, by 2.1 (i) and Theorem 4.3, $\mu((x^n \to y) \to u) \leq \mu((u^n \to y) \to u) \leq \mu(u)$. Consequently, we obtain

$$\mu((x^n \to y) \to y) \leq \mu((y \to x) \to x).$$ □

Theorem 4.10. *Let μ be a fuzzy n-fold positive implicative prefilter of L. If $\mu((x \to y)^n \to y) \leq \mu((y \to x) \to x)$ for any $x, y \in L$, then μ is a fuzzy n-fold implicative prefilter of L.*

Proof. Suppose that μ is a fuzzy n-fold positive implicative prefilter of L and $\mu((x \to y)^n \to y) \leq \mu((y \to x) \to x)$, for any $x, y \in L$. Then by Lemma 2.1 (ii) and (vi), we have $y \leq x^n \to y$ and $(x^n \to y) \to x \leq y \to x$ and so by Proposition 2.1 (i), we get that

$$\mu((x^n \to y) \to x) \leq \mu(y \to x). \tag{4.1}$$

Moreover, since by Lemma 2.1 (iii), $(x^n \to y) \to x \leq (x \to y) \to ((x^n \to y) \to y)$, by Proposition 2.1 (i), we get that $\mu((x^n \to y) \to x) \leq \mu((x \to y) \to ((x^n \to y) \to y))$ and so by (4.1), we have

$$\mu((x^n \to y) \to x) \leq \mu((x \to y) \to ((x^n \to y) \to y)) \wedge \mu(y \to x). \tag{4.2}$$

Now, since by Lemma 2.1 (i) and (vi), $(x \to y)^n \leq x \to y$ and $(x \to y) \to ((x^n \to y) \to y) \leq ((x \to y)^n \to ((x^n \to y) \to y))$, we conclude that

$$\mu((x \to y) \to ((x^n \to y) \to y)) \leq \mu((x \to y)^n \to ((x^n \to y) \to y)),$$

and since $x^n \leq x$ and $(x \to y)^n \leq x \to y \leq x^n \to y$, we get that $(x \to y)^n \to (x^n \to y) = 1$ and so $\mu((x \to y)^n \to (x^n \to y)) = \mu(1)$ and since $\mu((x \to y)^n \to y) \leq \mu((y \to x) \to x)$ and since μ is a fuzzy n-fold positive implicative prefilter of L, we get that

$$\begin{aligned}
\mu((x \to y)^n \to ((x^n \to y) \to y)) &= \mu((x \to y)^n \to ((x^n \to y) \to y)) \wedge \mu(1) \\
&= \mu((x \to y)^n \to [(x^n \to y) \to y]) \\
&\wedge \mu((x \to y)^n \to (x^n \to y)) \\
&\leq \mu((x \to y)^n \to y) \\
&\leq \mu((y \to x) \to x).
\end{aligned}$$

Hence, by (4.2), we conclude that

$$\mu((x^n \to y) \to x) \leq \mu((y \to x) \to x) \wedge \mu(y \to x) \leq \mu(x).$$

Therefore, by Theorem 4.3, μ is a fuzzy n-fold implicative prefilter of L. □

Theorem 4.11. *Let μ be fuzzy positive implicative prefilter of L with the weak exchange principle. Then the following are equivalent:*

(i) *μ is a fuzzy implicative prefilter of L;*

(ii) *$\mu((x \to y) \to y) \leq \mu((y \to x) \to x)$ for all $x, y \in L$.*

Proof. It follows from Theorem 4.9 and Theorem 4.10, whenever $n = 1$. □

Theorem 4.12. *Let L be an EQ-algebra with a bottom element* 0 *and μ be a fuzzy n-fold positive implicative prefilter of L. If $\mu(\neg(\neg x)^n) \leq \mu(x)$ for any $x \in L$, then μ is a fuzzy n-fold implicative prefilter of L.*

Proof. Let μ be a fuzzy n-fold positive implicative prefilter of L. Then for any $x \in L$, by Lemma 2.1 (iii), $\neg x^n \to x \leq (x \to 0) \to (\neg x^n \to 0) = \neg x \to (\neg x^n \to 0)$. Hence, by Proposition 2.1 (i), we have $\mu(\neg x^n \to x) \leq \mu(\neg x \to (\neg x^n \to 0))$ and since $(\neg x)^n \leq \neg x$, by Lemma 2.1 (vi), we get that $\neg x \to (\neg x^n \to 0) \leq (\neg x)^n \to (\neg x^n \to 0)$ and so $\mu(\neg x \to (\neg x^n \to 0)) \leq \mu((\neg x)^n \to (\neg x^n \to 0))$. Hence, $\mu(\neg x^n \to x) \leq \mu((\neg x)^n \to (\neg x^n \to 0))$. Now, since $x^n \leq x$, we have $\neg x \leq \neg x^n$ and so $(\neg x)^n \leq \neg x$, we conclude that $(\neg x)^n \leq \neg x^n$ and so $(\neg x)^n \to \neg x^n = 1$ and since μ is a fuzzy n-fold positive implicative prefilter, we conclude that

$$\begin{aligned}
\mu((\neg x)^n \to (\neg x^n \to 0)) &= \mu((\neg x)^n \to (\neg x^n \to 0)) \wedge \mu(1) \\
&= \mu((\neg x)^n \to (\neg x^n \to 0)) \wedge \mu((\neg x)^n \to \neg x^n) \\
&\leq \mu((\neg x)^n \to 0),
\end{aligned}$$

and since by hypothesis $\mu(\neg(\neg x)^n) = \mu((\neg x)^n \to 0) \leq \mu(x)$, we get that $\mu(\neg x^n \to x) \leq \mu(x)$. Therefore, by Theorem 4.5, μ is a fuzzy n-fold implicative prefilter of L. □

Theorem 4.13. *Let L be an EQ-algebra with a bottom element 0 and μ be a fuzzy n-fold implicative prefilter of L. Then $\mu(\neg\neg x^n) \leq \mu(x)$ for all $x \in L$.*

Proof. Let μ be a fuzzy n-fold implicative prefilter of L and $x \in L$. Then by Lemma 2.1 (vi), $\neg\neg x^n = \neg x^n \to 0 \leq \neg x^n \to x$ and so by Proposition 2.1 (i), $\mu(\neg\neg x^n) \leq \mu(\neg x^n \to x)$ and since by Theorem 4.5, $\mu(\neg x^n \to x) \leq \mu(x)$, we conclude that $\mu(\neg\neg x^n) \leq \mu(x)$. □

5. Conclusion

In this paper, the notion of fuzzy n-fold positive implicative and fuzzy n-fold implicative (pre)filters in EQ-algebras are introduced and several properties of them are stated. Using the concept of level subsets, some characterizations of fuzzy n-fold (positive) implicative (pre)filters are proved. Furthermore, we discussed the relationship between fuzzy n-fold positive implicative (pre)filters and fuzzy n-fold implicative (pre)filters and verified that under what conditions the fuzzy n-fold positive implicative (pre)filters and fuzzy n-fold implicative (pre)filters are equivalent in EQ-algebras. In this article, there are theorems and propositions that have been proved by adding some conditions to an EQ-algebra. One of the important questions for future research is how we can prove these theorems without these conditions or with less conditions. Also, how to define the notions of fuzzy n-fold fantastic filters in EQ-algebras? What is the relation between fuzzy n-fold fantastic filters and other types fuzzy filters?

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