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ORESME HYBRID NUMBERS AND HYBRATIONALS

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Abstract. In this paper we introduce and study Oresme hybrid numbers and hybrationals based on the known Oresme sequence. The main aim is to present these new concepts and to give some properties of Oresme hybrid numbers.

1. Introduction

Let p, q, n be integers. For $n \geq 0$ Horadam (see [2]) defined the numbers $W_n = W_n(W_0, W_1; p, q)$ by the recursive equation

$$W_{n+2} = p \cdot W_{n+1} - q \cdot W_n, \tag{1.1}$$

with fixed real numbers W_0, W_1. For the historical reasons these numbers were later called Horadam numbers.

For special W_0, W_1, p, q the equation (1.1) defines selected numbers of the Fibonacci type, e.g. Fibonacci numbers $F_n = W_n(0, 1; 1, -1)$, Pell numbers $P_n = W_n(0, 1; 2, -1)$, Jacobsthal numbers $J_n = W_n(0, 1; 1, -2)$.

In [3] Horadam extended the equation (1.1) considering values of p, q to be arbitrary rational numbers. Then taking $W_0 = 0$, $W_1 = \frac{1}{2}$, $p = 1$ and $q = \frac{1}{4}$ the equation (1.1) gives the known Oresme sequence $\{O_n\} = \{W_n(0, \frac{1}{2}; 1, \frac{1}{4})\}$, where O_n is the nth Oresme number. Consequently,

$$O_n = O_{n-1} - \frac{1}{4} O_{n-2}, \tag{1.2}$$

for $n \geq 2$ with $O_0 = 0$, $O_1 = \frac{1}{2}$.

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Solving the above recurrence equation we obtain Binet formula for Oresme numbers of the form

$$O_n = \frac{n}{2^n}. \tag{1.3}$$

Then Oresme sequence has the form $0, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, \ldots$

For Oresme numbers some identities can be found in [3], we recall some of them

$$\begin{aligned} O_{n+3} =& \frac{3}{4}O_{n+1} - \frac{1}{4}O_n, \\ O_{n+3} =& \frac{3}{4}O_{n+2} - \frac{1}{16}O_n, \\ \sum_{j=0}^{n} O_j =& 4\left(\frac{1}{2} - O_{n+2}\right). \end{aligned}$$

In some mathematical sources we can find that the Oresme sequence has a biological applications, see [3].

Oresme numbers were generalized by Cook in [1]. We use this concept for our future investigations.

Let $k \geq 2$, $n \geq 0$, be integers. Then k-Oresme numbers $\left\{O_n^{(k)}\right\}$ are defined by

$$O_n^{(k)} = O_{n-1}^{(k)} - \frac{1}{k^2}O_{n-2}^{(k)}, \tag{1.4}$$

for $n \geq 2$ with $O_0^{(k)} = 0$, $O_1^{(k)} = \frac{1}{k}$.

Clearly $W_n(0, \frac{1}{k}; 1, \frac{1}{k^2}) = O_n^{(k)}$ and $O_n^{(2)} = O_n$.

Although the equation (1.4) works for $k \geq 2$ Binet formulas for $O_n^{(k)}$ have to be given separately for $k = 2$ and $k \geq 3$. It follows from roots of the characteristic equation of (1.4). If $k \geq 3$ then Binet formula for k-Oresme numbers has the form

$$O_n^{(k)} = \frac{1}{\sqrt{k^2-4}}\left[\left(\frac{k+\sqrt{k^2-4}}{2k}\right)^n - \left(\frac{k-\sqrt{k^2-4}}{2k}\right)^n\right], \tag{1.5}$$

if $k^2 - 4 > 0$.

In [1] identities provided by Horadam in [3] are extended for some of k-Oresme numbers. We recall some of them for future investigations

$$O_{n+3}^{(k)} = \frac{k^2-1}{k^2}O_{n+1}^{(k)} - \frac{1}{k^2}O_n^{(k)}, \tag{1.6}$$

$$O_{n+3}^{(k)} = \frac{k^2-1}{k^2}O_{n+2}^{(k)} - \frac{1}{k^4}O_n^{(k)}, \tag{1.7}$$

$$\sum_{j=0}^{n} O_j^{(k)} = k^2\left(\frac{1}{k} - O_{n+2}^{(k)}\right) = k - k^2 O_{n+2}^{(k)}. \tag{1.8}$$

2. Oresme Hybrid Numbers

In [4] Özdemir introduced a new non-commutative number system called hybrid numbers. The set of hybrid numbers, denoted by $\mathbb{K}$, is defined by

$$\mathbb{K} = \{\mathbf{z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} \colon a, b, c, d \in \mathbb{R}\},$$

where

$$\mathbf{i}^2 = -1, \quad \varepsilon^2 = 0, \quad \mathbf{h}^2 = 1, \quad \mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}. \tag{2.1}$$

Two hybrid numbers

$$\mathbf{z_1} = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}, \quad \mathbf{z_2} = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h},$$

are equal if

$$a_1 = a_2, \quad b_1 = b_2, \quad c_1 = c_2, \quad d_1 = d_2.$$

The sum of two hybrid numbers is defined by

$$\mathbf{z_1} + \mathbf{z_2} = a_1 + a_2 + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\varepsilon + (d_1 + d_2)\mathbf{h}.$$

Addition operation is commutative and associative, zero is the null element. With respect to the addition operation, the inverse element of $\mathbf{z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is $-\mathbf{z} = -a - b\mathbf{i} - c\varepsilon - d\mathbf{h}$. Hence, $(\mathbb{K}, +)$ is Abelian group.

Using (2.1), we get the multiplication table (see Table 1).

Table 1.

$\cdot$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	-1	$1 - \mathbf{h}$	$\varepsilon + \mathbf{i}$
ε	$1 + \mathbf{h}$	0	$-\varepsilon$
$\mathbf{h}$	$-(\varepsilon + \mathbf{i})$	ε	1

The conjugate of a hybrid number $\mathbf{z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, denoted by $\overline{\mathbf{z}}$, is defined as $\overline{\mathbf{z}} = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}$. The real number

$$\mathcal{C}(\mathbf{z}) = \mathbf{z}\overline{\mathbf{z}} = \overline{\mathbf{z}}\mathbf{z} = a^2 + (b - c)^2 - c^2 - d^2$$

is called the character of the hybrid number $\mathbf{z}$.

Some interesting results for Horadam hybrid numbers, i.e., numbers defined in the following way

$$H_n = W_n + W_{n+1}\mathbf{i} + W_{n+2}\varepsilon + W_{n+3}\mathbf{h}, \tag{2.2}$$

were obtained in [5]. Tan and Ait-Amrane in *On a new generalization of Fibonacci hybrid numbers* (`https://arxiv.org/abs/2006.09727`) introduced the bi-periodic Horadam hybrid numbers which generalize the classical Horadam hybrid numbers. In this paper we define and study Oresme hybrid numbers and k-Oresme hybrid numbers.

Let $n \geq 0$ be an integer. Oresme hybrid sequence $\{OH_n\}$ we define by the following recurrence

$$OH_n = O_n + O_{n+1}\mathbf{i} + O_{n+2}\varepsilon + O_{n+3}\mathbf{h}, \tag{2.3}$$

where O_n denotes the nth Oresme number.

Using (2.3) we get

$$\begin{aligned} OH_0 =& \frac{1}{2}\mathbf{i} + \frac{2}{4}\varepsilon + \frac{3}{8}\mathbf{h}, \\ OH_1 =& \frac{1}{2} + \frac{2}{4}\mathbf{i} + \frac{3}{8}\varepsilon + \frac{4}{16}\mathbf{h}, \\ OH_2 =& \frac{2}{4} + \frac{3}{8}\mathbf{i} + \frac{4}{16}\varepsilon + \frac{5}{32}\mathbf{h}, \\ OH_3 =& \frac{3}{8} + \frac{4}{16}\mathbf{i} + \frac{5}{32}\varepsilon + \frac{6}{64}\mathbf{h}. \end{aligned} \tag{2.4}$$

In [5] it was determined the character of the nth Horadam hybrid number H_n.

Theorem 2.1 ([5])**.** *Let $n \geq 0$ be an integer. Then*

$$\begin{aligned} \mathcal{C}(H_n) =& W_n^2(1 - p^2q^2) + W_nW_{n+1}(2q + 2p^3q - 2pq^2) + \\ &+ W_{n+1}^2(1 - 2p - p^4 + 2p^2q - q^2). \end{aligned} \tag{2.5}$$

By (2.5) we get the following.

Corollary 2.1. *Let $n \geq 0$ be an integer. Then*

$$\mathcal{C}(OH_n) = \frac{15}{16}O_n^2 + \frac{14}{16}O_nO_{n+1} - \frac{25}{16}O_{n+1}^2 \tag{2.6}$$

and using (1.3) *we obtain*

$$\mathcal{C}(OH_n) = \frac{63n^2 - 22n - 25}{64 \cdot 2^{2n}}.$$

Theorem 2.2 (Binet formula for Oresme hybrid numbers)**.** *Let $n \geq 0$ be an integer. Then*

$$OH_n = \frac{n}{2^n} + \frac{n+1}{2^{n+1}}\mathbf{i} + \frac{n+2}{2^{n+2}}\varepsilon + \frac{n+3}{2^{n+3}}\mathbf{h}. \tag{2.7}$$

Proof. Using (2.3) and (1.3) we obtain the desired formula. □

Theorem 2.3 (Catalan identity for Oresme hybrid numbers)**.** *Let $n \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$OH_{n+r} \cdot OH_{n-r} - (OH_n)^2 = \frac{-65r^2}{64 \cdot 4^n} + \frac{-4r^2 + r}{4 \cdot 4^n}\mathbf{i} + \frac{-8r^2 + 3r}{16 \cdot 4^n}\varepsilon + \frac{-r^2 - r}{4 \cdot 4^n}\mathbf{h}.$$

Proof. For integers $n \geq 0$, $r \geq 0$ and $n \geq r$, using Binet formula for Oresme hybrid numbers, we have

$$OH_{n+r} = \frac{n+r}{2^{n+r}} + \frac{n+r+1}{2^{n+r+1}}\mathbf{i} + \frac{n+r+2}{2^{n+r+2}}\varepsilon + \frac{n+r+3}{2^{n+r+3}}\mathbf{h}$$

and

$$OH_{n-r} = \frac{n-r}{2^{n-r}} + \frac{n-r+1}{2^{n-r+1}}\mathbf{i} + \frac{n-r+2}{2^{n-r+2}}\varepsilon + \frac{n-r+3}{2^{n-r+3}}\mathbf{h}.$$

So, after calculations the result follows. □

Note that for $r = 1$ we obtain Cassini type identity for Oresme hybrid numbers.

Corollary 2.2 (Cassini identity for Oresme hybrid numbers)**.** *Let $n \geq 1$ be an integer. Then*

$$OH_{n+1} \cdot OH_{n-1} - (OH_n)^2 = \frac{-65}{64 \cdot 4^n} + \frac{-3}{4 \cdot 4^n}\mathbf{i} + \frac{-5}{16 \cdot 4^n}\varepsilon + \frac{-2}{4 \cdot 4^n}\mathbf{h}.$$

Let $k \geq 2$, $n \geq 0$, be integers. Then k-Oresme hybrid sequence $\left\{OH_n^{(k)}\right\}$ we define by the following recurrence

$$OH_n^{(k)} = O_n^{(k)} + O_{n+1}^{(k)}\mathbf{i} + O_{n+2}^{(k)}\varepsilon + O_{n+3}^{(k)}\mathbf{h}, \tag{2.8}$$

where $O_n^{(k)}$ denotes the nth k-Oresme number.

Theorem 2.4. *Let $n \geq 0$, $k \geq 2$, be integers. Then*

(i) $OH_n^{(k)} + \overline{OH_n^{(k)}} = 2O_n^{(k)}$;

(ii) $\mathcal{C}(OH_n^{(k)}) = 2O_n^{(k)} \cdot OH_n^{(k)} - \left(OH_n^{(k)}\right)^2$;

(iii) $OH_{n+3}^{(k)} = \frac{k^2-1}{k^2}OH_{n+1}^{(k)} - \frac{1}{k^2}OH_n^{(k)}$;

(iv) $OH_{n+3}^{(k)} = \frac{k^2-1}{k^2}OH_{n+2}^{(k)} - \frac{1}{k^4}OH_n^{(k)}$;

(v) $\sum\limits_{j=0}^{n} OH_j^{(k)} = k^2\left(OH_1^{(k)} - OH_{n+2}^{(k)}\right)$.

Proof. (i) By the definition of the conjugate of a hybrid number we obtain

$$\begin{aligned} OH_n^{(k)} + \overline{OH_n^{(k)}} =& O_n^{(k)} + O_{n+1}^{(k)}\mathbf{i} + O_{n+2}^{(k)}\varepsilon + O_{n+3}^{(k)}\mathbf{h} + O_n^{(k)} - O_{n+1}^{(k)}\mathbf{i} - O_{n+2}^{(k)}\varepsilon - O_{n+3}^{(k)}\mathbf{h} \\ =& 2O_n^{(k)}. \end{aligned}$$

(ii) By formula (2.3) and Table 1 we have

$$\begin{aligned} \left(OH_n^{(k)}\right)^2 =& \left(O_n^{(k)}\right)^2 - \left(O_{n+1}^{(k)}\right)^2 + \left(O_{n+3}^{(k)}\right)^2 \\ &+ 2O_n^{(k)}O_{n+1}^{(k)}\mathbf{i} + 2O_n^{(k)}O_{n+2}^{(k)}\varepsilon + 2O_n^{(k)}O_{n+3}^{(k)}\mathbf{h} \\ &+ O_{n+1}^{(k)}O_{n+2}^{(k)}(\mathbf{i}\varepsilon + \varepsilon\mathbf{i}) + O_{n+1}^{(k)}O_{n+3}^{(k)}(\mathbf{ih} + \mathbf{hi}) + O_{n+2}^{(k)}O_{n+3}^{(k)}(\varepsilon\mathbf{h} + \mathbf{h}\varepsilon) \\ =& \left(O_n^{(k)}\right)^2 - \left(O_{n+1}^{(k)}\right)^2 + \left(O_{n+3}^{(k)}\right)^2 + 2O_{n+1}^{(k)}O_{n+2}^{(k)} \\ &+ 2\left(O_n^{(k)}O_{n+1}^{(k)}\mathbf{i} + O_n^{(k)}O_{n+2}^{(k)}\varepsilon + O_n^{(k)}O_{n+3}^{(k)}\mathbf{h}\right) \\ =& 2O_n^{(k)} \cdot OH_n^{(k)} - \left(O_n^{(k)}\right)^2 - \left(O_{n+1}^{(k)}\right)^2 + 2O_{n+1}^{(k)}O_{n+2}^{(k)} + \left(O_{n+3}^{(k)}\right)^2 \\ =& 2O_n^{(k)} \cdot OH_n^{(k)} - \mathcal{C}(OH_n^{(k)}). \end{aligned}$$

Hence, we get the result.

(iii) Using (1.6) we have

$$\begin{aligned}
OH_{n+3}^{(k)} =& O_{n+3}^{(k)} + O_{n+4}^{(k)}\mathbf{i} + O_{n+5}^{(k)}\varepsilon + O_{n+6}^{(k)}\mathbf{h}\\
=& \left(\frac{k^2-1}{k^2}O_{n+1}^{(k)} - \frac{1}{k^2}O_{n}^{(k)}\right) + \left(\frac{k^2-1}{k^2}O_{n+2}^{(k)} - \frac{1}{k^2}O_{n+1}^{(k)}\right)\mathbf{i}\\
&+ \left(\frac{k^2-1}{k^2}O_{n+3}^{(k)} - \frac{1}{k^2}O_{n+2}^{(k)}\right)\varepsilon + \left(\frac{k^2-1}{k^2}O_{n+4}^{(k)} - \frac{1}{k^2}O_{n+3}^{(k)}\right)\mathbf{h}\\
=& \frac{k^2-1}{k^2}OH_{n+1}^{(k)} - \frac{1}{k^2}OH_{n}^{(k)}.
\end{aligned}$$

(iv) Using (1.7) and proceeding analogously as in (iii) we obtain (iv).
(v) We have

$$\begin{aligned}
\sum_{j=0}^{n} OH_j^{(k)} =& OH_0^{(k)} + OH_1^{(k)} + \cdots + OH_n^{(k)}\\
=& O_0^{(k)} + O_1^{(k)}\mathbf{i} + O_2^{(k)}\varepsilon + O_3^{(k)}\mathbf{h} + O_1^{(k)} + O_2^{(k)}\mathbf{i} + O_3^{(k)}\varepsilon + O_4^{(k)}\mathbf{h}\\
&+ \cdots + O_n^{(k)} + O_{n+1}^{(k)}\mathbf{i} + O_{n+2}^{(k)}\varepsilon + O_{n+3}^{(k)}\mathbf{h}\\
=& O_0^{(k)} + O_1^{(k)} + \cdots + O_n^{(k)}\\
&+ \left(O_1^{(k)} + O_2^{(k)} + \cdots + O_{n+1}^{(k)} + O_0^{(k)} - O_0^{(k)}\right)\mathbf{i}\\
&+ \left(O_2^{(k)} + O_3^{(k)} + \cdots + O_{n+2}^{(k)} + O_0^{(k)} + O_1^{(k)} - O_0^{(k)} - O_1^{(k)}\right)\varepsilon\\
&+ \left(O_3^{(k)} + O_4^{(k)} + \cdots + O_{n+3}^{(k)} + O_0^{(k)} + O_1^{(k)} + O_2^{(k)}\right.\\
&\left.- O_0^{(k)} - O_1^{(k)} - O_2^{(k)}\right)\mathbf{h}.
\end{aligned}$$

Using (1.8) we obtain

$$\begin{aligned}
\sum_{j=0}^{n} OH_j^{(k)} &= k - k^2 O_{n+2}^{(k)} + \left(k - k^2 O_{n+3}^{(k)} - 0\right)\mathbf{i}\\
&\quad + \left(k - k^2 O_{n+4}^{(k)} - 0 - \frac{1}{k}\right)\varepsilon + \left(k - k^2 O_{n+5}^{(k)} - 0 - \frac{1}{k} - \frac{1}{k}\right)\mathbf{h}\\
&= \left(k + k\mathbf{i} + \frac{k^2-1}{k}\varepsilon + \frac{k^2-2}{k}\mathbf{h}\right) - k^2 OH_{n+2}^{(k)}\\
&= k^2\left(\frac{1}{k} + \frac{1}{k}\mathbf{i} + \frac{k^2-1}{k^3}\varepsilon + \frac{k^2-2}{k^3}\mathbf{h}\right) - k^2 OH_{n+2}^{(k)}\\
&= k^2\left(OH_1^{(k)} - OH_{n+2}^{(k)}\right).
\end{aligned}$$

□

3. Oresme Hybrationals

Cerda-Morales in *Oresme polynomials and their derivatives* (`https://arxiv.org/abs/1904.01165v1`) extended the sequence of k-Oresme numbers to the sequence of rational functions $\{O_n(x)\}$ by replacing k with a real variable x. These rational functions were named as Oresme polynomials.

Let x be a nonzero real variable. The sequence of Oresme polynomials is recursively defined as follows:

$$O_n(x) = \begin{cases} 0, & \text{if } n = 0, \\ \frac{1}{x}, & \text{if } n = 1 \\ O_{n-1}(x) - \frac{1}{x^2} O_{n-2}(x), & \text{if } n \geq 2. \end{cases} \tag{3.1}$$

Solving the characteristic equation of Oresme polynomials recurrence relation

$$r^2 - r + \frac{1}{x^2} = 0,$$

we obtain Binet formula

$$O_n(x) = \frac{1}{\sqrt{x^2-4}} \left[\left(\frac{x+\sqrt{x^2-4}}{2x} \right)^n - \left(\frac{x-\sqrt{x^2-4}}{2x} \right)^n \right],$$

for $x^2 - 4 > 0$, $x \neq 0$, and

$$O_n(x) = \frac{\mathbf{i}}{\sqrt{4-x^2}} \left[\left(\frac{x-\sqrt{4-x^2}\,\mathbf{i}}{2x} \right)^n - \left(\frac{x+\sqrt{4-x^2}\,\mathbf{i}}{2x} \right)^n \right],$$

for $x^2 - 4 < 0$. Moreover, $O_n(2) = O_n$ and $O_n(-2) = -O_n$.

For $n \geq 0$ and nonzero real variable x Oresme hybrationals are defined by

$$OH_n(x) = O_n(x) + O_{n+1}(x)\mathbf{i} + O_{n+2}(x)\varepsilon + O_{n+3}(x)\mathbf{h},$$

where $O_n(x)$ is the nth Oresme polynomial and $\mathbf{i}$, ε, $\mathbf{h}$ are hybrid units.

For $x = k$ we obtain k-Oresme hybrid numbers.

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