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# EXISTENCE RESULT FOR FRACTIONAL DIFFERENTIAL EQUATION ON UNBOUNDED DOMAIN

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Abstract. In this article, we establish certain sufficient conditions to show the existence of solutions of boundary value problem for fractional differential equations on the half-line in a Fréchet space. The main result is based on Tykhonoff fixed point theorem combining with a suitable measure of non-compactness. An example is given to illustrate our approach.

## 1. Introduction

The theoretical study of fractional differential equations has recently acquired great importance in applied mathematics and the modeling of many phenomena in various sciences, let us quote for example [11, 12, 15, 17]. The monographs [14, 16, 18, 20] contain basic concepts and theory in fractional differential equations and fractional calculus.

Very recently, excellent works have been done to study fractional differential equations with various conditions which resides in the existence and uniqueness theorem by utilizing some analytical and numerical methods and certain basic tools from functional analysis, we refer the reader to [1, 4–9].

Several results existence of these problems were obtained on unbounded domains like $[0, +\infty)$ involving classical methods, for example, Xinwei Su discussed in the work [19] the existence of solutions of the following problem

$$\begin{cases} D^{\alpha}_{0^+}y(t) = f(t, y(t)), & t \in J = (0, +\infty), 1 < \alpha \leq 2, \\ y(0) = 0, & D^{\alpha-1}_{0^+}y(\infty) = y_{\infty}, \end{cases}$$

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where $D^{\alpha}_{0+}$ is the Riemann-Liouville fractional derivative of order $\alpha$, the state $y(\cdot)$ takes value in a Banach space $E$, $f : J \times E \to E$ is a continuous function and $y_\infty \in E$. The main approach is based on Dardo's fixed point theorem. The contents of this article are an extension of their work for a class generally.

This article studies the existence of solutions of boundary value problem for fractional differential equations on unbounded interval. We consider the following problem

$$D^{\alpha}_{0+}y(t) = f(t, y(t)), \quad t \in J = (0, +\infty), \tag{1.1}$$

$$I^{2-\alpha}_{0+}y(0^+) = y_0, \tag{1.2}$$

$$D^{\alpha-1}_{0+}y(\infty) = y_\infty, \tag{1.3}$$

where $D^{\delta}_{0+}$ denotes Riemann-Liouville fractional derivative for $\delta \in \{\alpha, \alpha - 1\}$ with $1 < \alpha \leq 2$, $I^{2-\alpha}_{0+}$ denotes the left-sided Riemann-Liouville fractional integral, $E$ is a real Banach space with the norm $\|\cdot\|$, $y_0, y_\infty \in E$ and $f : (0, \infty) \times E \to E$ a function satisfying some specified conditions (see Section 3).

The present work is organized in the following way. In Section 2, we give some general results and preliminaries and in Section 3, we show the existence solution for the problem (1.1)–(1.3) by using the Tykhonoff fixed point theorem combined with the technique of measure of non-compactness of Kuratowski. Finally an illustrative example will be presented in the last section.

## 2. Backgrounds

We introduce in this section some notation and technical results which are used throughout this paper. Let $I \subset J$ be a compact interval and denote by $C(I, E)$ the Banach space of continuous functions $y : I \to E$ with the usual norm

$$\|y\|_\infty = \sup\{\|y(t)\| \mid t \in I\}.$$

$L^1(I, E)$ denotes the space of $E$-valued Bochner integrable functions on $I$ with the norm

$$\|f\|_{L^1} = \int_I \|f(t)\| dt.$$

We consider the following Fréchet space

$$C_\alpha([0, \infty), E) = \left\{ y \in C(J, E) \mid \lim_{t \to 0^+} t^{2-\alpha} y(t) \text{ exists and is finite} \right\},$$

equipped with the family of seminorms

$$\|y\|_T = \sup_{t \in [0,T]} \left\{ \frac{t^{2-\alpha}}{1 + t^\alpha} \|y(t)\| \mid T \geq 0 \right\}.$$

For $y \in C_\alpha((0, \infty), E)$, we define $y_\alpha$ by

$$y_\alpha(t) = \begin{cases} \dfrac{t^{2-\alpha}}{1 + t^\alpha} y(t), & \text{if } t \in (0, \infty), \\ \lim\limits_{t \longrightarrow 0} t^{2-\alpha} y(t), & \text{if } t = 0. \end{cases}$$

It is clear that $y_\alpha \in C([0,\infty), E)$.

We begin with the definition of the measure of non-compactness in the sense of Kuratowski and its properties. For more details, we refer the reader [2, 3, 13].

**Definition 2.1.** The Kuratowski measure of non-compactness $\gamma$ is defined on each bounded subset $\Omega$ of $E$ by

$$\gamma(\Omega) = \inf\{\varepsilon > 0 \mid \Omega \text{ admits a finite cover by sets of diameter } \leq \varepsilon\}.$$

**Lemma 2.1** ([3]). *Let $\{D_n\}_0^\infty$ be a sequence of nonempty, bounded and closed subsets of $E$, such that for all positive integer $n$, $D_{n+1} \subset D_n$. If $\lim_{n\to\infty} \gamma(D_n) = 0$, then the set*

$$D_\infty = \bigcap_{n=0}^{\infty} D_n$$

*is nonempty and compact.*

**Lemma 2.2** ([2]). *Let $E$ be a Banach space and $A, B$ be two bounded subsets of $E$. The following properties hold:*

($i_1$) $\gamma(A) = 0$ *if and only if $A$ is relatively compact;*
($i_2$) $\gamma(A) = \gamma(\overline{A})$, *where $\overline{A}$ denotes the closure of $A$;*
($i_3$) $\gamma(A + B) \leq \gamma(A) + \gamma(B)$;
($i_4$) $A \subset B$ *implies* $\gamma(A) \leq \gamma(B)$;
($i_5$) $\gamma(a.A) = |a|.\gamma(A)$ *for all* $a \in E$;
($i_6$) $\gamma(\{a\} \cup A) = \gamma(A)$ *for all* $a \in E$;
($i_7$) $\gamma(A) = \gamma(Conv(A))$, *where $Conv(A)$ denotes the convex hull of $A$.*

**Lemma 2.3** ([2]). *If $\Omega$ is a bounded and equicontinuous subset of $C(I, E)$, then $\gamma(\Omega(t))$ is continuous on $I$ and*

$$\gamma_C(\Omega) = \max_{t\in I} \gamma(\Omega(t)), \quad \gamma\left(\left\{\int_I x(t)dt : x \in \Omega\right\}\right) \leq \int_I \gamma(\Omega(t))dt,$$

*where $\Omega(t) = \{x(t) \mid x \in \Omega\}$ and $\gamma_C$ is the non-compactness measure on the space $C(I, E)$.*

The following theorem is due to Tykhonoff.

**Theorem 2.1** ([10]). *Let $F$ be a locally convex space, $K$ a compact convex subset of $F$ and $N : K \to K$ a continuous map. Then $N$ has at least one fixed point in $K$.*

Let us now give some definitions from the theory of fractional calculus.

**Definition 2.2** ([14]). Let $\Gamma$ be the gamma function, $\alpha$ a non-negative real number and $h \in C(J, E)$.

(1) The Riemann-Liouville fractional integral of the function $h$ of order $\alpha$ is defined by

$$I_{0+}^\alpha h(t) = g_\alpha(t) * h(t) = \int_0^t g_\alpha(t-s)h(s)ds, \quad t > 0,$$

where $*$ denotes convolution and $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$.

(2) The Riemann-Liouville fractional derivative of the function $h$ of order $\alpha$ is defined by

$$D_{0+}^{\alpha}h(t) = \frac{d^n}{dt^n}(g_{n-\alpha}(t) * h(t)),$$

for all $t > 0$, where $n$ is the least integer greater than or equal to $\alpha$.

*Remark* 2.1. For $\alpha > 0,\ k > -1$, we have

$$I_{0+}^{\alpha}t^k = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}t^{\alpha+k} \quad \text{and} \quad D_{0+}^{\alpha}t^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha},\ t > 0,$$

giving in particular $D_{0+}^{\alpha}t^{\alpha-m} = 0,\ m = 1,\dots,n$, where $n$ is the smallest integer greater than or equal to $\alpha$.

*Remark* 2.2. If $h$ is suitabe function (see for instance [14, 16, 18]), we have the composition relations $D_{0+}^{\alpha}I_{0+}^{\alpha}h(t) = h(t),\ \alpha > 0$, and $D_{0+}^{\alpha}I_{0+}^{k}h(t) = I_{0+}^{k-\alpha}h(t),\ k > \alpha > 0,\ t > 0$.

## 3. Main Result

We need to introduce the following four hypotheses to present our main result at the end of this section.

$(H_1)$ $f : J \times E \to E$ is a Carathéodory function.

$(H_2)$ There exists nonnegative functions $a, b \in C(J, \mathbb{R}^+)$ such that

$$\|f(t,u)\| \leq a(t) + t^{2-\alpha}b(t)\|u\|, \quad \text{for all } t \in J \text{ and } u \in E,$$

where

$$\int_0^{\infty}(1+t^{\alpha})b(t)dt < \Gamma(\alpha), \quad \int_0^{\infty} a(t)dt < \infty.$$

$(H_3)$ There exists a locally integrable function $\ell \in L^1(J, \mathbb{R}^+)$ such that, for each nonempty, bounded set, we have $\Omega \subset C_{\alpha}(J, E)$

$$\gamma(f(t,\Omega(t))) \leq \ell(t)\gamma(t^{2-\alpha}\Omega(t)), \quad \text{for all } t \in J,$$

where

$$\int_0^{\infty}(1+s^{\alpha})\ell(s)ds < \Gamma(\alpha). \tag{3.1}$$

$(H_4)$ There exists $R > 0$ such that

$$R > \frac{\|y_{\infty}\| + (\alpha-1)\|y_0\| + \int_0^{\infty} a(t)dt}{\Gamma(\alpha) - \int_0^{\infty}(1+t^{\alpha})b(t)dt}.$$

**Definition 3.1.** A function $y \in C_{\alpha}([0, +\infty))$ is said to be a solution of the problem (1.1)–(1.3) if $y$ satisfies the equation $D_{0+}^{\alpha}y(t) = f(t, y(t))$ and the conditions (1.2)–(1.3).

**Lemma 3.1.** *Let $1 < \alpha < 2$ and let $h : J \to E$ be continuous. If $y$ is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)}\left[y_\infty - \int_0^\infty h(t)dt\right]t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha-1)}t^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds, \tag{3.2}$$

*then it is also a solution of the problem*

$$D_{0+}^{\alpha} y(t) = h(t), \quad t \in J = (0, +\infty), \tag{3.3}$$

$$I_{0+}^{2-\alpha} y(0^+) = y_0, \tag{3.4}$$

$$D_{0+}^{\alpha-1} y(\infty) = y_\infty. \tag{3.5}$$

*Proof.* Suppose that $y$ is a solution of the integral equation (3.2). Applying $I_{0+}^{2-\alpha}$ to both sides of (3.2) and using Remark 2.1, we obtain

$$I_{0+}^{2-\alpha} y(t) = \left(y_\infty - \int_0^\infty h(t))dt\right) t + y_0 + I_{0+}^{2} h(t).$$

As $t \to 0$, we get

$$I_{0+}^{2-\alpha} y(0^+) = y_0.$$

Now, by applying $D_{0+}^{\alpha-1}$ to both sides of (3.2) and by using Remark 2.1, Remark 2.2, we have

$$D_{0+}^{\alpha-1} y(t) = y_\infty - \int_0^\infty h(t)dt + I_{0+}^{1} h(t).$$

As $t \to \infty$, we get

$$D_{0+}^{\alpha-1} y(\infty) = y_\infty.$$

Next, by applying $D_{0+}^{\alpha}$ to both sides of (3.2) and by using Remark 2.1, Remark 2.2, we obtain $D_{0+}^{\alpha} y(t) = h(t)$. The results are proved completely. □

Consider the operator $N : C_\alpha([0,\infty), E) \to C_\alpha([0,\infty), E)$ defined by

$$Ny(t) = \frac{y_\infty - \int_0^\infty f(t, y(t))dt}{\Gamma(\alpha)} t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha-1)}t^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s))ds.$$

Let

$$B = \{y \in C_\alpha([0,\infty), E) \mid \|y\|_T \le R\}.$$

*Remark* 3.1. **(1)** Clearly the operator $N$ is well defined.

**(2)** There exists a positive real number $M$ such that

$$\int_0^\infty \|f(t, y(t))\|dt \le M, \quad \text{for any } y \in B.$$

**Lemma 3.2.** *If the conditions $(H_1)$ and $(H_2)$ are valid, then*

**(1)** *the operator $N$ is bounded and continuous on the subset $B$;*

**(2)** *the subset $(NB)_\alpha = \{(Ny)_\alpha \mid y \in B\}$ is equicontinuous on the compact interval $[0, T]$, $T > 0$;*

**(3)** *for given $\varepsilon > 0$, there exists a constant $N_1 > 0$ such that*

$$\left\| \frac{t_1^{2-\alpha} N(y)(t_1)}{1+t_1^\alpha} - \frac{t_2^{2-\alpha} N(y)(t_2)}{1+t_2^\alpha} \right\| < \varepsilon, \quad \textit{for any } t_1, t_2 \geq N_1 \textit{ and } y(\cdot) \in B.$$

*Proof.* In order to prove (**1**), let $y \in B$ and $t \in [0, T],\ T > 0$, from $(H_2)$, we have

$$\begin{aligned} \frac{t^{2-\alpha}\|N(y)(t)\|}{1+t^\alpha} \leq & \frac{\|y_\infty\|}{\Gamma(\alpha)} + \frac{\|y_0\|}{\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \int_0^\infty \|f(s, y(s))\| ds \\ \leq & \frac{\|y_\infty\|}{\Gamma(\alpha)} + \frac{\|y_0\|}{\Gamma(\alpha-1)} + \frac{R}{\Gamma(\alpha)} \int_0^\infty (1+t^\alpha) b(t) dt + \frac{1}{\Gamma(\alpha)} \int_0^\infty a(t) dt. \end{aligned}$$

Hence, $N$ is bounded on the subset $B$. Next, we will prove that $N$ is continuous. We have

$$Ny(t) = \frac{y_\infty - \int_0^\infty f(t, y(t)) dt}{\Gamma(\alpha)} t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds.$$

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $B$, such that $y_n \to y$ in $B$. Let $T > 0$ and $\varepsilon > 0$, from $(H_1)$ and $(H_2)$, there exists $L > T$ such that

$$\int_L^\infty a(t) dt < \frac{\Gamma(\alpha)}{6} \varepsilon, \quad \int_L^\infty (1+t^\alpha) b(t) dt < \frac{\Gamma(\alpha)}{6M} \varepsilon,$$

and there exists $\widetilde{N} \in \mathbb{N}$ such that, for all $n \geq \widetilde{N}$, we have

$$\int_0^\infty \|f(s, y_n(s)) - f(s, y(s))\| ds < \frac{\Gamma(\alpha)}{3} \varepsilon.$$

Therefore, for $t \in [0, T],\ T > 0$ and $n > \widetilde{N}$, we have

$$\begin{aligned} & \frac{t^{2-\alpha}}{1+t^\alpha} \|N(y_n)(t) - N(y)(t)\| \\ \leq & \frac{1}{\Gamma(\alpha)} \int_0^t \|f(s, y_n(s)) - f(s, y(s))\| ds + \frac{1}{\Gamma(\alpha)} \int_t^\infty \|f(s, y_n(s)) - f(s, y(s))\| ds \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^t \|f(s, y_n(s)) - f(s, y(s))\| ds \\ & + \frac{1}{\Gamma(\alpha)} \left[ \int_t^L \|f(s, y_n(s)) - f(s, y(s))\| ds + \int_L^\infty \|f(s, y_n(s)) - f(s, y(s))\| ds \right] \\ \leq & \frac{1}{\Gamma(\alpha)} \int_0^L \|f(s, y_n(s)) - f(s, y(s))\| ds + \frac{2M}{\Gamma(\alpha)} \int_L^\infty (1+s^\alpha) b(s) ds + \frac{2}{\Gamma(\alpha)} \int_L^\infty a(s) ds \\ \leq & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Then,

$$\|Ny_n - Ny\|_T \to 0 \quad \text{as} \quad n \to \infty.$$

We will prove (**2**). Let $y \in B$ and $t_1, t_2 \in [0, T],\ T > 0$ where $t_1 > t_2$. Then

$$\left\| \frac{t_1^{2-\alpha} N(y)(t_1)}{1+t_1^\alpha} - \frac{t_2^{2-\alpha} N(y)(t_2)}{1+t_2^\alpha} \right\|$$

$$
\begin{aligned}
\leq & \frac{\|y_\infty\|+M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha}-\frac{t_2}{1+t_2^\alpha}\right|+\frac{\|y_0\|}{\Gamma(\alpha-1)}\left|\frac{1}{1+t_1^\alpha}-\frac{1}{1+t_2^\alpha}\right| \\
& +\frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,y(s))ds-\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))ds\right\| \\
\leq & \frac{\|y_\infty\|+M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha}-\frac{t_2}{1+t_2^\alpha}\right|+\frac{\|y_0\|}{\Gamma(\alpha-1)}\left|\frac{1}{1+t_1^\alpha}-\frac{1}{1+t_2^\alpha}\right| \\
& +\frac{1}{\Gamma(\alpha)}\int_0^{t_1}|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|\|f(s,y(s))\|ds \\
& +\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|f(s,y(s))\|ds \\
\leq & \frac{\|y_\infty\|+M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha}-\frac{t_2}{1+t_2^\alpha}\right|+\frac{\|y_0\|}{\Gamma(\alpha-1)}\left|\frac{1}{1+t_1^\alpha}-\frac{1}{1+t_2^\alpha}\right| \\
& +\frac{1}{\Gamma(\alpha)}\int_0^{t_1}|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|a(s)ds \\
& +\frac{r}{\Gamma(\alpha)}\int_0^{t_1}|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|(1+s^\alpha)b(s)ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}a(s)ds \\
& +\frac{r}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(1+s^\alpha)b(s)ds \\
\leq & \frac{\|y_\infty\|+M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha}-\frac{t_2}{1+t_2^\alpha}\right|+\frac{\|y_0\|}{\Gamma(\alpha-1)}\left|\frac{1}{1+t_1^\alpha}-\frac{1}{1+t_2^\alpha}\right| \\
& +\frac{a^*+b^*r}{\Gamma(\alpha)}\int_0^{t_1}\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}ds\right)+\frac{a^*+b^*r}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
& +\frac{2b^*r}{\Gamma(\alpha)}\left(\int_0^{t_2}(t_2-s)^{\alpha-1}s^\alpha ds-\int_0^{t_1}(t_1-s)^{\alpha-1}s^\alpha ds\right) \\
\leq & \frac{\|y_\infty\|+M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha}-\frac{t_2}{1+t_2^\alpha}\right|+\frac{\|y_0\|}{\Gamma(\alpha-1)}\left|\frac{1}{1+t_1^\alpha}-\frac{1}{1+t_2^\alpha}\right| \\
& +\frac{a^*+b^*r}{\Gamma(1+\alpha)}\left(t_2^\alpha-t_1^\alpha-(t_2-t_1)^\alpha\right)+\frac{a^*+b^*r}{\Gamma(1+\alpha)}(t_2-t_1)^\alpha \\
& +\frac{2b^*r\mathcal{B}(\alpha,\alpha+1)}{\Gamma(\alpha)}(t_2^{2\alpha}-t_1^{2\alpha}),
\end{aligned}
$$

where $a^*=\max_{t\in[0,T]}a(t)$ and $b^*=\max_{t\in[0,T]}b(t)$. As $t_2\to t_1$ the right-hand side of the above inequality tends to zero. Then $(NB)_\alpha$ is equicontinuous on $[0,T]$.

Next, we verify assertion (**3**). Let $\varepsilon>0$, we have

$$
\begin{aligned}
& \left\|\frac{t_1^{2-\alpha}N(y)(t_1)}{1+t_1^\alpha}-\frac{t_2^{2-\alpha}N(y)(t_2)}{1+t_2^\alpha}\right\| \\
\leq & \frac{\|y_\infty\|+M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha}-\frac{t_2}{1+t_2^\alpha}\right|+\frac{\|y_0\|}{\Gamma(\alpha-1)}\left|\frac{1}{1+t_1^\alpha}-\frac{1}{1+t_2^\alpha}\right|
\end{aligned}
$$

$$+\frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}f(s,y(s))ds-\int_0^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}f(s,y(s))ds\right\|.$$

It is sufficient to prove that

$$\left\|\int_0^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}f(s,y(s))ds-\int_0^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}f(s,y(s))ds\right\|\leq\varepsilon.$$

Remark 3.1 yields that, there exists $N_0>0$ such that

$$\int_{N_0}^{\infty}\|f(t,y(t))\|dt\leq\frac{\varepsilon}{3},\quad\text{for all } y\in B. \tag{3.6}$$

On the other hand, since $\lim_{t\to\infty}\frac{t^{2-\alpha}(t-N_0)^{\alpha-1}}{1+t^\alpha}=0$, there exists $N_1>N_0$ such that, for all $t_1,t_2>N_1$ and $s\in[0,N_1]$, we have

$$\left|\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}-\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\right|<\frac{\varepsilon}{3M}. \tag{3.7}$$

Now taking $t_1,t_2\geq N_1$, from (3.6), (3.7), we can arrive at

$$\begin{aligned}&\left\|\int_0^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}f(s,y(s))ds-\int_0^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}f(s,y(s))ds\right\|\\
\leq&\int_0^{N_1}\left|\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}-\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\right|\|f(s,y(s))\|ds\\
&+\int_{N_1}^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\|f(s,y(s))\|ds+\int_{N_1}^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}\|f(s,y(s))\|ds\\
<&\frac{\varepsilon}{3M}\int_0^\infty\|f(s,y(s))\|ds+2\int_{N_1}^\infty\|f(s,y(s))\|ds<\varepsilon.\end{aligned}$$

□

**Theorem 3.1.** *Suppose that the conditions* $(H_1)$, $(H_2)$, $(H_3)$ *and* $(H_4)$ *are valid. Then the problem* (1.1)–(1.3) *has at least one solution.*

*Proof.* We shall prove that $N$ satisfies the conditions of Tykhonoff fixed point theorem 2.1. From Lemma 3.2, the operator $N$ is continuous on $B$. We can derive that $N:B\to B$. Indeed, for any $y\in B$ and $t\in[0,T]$, $T>0$, and by condition $(H_1)$ and $(H_4)$, we get

$$\begin{aligned}\|t^{2-\alpha}N(y)(t)\|&\leq\frac{\|y_\infty\|}{\Gamma(\alpha)}+\frac{\|y_0\|}{\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha)}\int_0^\infty\|f(t,y(t))\|dt\\
&\leq\frac{(1}{\Gamma(\alpha)}\left(\|y_\infty\|+(\alpha-1)\|y_0\|+\int_0^\infty a(t)dt+R\int_0^\infty(1+t^\alpha))b(t)dt\right)\\
&<R.\end{aligned}$$

Let $\gamma_\alpha$ be the measure of non-compacteness of Kuratowski defined on the family of bounded subsets of the space $C_\alpha(J,E)$. We have

$$\gamma_\alpha(NB)=\sup_{T>0}\left\{\sup_{t\in[0,T]}\gamma\left(\frac{t^{\alpha-2}}{1+t^\alpha}N(B)(t)\right)\right\}.$$

For demostrations, see [19, Lemma 3.4].

We define the sequence of sets $\{D_n\}_{n=0}^{\infty}$ by

$$\begin{cases} D_0 = B, \\ D_{n+1} = Conv((N(D_n))), \quad n = 0, 1, \ldots, \\ D_\infty = \cap_{n=0}^{\infty} D_n. \end{cases}$$

We have $D_{n+1} \subset D_n$, for each $n$. Finally, we need to prove the following relation

$$\lim_{n\to\infty} \gamma_\alpha(D_n) = 0.$$

Suppose that $T$ is sufficiently large. For each $y \in B$, we consider

$$\begin{aligned} N_T(y)(t) =& \frac{y_\infty}{\Gamma(\alpha)} t^{\alpha-1} + \frac{y_0}{\Gamma(\alpha)} t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t [t^{\alpha-1} - (t-s)^{\alpha-1}] f(s, y(s)) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_t^T (t-s)^{\alpha-1} f(s, y(s)) ds. \end{aligned}$$

Then, from $(H_2)$, we obtain that

$$\begin{aligned} \frac{t^{2-\alpha}}{1+t^\alpha} \|N_T(y)(t) - N(y)(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_T^\infty \|f(t, y(t))\| dt \\ &\le \frac{1}{\Gamma(\alpha)} \left( \int_T^\infty a(t) dt + R \int_T^\infty (1+t^\alpha) b(t) dt \right), \end{aligned}$$

this shows that

$$H_d \left( \frac{t^{2-\alpha} N_T(B)(t)}{1+t^\alpha}, \frac{t^{2-\alpha} N(B)(t)}{1+t^\alpha} \right) \to 0 \quad \text{as} \quad T \to \infty, t \in J.$$

Where $H_d$ denotes the Hausdorff metric in space $E$. By Property of non-compactness measure, we get

$$\lim_{T\to\infty} \gamma \left( \frac{t^{2-\alpha} N_T(B)(t)}{1+t^\alpha} \right) = \gamma \left( \frac{t^{2-\alpha} N(B)(t)}{1+t^\alpha} \right). \tag{3.8}$$

Let $\varepsilon > 0$, from (3.8), then exists $\widetilde{T} > 0$ such that, for $T \ge \widetilde{T}$, we have

$$\gamma \left( \frac{t^{2-\alpha} N(B)(t)}{1+t^\alpha} \right) < \varepsilon + \gamma \left( \frac{t^{2-\alpha} N_T(B)(t)}{1+t^\alpha} \right).$$

Using Lemma 2.3, Lemma 3.2 and assumption $(H_3)$, for each $n \in \mathbb{N}$ and $t \in [0, T]$, $T > \widetilde{T}$, we get

$$\begin{aligned} \gamma \left( \frac{t^{2-\alpha} N(D_{n+1})(t)}{1+t^\alpha} \right) &\le \varepsilon + \frac{1}{\Gamma(\alpha)} \int_0^T (1+s^\alpha) \ell(s) \gamma \left( \frac{s^{2-\alpha} D_{n+1}(s)}{1+s^\alpha} \right) ds \\ &\le \varepsilon + \frac{1}{\Gamma(\alpha)} \int_0^T (1+s^\alpha) \ell(s) \sup_{s\in[0,T]} \gamma \left( \frac{s^{2-\alpha} N(D_n)(s)}{1+s^\alpha} \right) ds. \end{aligned}$$

Then

$$\sup_{s\in[0,T]} \gamma\left(\frac{t^{2-\alpha}N(D_{n+1})(t)}{1+t^\alpha}\right) \leq \varepsilon + \frac{1}{\Gamma(\alpha)}\int_0^T (1+s^\alpha)\ell(s) \sup_{s\in[0,T]} \gamma\left(\frac{s^{2-\alpha}N(D_n)(s)}{1+s^\alpha}\right) ds.$$

$$\leq \varepsilon + \frac{\tau}{\Gamma(\alpha)} \sup_{s\in[0,T]} \gamma\left(\frac{s^{2-\alpha}N(D_n)(s)}{1+s^\alpha}\right),$$

where

$$\tau = \int_0^\infty (1+s^\alpha)\ell(s)ds.$$

Consequently, since $\varepsilon$ is arbitrary, we obtain

$$\gamma_\alpha(D_{n+1}) \leq \frac{\tau\gamma_\alpha(D_n)}{\Gamma(\alpha)}, \quad \text{for each } n\in\mathbb{N}.$$

By induction, we can show that

$$\gamma_\alpha(D_{n+1}) \leq \left(\frac{\tau}{\Gamma(\alpha}\right)^{n+1} \gamma_\alpha(D_0), \quad \text{for each } n\in\mathbb{N}.$$

Hence, by (3.1), we get

$$\lim_{n\to\infty} \gamma_\alpha(D_n) = 0.$$

Taking into account Lemma 2.1, we infer that $D_\infty = \bigcap_{n=0}^{\infty} D_n$ is nonempty, convex and compact. From Theorem 2.1, we conclude that $N : D_\infty \to D_\infty$ has a fixed point $y \in D_\infty$, which is a solution of problem (1.1)–(1.3). □

## 4. Example

As an application of our results, we consider the following fractional differential equation

$$D^{\frac{3}{2}} y(t) = \left(\frac{\sqrt{t}y_n(t)}{(1+t^{\frac{3}{2}})e^{5t}} + \frac{\sin(t)}{1+t^2}\right)_{n=1}^{\infty}, \quad t\in J=(0,+\infty), \tag{4.1}$$

$$I_{0^+}^{\frac{1}{2}} y(t) = y_0, \tag{4.2}$$

$$D_{0^+}^{\frac{1}{2}} y(\infty) = y_\infty. \tag{4.3}$$

Let

$$E = \{(y_1, y_2, \ldots, y_n, \ldots) \mid \sup |y_n| < \infty\},$$

with the norm $\|y\| = \sup_n |y_n|$, then $E$ is a Banach space and problem (4.1)–(4.3) can be regarded as a problem of the form (1.1)–(1.3), with

$$\alpha = \frac{3}{2} \quad \text{and} \quad f(t,y(t)) = (f(t,y_1(t)), \ldots, f(t,y_n(t)), \ldots),$$

where

$$f(t,y_n(t)) = \frac{\sqrt{t}y_n(t)}{(1+t^{\frac{3}{2}})e^{5t}} + \frac{\sin(t)}{1+t^2}, \quad n\in\mathbb{N}^*.$$

We shall verify the conditions $(H_2)$–$(H_4)$. Evidently, $f$ is Carathéodory function in $J \times E$ and

$$\|f(t, y(t))\| \leq \frac{\sqrt{t}}{(1+t^{\frac{3}{2}})e^{5t}}\|y(t)\| + \frac{1}{1+t^2}.$$

With the aid of simple computation, we find that

$$\int_0^\infty e^{-5t}dt = \frac{1}{5} < \Gamma\left(\frac{3}{2}\right) \quad \text{and} \quad \int_0^\infty \frac{1}{1+t^2}dt = \frac{\pi}{2} < \infty.$$

Finally, we verify the condition $(H_3)$. For any bounded set $B \subset E$, we have

$$f(t, B(t)) = \frac{\sqrt{t}}{(1+t^{\frac{3}{2}})e^{5t}}B(t) + \left\{\frac{\sin(t)}{1+t^2}\right\}.$$

Then

$$\gamma(f(t, B(t)) \leq \frac{\sqrt{t}}{(1+t^{\frac{3}{2}})e^{5t}}\gamma(B(t)).$$

Since $\int_0^\infty e^{-5t}dt = \frac{1}{5} < \Gamma(\frac{3}{2})$, we conclude that the condition $(H_3)$ is satisfied. Therefore, Theorem 3.1 ensures that the Problem (4.1)–(4.3) has a solution.

## 5. Conclusion

We hope that we have given some result as far as we know not existing in the literature concerning existence solution for Riemann-Liouville fractional differential equation on the half line involving the discontinuity of the state $y$ at $0^+$, to overcome this obstruction we have defined a special weight space of continuous function $C_\alpha([0, +\infty))$. The constructed space is in a natural way. In this work we have assumed a more general growth condition $(H_1)$ unlike what is in the literature, condition $(H_2)$ being supposed to overcome the equiconvergence at infinity, condition $(H_3)$ ensure the proof of Tykhonoff fixed point theorem, these conditions are optimal in the sense that no condition implies the other. We make use in our approach Tykhonoff fixed point theorem combining with analysis functional tools and a suitable measure of non-compactness. The paper is ended by an example to illustrate the main result.

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