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ON NORMALIZED SIGNLESS LAPLACIAN RESOLVENT ENERGY

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Abstract. Let G be a simple connected graph with n vertices. Denote by $\mathcal{L}^+(G) = D(G)^{-1/2} Q(G) D(G)^{-1/2}$ the normalized signless Laplacian matrix of graph G, where $Q(G)$ and $D(G)$ are the signless Laplacian and diagonal degree matrices of G, respectively. The eigenvalues of matrix $\mathcal{L}^+(G)$, $2 = \gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+ \geq 0$, are normalized signless Laplacian eigenvalues of G. In this paper, we introduce the normalized signless Laplacian resolvent energy of G as $ERNS(G) = \sum_{i=1}^{n} \frac{1}{3-\gamma_i^+}$. We also obtain some lower and upper bounds for $ERNS(G)$ as well as its relationships with other energies and signless Kemeny's constant.

1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple connected graph with n vertices and m edges, where $|V| = n$ and $|E| = m$. Denote by d_i the degree of the vertex v_i of G, $i = 1, 2, \ldots, n$. If v_i and v_j are two adjacent vertices of G, then we denote this by $i \sim j$.

Let $A(G)$ be the adjacency matrix of G. Eigenvalues of $A(G)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, are said to be the (ordinary) eigenvalues of G [11]. Then the energy of the graph G is defined as [15]

$$E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

Various properties and bounds on $E(G)$ may be found in the monographs [19, 22] and references cited therein.

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In line with concept of graph energy, the resolvent energy of G is put forward in [18] as

$$ER(G) = \sum_{i=1}^{n} \frac{1}{n - \lambda_i}.$$

For the basic properties and bounds of $ER(G)$, the reader may refer to [1,13,34,35].

Let $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ denote the diagonal degree matrix of G. The Laplacian and signless Laplacian matrices of G are, respectively, defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$. Denote by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ and $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$ the eigenvalues of $L(G)$ and $Q(G)$, respectively [26]. Recently, Laplacian resolvent and signless Laplacian resolvent energies of G are, respectively, introduced as [7]

$$RL(G) = \sum_{i=1}^{n} \frac{1}{n + 1 - \mu_i}$$

and

$$RQ(G) = \sum_{i=1}^{n} \frac{1}{2n - 1 - q_i}.$$

Since graph G is connected, the matrix $D(G)^{-1/2}$ is well defined. Then, the normalized Laplacian matrix of G is defined by [10]

$$\mathcal{L}(G) = D(G)^{-1/2} L(G) D(G)^{-1/2} = I_n - R(G),$$

where I_n is the $n \times n$ unity matrix and $R(G)$ is the Randić matrix [2]. The following properties for the normalized Laplacian eigenvalues, $\gamma_1^- \geq \gamma_2^- \geq \cdots \geq \gamma_{n-1}^- > \gamma_n^- = 0$, are valid [36]

$$\sum_{i=1}^{n-1} \gamma_i^- = n \quad \text{and} \quad \sum_{i=1}^{n-1} (\gamma_i^-)^2 = n + 2R_{-1}(G), \tag{1.1}$$

where

$$R_{-1}(G) = \sum_{i \sim j} \frac{1}{d_i d_j},$$

is a kind of topological index of G called as general Randić index [8,31].

The matrix $\mathcal{L}^+(G) = D(G)^{-1/2} Q(G) D(G)^{-1/2} = I_n + R(G)$ is defined to be the normalized signless Laplacian matrix of G [10]. Some well known identities concerning the normalized signless Laplacian eigenvalues, $\gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+ \geq 0$, are [9]

$$\sum_{i=1}^{n} \gamma_i^+ = n \quad \text{and} \quad \sum_{i=1}^{n} (\gamma_i^+)^2 = n + 2R_{-1}(G). \tag{1.2}$$

For $i = 1, 2, \ldots, n$, the following relations (see [14,24]) exist

$$\gamma_i^- = 1 - \rho_{n-i+1} \quad \text{and} \quad \gamma_i^+ = 1 + \rho_i. \tag{1.3}$$

Here, $1 = \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ are the Randić eigenvalues of G [2,24].

Motivated by the definitions of graph resolvent energies ER, RL and RQ and considering the fact that $\gamma_i^- \leq 2$, $1 \leq i \leq n$, Sun and Das [33] defined the normalized Laplacian resolvent energy of G as

$$ERN\,(G) = \sum_{i=1}^{n} \frac{1}{3 - \gamma_i^-}.$$

Since the property $\gamma_i^+ \leq 2$, $1 \leq i \leq n$, is also satisfied by the normalized signless Laplacian eigenvalues, we now introduce the normalized signless Laplacian resolvent energy of G as follows

$$ERNS\,(G) = \sum_{i=1}^{n} \frac{1}{3 - \gamma_i^+}.$$

Notice that in the case of bipartite graph the normalized Laplacian and normalized signless Laplacian eigenvalues coincide [3]. From hence, for bipartite graphs, $ERN\,(G)$ is equal to $ERNS\,(G)$.

Before we proceed, let us recall another graph invariant closely related to normalized Laplacian eigenvalues and so called Kemeny's constant. It is defined as [6]

$$K\,(G) = \sum_{i=1}^{n-1} \frac{1}{\gamma_i^-}.$$

For more information on $K\,(G)$, see [21, 27].

Since for connected non-bipartite graphs $\gamma_i^+ > 0$ for $i = 1, 2, \dots, n$, [4], very recently, in an analogous manner with Kemeny's constant, signless Kemeny's constant of connected non-bipartite graphs is considered as [28]

$$K^+\,(G) = \sum_{i=1}^{n} \frac{1}{\gamma_i^+}.$$

In [28], it is also emphasized that $K\,(G)$ coincides with $K^+\,(G)$ in the case of bipartite graphs.

In this paper, we obtain some lower and upper bounds for $ERNS\,(G)$ as well as its relationships with other energies and $K^+\,(G)$.

2. Lemmas

We now recall some known results on graph spectra and analytical inequalities that will be used in our main results.

Lemma 2.1 ([14]). *For any connected graph G, the largest normalized signless Laplacian eigenvalue is $\gamma_1^+ = 2$.*

Lemma 2.2 ([14]). *Let G be a graph of order $n \geq 2$ with no isolated vertices. Then*

$$\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+ = \frac{n-2}{n-1}$$

if and only if $G \cong K_n$.

Lemma 2.3 ([4]). *If G is a connected non–bipartite graph of order n, then $\gamma_i^+ > 0$ for $i = 1, 2, \ldots, n$.*

Lemma 2.4 ([3]). *If G is a bipartite graph, then the eigenvalues of $\mathcal{L}$ and $\mathcal{L}^+$ coincide.*

Lemma 2.5 ([23]). *Let G be a connected graph of order n. Then $\gamma_2^- \geq 1$, the equality holds if and only if G is a complete bipartite graph.*

Lemma 2.6 ([12]). *Let G be a connected graph with $n > 2$ vertices. Then $\gamma_2^- = \gamma_3^- = \cdots = \gamma_{n-1}^-$ if and only if $G \cong K_n$ or $G \cong K_{p,q}$.*

Lemma 2.7 ([10]). *Let G be a bipartite graph with n vertices Then for $i = 1, 2, \ldots, n$, $\gamma_i^- + \gamma_{n-i+1}^- = 2$.*

Lemma 2.8 ([24]). *For any connected graph G, the largest Randić eigenvalue is $\rho_1 = 1$.*

Lemma 2.9 ([2]). *Let G be a graph with n vertices and Randić matrix $R(G)$. Then*

$$tr\left(R(G)^2\right) = 2R_{-1}$$

and

$$tr\left(R(G)^3\right) = 2\sum_{i\sim j} \frac{1}{d_i d_j}\left(\sum_{k\sim i,\ k\sim j} \frac{1}{d_k}\right).$$

Lemma 2.10 ([30]). *Let $x = (x_i)$ and $a = (a_i)$ be two sequences of positive real numbers, $i = 1, 2, \ldots, n$. Then for any $r \geq 0$*

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum\limits_{i=1}^{n} a_i\right)^{r}}. \tag{2.1}$$

Equality holds if and only if $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

Lemma 2.11 ([20]). *Let $a = (a_i)$ and $p = (p_i)$ be two sequences of positive real numbers such that $\sum\limits_{i=1}^{n} p_i = 1$ and $0 < r \leq a_i \leq R < +\infty$, $i = 1, 2, \ldots, n$, $r, R \in \mathbb{R}$. Then*

$$\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} \frac{p_i}{a_i} \leq \frac{1}{4}\left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}}\right)^2. \tag{2.2}$$

Equality holds if and only if $R = a_1 = a_2 = \cdots = a_n = r$.

3. Lower and Upper Bounds on $ERNS(G)$

In this section, we establish some lower and upper bounds for $ERNS(G)$.

Theorem 3.1. *Let G be a connected non-bipartite graph with $n \geq 3$ vertices. Then, for any real α, such that $\gamma_2^+ \geq \alpha \geq \frac{n-2}{n-1}$,*

$$ERNS\left(G\right) \geq 1 + \frac{1}{3-\alpha} + \frac{\left(n-2\right)^2}{2n-4+\alpha}. \tag{3.1}$$

If $\alpha = \frac{n-2}{n-1}$, equality holds if and only if $G \cong K_n$.

Proof. By arithmetic-harmonic mean inequality [29], we have

$$\sum_{i=3}^{n} a_i \sum_{i=3}^{n} \frac{1}{a_i} \geq \left(n-2\right)^2,$$

where $a_i > 0$, $i = 3, 4, \ldots, n$, are arbitrary real numbers. For $a_i = 3 - \gamma_i^+$, $i = 3, 4, \ldots, n$, the above inequality transforms into

$$\sum_{i=3}^{n} \left(3-\gamma_i^+\right) \sum_{i=3}^{n} \frac{1}{3-\gamma_i^+} \geq \left(n-2\right)^2,$$

that is

$$\sum_{i=1}^{n} \frac{1}{3-\gamma_i^+} \geq \frac{1}{3-\gamma_1^+} + \frac{1}{3-\gamma_2^+} + \frac{\left(n-2\right)^2}{\sum_{i=3}^{n}\left(3-\gamma_i^+\right)}.$$

Then, it follows from the above, (1.2) and Lemma 2.1 that

$$ERNS\left(G\right) \geq 1 + \frac{1}{3-\gamma_2^+} + \frac{\left(n-2\right)^2}{2n-4+\gamma_2^+}. \tag{3.2}$$

Now, consider the function defined as follows

$$f\left(x\right) = \frac{1}{3-x} + \frac{\left(n-2\right)^2}{2n-4+x}.$$

It can be easily seen that f is increasing for $x \geq \frac{n-2}{n-1}$. Then for any real α, $\gamma_2^+ \geq \alpha \geq \frac{n-2}{n-1}$,

$$f\left(\gamma_2^+\right) \geq f\left(\alpha\right) = \frac{1}{3-\alpha} + \frac{\left(n-2\right)^2}{2n-4+\alpha}.$$

Based on this inequailty and (3.2), we obtain the lower bound (3.1). Equality in (3.1) holds if and only if

$$\gamma_2^+ = \alpha \quad \text{and} \quad \gamma_3^+ = \cdots = \gamma_n^+.$$

If $\alpha = \frac{n-2}{n-1}$, then from the above and Lemma 2.2, one can easily conclude that the equality in (3.1) holds if and only if $G \cong K_n$. □

Corollary 3.1. *Let G be a connected non-bipartite graph with $n \geq 3$ vertices.Then*

$$ERNS\left(G\right) \geq 1 + \frac{\left(n-1\right)^2}{2n-1}.$$

Equality holds if and only if $G \cong K_n$.

Considering the techniques in Theorem 3.1 with Lemmas 2.1, 2.4 and 2.6, we obtain the following result for bipartite graphs.

Theorem 3.2. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then, for any real α, such that $\gamma_2^+ = \gamma_2^- \geq \alpha \geq 1$*

$$ERNS\,(G) = ERN\,(G) \geq \frac{4}{3} + \frac{1}{3-\alpha} + \frac{(n-3)^2}{2n-7+\alpha}.$$

If $\alpha = 1$, equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.

In [5], it was obtained that

$$\gamma_2^+ = \gamma_2^- \geq 1 + \sqrt{\frac{2\,(R_{-1}\,(G)-1)}{n-2}}. \tag{3.3}$$

From Theorem 3.2 and (3.3), we directly have the following.

Corollary 3.2. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS\,(G) = ERN\,(G) \geq \frac{4}{3} + \frac{1}{2-\sqrt{\frac{2(R_{-1}(G)-1)}{n-2}}} + \frac{(n-3)^2}{2n-6+\sqrt{\frac{2(R_{-1}(G)-1)}{n-2}}}. \tag{3.4}$$

From Theorem 3.2 and Lemma 2.5, we have the following result. It was proven in Theorem 3.8 of [33].

Corollary 3.3 ([33]). *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS\,(G) = ERN\,(G) \geq \frac{n}{2} + \frac{1}{3}. \tag{3.5}$$

Equality holds if and only if $G \cong K_{p,q}$.

Remark 3.1. Note that the lower bound (3.4) is stronger than the lower bound (3.5).

Theorem 3.3. *Let G be a connected graph with $n \geq 3$ vertices. Then*

$$ERNS(G) \geq \frac{1}{3}\left(n+2+\frac{(n-2)^2}{2(n-1-R_{-1}(G))}\right). \tag{3.6}$$

Equality holds if and only if $G \cong K_n$ or $G \cong K_{p,q}$, $p+q=n$.

Proof. Suppose G is a connected non–bipartite graph with $n \geq 3$ vertices. Then, by Lemmas 2.1 and 2.3, $\gamma_1^+ = 2$ and $\gamma_i^+ > 0$, $i = 2,3,\dots,n$. For $r=1$ the inequality (2.1) transforms into

$$\sum_{i=2}^{n} \frac{x_i^2}{a_i} \geq \frac{\left(\sum\limits_{i=2}^{n} x_i\right)^2}{\sum\limits_{i=2}^{n} a_i}. \tag{3.7}$$

Setting $x_i = \gamma_i^+$, $a_i = \gamma_i^+(3-\gamma_i^+)$, $i = 2,3,\dots,n$, in (3.7) and using (1.2) and Lemma 2.1, we have

$$\sum_{i=2}^{n} \frac{\left(\gamma_i^+\right)^2}{\gamma_i^+(3-\gamma_i^+)} \geq \frac{\left(\sum\limits_{i=2}^{n} \gamma_i^+\right)^2}{\sum\limits_{i=2}^{n} \gamma_i^+(3-\gamma_i^+)} = \frac{(n-2)^2}{2(n-1-R_{-1}(G))}. \tag{3.8}$$

On the other hand, from the above and Lemma 2.1, we also have

$$\begin{aligned}\sum_{i=2}^{n} \frac{\left(\gamma_i^+\right)^2}{\gamma_i^+(3-\gamma_i^+)} &= \sum_{i=2}^{n} \frac{\gamma_i^+}{3-\gamma_i^+} = \sum_{i=2}^{n} \frac{\gamma_i^+ - 3 + 3}{3-\gamma_i^+} \\ &= -(n-1) + 3\left(ERNS(G) - 1\right) \\ &= 3ERNS(G) - n - 2. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), the inequality (3.6) is obtained.

Equality in (3.8) holds if and only if

$$\frac{1}{3-\gamma_2^+} = \frac{1}{3-\gamma_3^+} = \cdots = \frac{1}{3-\gamma_n^+},$$

that is $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+$. By Lemma 2.2, when G is non–bipartite graph, equality in (3.6) holds if and only if $G \cong K_n$.

Now, suppose G is a connected bipartite graph with $n \geq 3$ vertices. Then, by Lemmas 2.1 and 2.4, $\gamma_1^+ = 2$ and $\gamma_2^+ \geq \gamma_3^+ \geq \cdots \geq \gamma_{n-1}^+ > \gamma_n^+ = 0$. The inequality (2.1) can be considered as

$$\sum_{i=2}^{n-1} \frac{x_i^2}{a_i} \geq \frac{\left(\sum\limits_{i=2}^{n-1} x_i\right)^2}{\sum\limits_{i=2}^{n-1} a_i}.$$

Taking $x_i = \gamma_i^+$, $a_i = \gamma_i^+(3-\gamma_i^+)$, $i = 2,3,\dots,n-1$, in the above inequality and considering (1.2), we get

$$\sum_{i=2}^{n-1} \frac{\left(\gamma_i^+\right)^2}{\gamma_i^+(3-\gamma_i^+)} \geq \frac{\left(\sum\limits_{i=2}^{n-1} \gamma_i^+\right)^2}{\sum\limits_{i=2}^{n-1} \gamma_i^+(3-\gamma_i^+)} = \frac{(n-2)^2}{2(n-1-R_{-1}(G))}. \tag{3.10}$$

Observe that

$$\begin{aligned}\sum_{i=2}^{n-1} \frac{\left(\gamma_i^+\right)^2}{\gamma_i^+(3-\gamma_i^+)} &= \sum_{i=2}^{n-1} \frac{\gamma_i^+}{3-\gamma_i^+} = \sum_{i=2}^{n-1} \frac{\gamma_i^+ - 3 + 3}{3-\gamma_i^+} = \\ &= -(n-2) + 3\left(ERNS(G) - 1 - \frac{1}{3}\right) = \\ &= 3ERNS(G) - n - 2\,. \end{aligned}$$

From the above and inequality (3.10) we arrive at (3.6).

Equality in (3.10) holds if and only if

$$\frac{1}{3-\gamma_2^+} = \frac{1}{3-\gamma_3^+} = \cdots = \frac{1}{3-\gamma_{n-1}^+},$$

that is when $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_{n-1}^+$. Since G is a bipartite graph, by Lemmas 2.4 and 2.6, equality in (3.6) holds if and only if $G \cong K_{p,q}$, $p+q=n$. □

Corollary 3.4. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) = ERN(G) \geq \frac{1}{3}\left(n+2+\frac{(n-2)^2}{2(n-1-R_{-1}(G))}\right).$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.

Theorem 3.4. *Let G be a connected non-bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) \leq 1 + \frac{4n-5-(n-1)(\gamma_2^+ + \gamma_n^+)}{(3-\gamma_2^+)(3-\gamma_n^+)}. \tag{3.11}$$

Equality holds if and only if $\gamma_i^+ \in \{\gamma_2^+, \gamma_n^+\}$, for $i = 2,3,\ldots,n$.

Proof. For every $i = 2,3,\ldots,n$, the following inequalities are valid

$$(3-\gamma_2^+ - 3 + \gamma_i^+)(3-\gamma_n^+ - 3 + \gamma_i^+) \leq 0,$$

$$(3-\gamma_i^+)^2 + (3-\gamma_2^+)(3-\gamma_n^+) \leq (6-\gamma_2^+-\gamma_n^+)(3-\gamma_i^+),$$

$$(3-\gamma_i^+) + \frac{(3-\gamma_2^+)(3-\gamma_n^+)}{3-\gamma_i^+} \leq 6-\gamma_2^+-\gamma_n^+. \tag{3.12}$$

After summation of (3.12) over i, $i = 2,3,\ldots,n$, we obtain

$$\sum_{i=2}^{n}(3-\gamma_i^+) + (3-\gamma_2^+)(3-\gamma_n^+)\sum_{i=2}^{n}\frac{1}{3-\gamma_i^+} \leq (6-\gamma_2^+-\gamma_n^+)\sum_{i=2}^{n}1,$$

that is

$$2n-1+(3-\gamma_2^+)(3-\gamma_n^+)(ERNS(G)-1) \leq (6-\gamma_2^+-\gamma_n^+)(n-1), \tag{3.13}$$

from which (3.11) is obtained.

Equality in (3.12) holds if and only if $\gamma_i^+ \in \{\gamma_2^+, \gamma_n^+\}$, for every $i = 2,3,\ldots,n$, which implies that equality in (3.11) holds if and only if $\gamma_i^+ \in \{\gamma_2^+, \gamma_n^+\}$, for every $i = 2,3,\ldots,n$. □

Corollary 3.5. *Let G be a connected non-bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) \leq 1 + \frac{\Big((n-1)(6-\gamma_2^+-\gamma_n^+)\Big)^2}{4(2n-1)(3-\gamma_2^+)(3-\gamma_n^+)}. \tag{3.14}$$

Equality holds if and only if $G \cong K_n$.

Proof. After applying the arithmetic-geometric mean inequality, AM-GM, on (3.13) we obtain

$$2\sqrt{(2n-1)(3-\gamma_2^+)(3-\gamma_n^+)(ERNS(G)-1)} \leq (6-\gamma_2^+-\gamma_n^+)(n-1)\,,$$

from which (3.14) is obtained. □

The proof of the next theorem is fully analogous to that of Theorem 3.4, thus omitted.

Theorem 3.5. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) = ERN(G) \leq \frac{4}{3} + \frac{2(n-2)}{(3-\gamma_2^+)(3-\gamma_{n-1}^+)}\,.$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.

Theorem 3.6. *Let G be a connected non-bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS\,(G) \leq \frac{1}{3}\left(n+2+\frac{\Big((n-2)(6-\gamma_2^+-\gamma_n^+)\Big)^2}{8\,(n-1-R_{-1}\,(G))\left(3-\gamma_2^+\right)(3-\gamma_n^+)}\right). \tag{3.15}$$

Equality holds if and only if $G \cong K_n$.

Proof. Setting $p_i = \frac{\gamma_i^+}{n-2}$, $a_i = \frac{1}{3-\gamma_i^+}$, $i = 2,3,\ldots,n$, $R = \frac{1}{3-\gamma_2^+}$, $r = \frac{1}{3-\gamma_n^+}$ in (2.2), we have that

$$\sum_{i=2}^{n}\left(\frac{\gamma_i^+}{n-2}\right)\left(\frac{1}{3-\gamma_i^+}\right)\sum_{i=2}^{n}\left(\frac{\gamma_i^+}{n-2}\right)\left(3-\gamma_i^+\right) \leq \frac{1}{4}\left(\sqrt{\frac{3-\gamma_n^+}{3-\gamma_2^+}}+\sqrt{\frac{3-\gamma_2^+}{3-\gamma_n^+}}\right)^2.$$

Considering this with (1.2) and (3.9) and Lemma 2.1, we obtain that

$$\frac{2\,(n-1-R_{-1}\,(G))}{(n-2)^2}\,(3ERNS\,(G)-n-2) \leq \frac{1}{4}\left(\frac{\left(6-\gamma_2^+-\gamma_n^+\right)^2}{\left(3-\gamma_2^+\right)(3-\gamma_n^+)}\right).$$

From the above result, we arrive at the upper bound (3.15). The equality in (3.15) holds if and only if

$$\frac{1}{3-\gamma_2^+} = \frac{1}{3-\gamma_3^+} = \cdots = \frac{1}{3-\gamma_n^+},$$

that is

$$\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+.$$

Thus, in view of Lemma 2.2, we conclude that the equality in (3.15) holds if and only if $G \cong K_n$. □

Using the techniques in Theorem 3.6 with Lemmas 2.1, 2.4, 2.6, 2.7 and 2.11, we have the following.

Theorem 3.7. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS\,(G) = ERN(G) \leq \frac{1}{3}\left(n+2+\frac{2\,(n-2)^2}{(n-1-R_{-1}\,(G))\left(3-\gamma_2^+\right)\left(3-\gamma_{n-1}^+\right)}\right).$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.

4. Relations Between $ERNS(G)$ and other Energies

One of the chemically/mathematically most important graph spectrum–based invariants in graph theory is the concept of graph energy introduced in [15]. Due to the evident success of graph energy, a number of graph energies and energy-like graph invariants have been put forward in the literature. We first recall some of them.

For a graph G, in full analogy with the graph energy [15], Randić (normalized Laplacian or normalized signless Laplacian) energy is defined as [2, 8, 17]

$$RE\,(G) = \sum_{i=1}^{n} |\rho_i|\,,$$

where $1 = \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ are the Randić eigenvalues of G [2, 24].

In analogous manner with Laplacian energy-like invariant [25], Laplacian incidence energy is introduced as [32]

$$LIE\,(G) = \sum_{i=1}^{n-1} \sqrt{\gamma_i^-}$$

and by analogy with incidence energy [16], the Randić (normalized) incidence energy is put forward in [9, 14] as

$$I_RE\,(G) = \sum_{i=1}^{n} \sqrt{\gamma_i^+}.$$

Here, $\gamma_1^- \geq \gamma_2^- \geq \cdots \geq \gamma_{n-1}^- > \gamma_n^- = 0$ and $2 = \gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+ \geq 0$ are, respectively, the normalized Laplacian and normalized signless Laplacian eigenvalues of G [10, 14]. Note that LIE is equal to I_RE, for bipartite graphs [3].

Now, we are ready to give some relationships between $ERNS\,(G)$ and other energies emphasized in the above.

Theorem 4.1. *Let G be a connected non-bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS\,(G) \geq 1 + \frac{(RE\,(G)-1)^2}{4R_{-1} - 2\sum\limits_{i\sim j}\frac{1}{d_id_j}\left(\sum\limits_{k\sim i,\;k\sim j}\frac{1}{d_k}\right) - 1}. \tag{4.1}$$

Equality is achieved for $G \cong K_n$.

Proof. For $x_i = \left|\gamma_i^+ - 1\right|$ and $a_i = \frac{1}{3-\gamma_i^+}$, $i = 2, 3, \ldots, n$, the inequality (3.7) becomes

$$\sum_{i=2}^{n} \left(\gamma_i^+ - 1\right)^2 \left(3 - \gamma_i^+\right) \geq \frac{\left(\sum\limits_{i=2}^{n} \left|\gamma_i^+ - 1\right|\right)^2}{\sum\limits_{i=2}^{n} \frac{1}{3-\gamma_i^+}}. \tag{4.2}$$

From (1.3) and Lemmas 2.8 and 2.9, we have

$$\begin{aligned}\sum_{i=2}^{n} \left(\gamma_i^+ - 1\right)^2 \left(3 - \gamma_i^+\right) =& \sum_{i=2}^{n} \rho_i^2 \left(2 - \rho_i\right)\\ =& 2\sum_{i=2}^{n} \rho_i^2 - \sum_{i=2}^{n} \rho_i^3\\ =& 2\left(2R_{-1} - 1\right) - \left(2\sum_{i\sim j} \frac{1}{d_i d_j}\left(\sum_{k\sim i,\, k\sim j} \frac{1}{d_k}\right) - 1\right)\\ =& 4R_{-1} - 2\sum_{i\sim j} \frac{1}{d_i d_j}\left(\sum_{k\sim i,\, k\sim j} \frac{1}{d_k}\right) - 1. \qquad (4.3)\end{aligned}$$

Then by (4.2) and (4.3) and Lemma 2.1, we get that

$$4R_{-1} - 2\sum_{i\sim j} \frac{1}{d_i d_j}\left(\sum_{k\sim i,\, k\sim j} \frac{1}{d_k}\right) - 1 \geq \frac{\left(RE\left(G\right) - 1\right)^2}{ERNS\left(G\right) - 1}.$$

From the above, the inequality (4.1) follows. One can easily check that the equality in (4.1) is achieved for $G \cong K_n$. □

Theorem 4.2. *Let G be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) \geq \frac{1}{3}\left(n + 2 + \frac{(I_R E(G) - \sqrt{2})^2}{2n-1}\right). \tag{4.4}$$

Equality holds if and only if $G \cong K_n$.

Proof. Taking $x_i = \sqrt{\gamma_i^+}$, $a_i = 3 - \gamma_i^+$, $i = 2, 3, \ldots, n$, in (3.7)

$$\sum_{i=2}^{n} \frac{\gamma_i^+}{3 - \gamma_i^+} \geq \frac{\left(\sum_{i=2}^{n} \sqrt{\gamma_i^+}\right)^2}{\sum_{i=2}^{n}(3 - \gamma_i^+)}.$$

Considering this with (1.2) and (3.9) and Lemma 2.1

$$3ERNS(G) - n - 2 \geq \frac{\left(I_R E(G) - \sqrt{2}\right)^2}{2n-1}.$$

From the above we obtain (4.4). Equality in (4.4) holds if and only if

$$\frac{\sqrt{\gamma_2^+}}{3 - \gamma_2^+} = \frac{\sqrt{\gamma_3^+}}{3 - \gamma_3^+} = \cdots = \frac{\sqrt{\gamma_n^+}}{3 - \gamma_n^+},$$

that is if and only if

$$\left(\sqrt{\gamma_i^+}-\sqrt{\gamma_j^+}\right)\left(3+\sqrt{\gamma_i^+\gamma_j^+}\right)=0,\quad i\neq j,$$

which implies that equality in (4.4) holds if and only if $G\cong K_n$. □

Theorem 4.3. *Let G be a connected bipartite graph with $n\geq 3$ vertices. Then*

$$ERNS(G)=ERN(G)\geq \frac{1}{3}\left(n+2+\frac{\left(LIE(G)-\sqrt{2}\right)^2}{2(n-2)}\right).$$

Equality holds if $G\cong K_{p,q}$, $p+q=n$.

5. Relationships Between $ERNS(G)$ and $K^+(G)$

In this section, we present some relationships between $ERNS(G)$ and $K^+(G)$.

Theorem 5.1. *Let G be a connected non-bipartite graph wit $n\geq 3$ vertices. Then*

$$ERNS(G)\geq \frac{3}{2}-K^+(G)+\frac{3(n-1)^2}{2(n-1-R_{-1}(G))}. \tag{5.1}$$

Equality holds if and only if $G\cong K_n$.

Proof. The arithmetic-harmonic mean inequality can be considered as [29]

$$\sum_{i=2}^n a_i\sum_{i=2}^n\frac{1}{a_i}\geq (n-1)^2. \tag{5.2}$$

For $a_i=\gamma_i^+(3-\gamma_i^+)$, $i=2,3,\dots,n$, the above inequality transforms into

$$\sum_{i=2}^n\gamma_i^+(3-\gamma_i^+)\sum_{i=2}^n\frac{1}{\gamma_i^+(3-\gamma_i^+)}\geq (n-1)^2. \tag{5.3}$$

From the above, (1.2) and Lemma 2.1,

$$\sum_{i=2}^n\frac{1}{\gamma_i^+(3-\gamma_i^+)}\geq \frac{(n-1)^2}{2(n-1-R_{-1}(G))}. \tag{5.4}$$

On the other hand, by Lemma 2.1, we have that

$$\begin{aligned}\sum_{i=2}^n\frac{1}{\gamma_i^+(3-\gamma_i^+)}&=\frac{1}{3}\left(\sum_{i=2}^n\frac{1}{\gamma_i^+}+\sum_{i=2}^n\frac{1}{3-\gamma_i^+}\right)\\&=\frac{1}{3}\left(K^+(G)-\frac{1}{2}+ERNS(G)-1\right)\\&=\frac{1}{3}\left(K^+(G)-\frac{3}{2}+ERNS(G)\right)\end{aligned}$$

Combining this with (5.4) we arrive at (5.1). Equality in (5.3) holds if and only if $\gamma_2^+(3-\gamma_2^+)=\gamma_3^+(3-\gamma_3^+)=\cdots=\gamma_n^+(3-\gamma_n^+)$. Suppose $i\neq j$. Then, from the identity $\gamma_i^+(3-\gamma_i^+)=\gamma_j^+(3-\gamma_j^+)$, follows that $(\gamma_i^+-\gamma_j^+)(3-\gamma_i^+-\gamma_j^+)=0$. Thus, we conclude that equality in (5.3) holds if and only if $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+$. Having this in mind and Lemma 2.2, we conclude that equality in (5.1) holds if and only if $G \cong K_n$. $\square$

Using the similar idea in Theorem 5.1 with Lemmas 2.1, 2.4 and 2.6, we get the following.

Theorem 5.2. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) = ERN(G) \geq \frac{11}{6} - K(G) + \frac{3(n-2)^2}{2(n-1-R_{-1}(G))}.$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.

Theorem 5.3. *Let G be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) \geq \frac{1}{3}\left(n+2+\frac{2(n-1)^2}{6K^+(G)-2n-1}\right). \tag{5.5}$$

Equality holds if and only if $G \cong K_n$.

Proof. For $a_i = \frac{\gamma_i^+}{3-\gamma_i^+}$, $i = 2, 3, \dots, n$, the inequality (5.2) transforms into

$$\sum_{i=2}^{n} \frac{\gamma_i^+}{3-\gamma_i^+} \sum_{i=2}^{n} \frac{3-\gamma_i^+}{\gamma_i^+} \geq (n-1)^2. \tag{5.6}$$

On the other hand, by Lemma 2.1, we have that

$$\sum_{i=2}^{n} \frac{3-\gamma_i^+}{\gamma_i^+} = 3\left(K^+(G) - \frac{1}{2}\right) - (n-1) = 3K^+(G) - n - \frac{1}{2}.$$

From the above, (3.9) and (5.6), we obtain (5.5). Equality in (5.6) holds if and only if

$$\frac{\gamma_2^+}{3-\gamma_2^+} = \frac{\gamma_3^+}{3-\gamma_3^+} = \cdots = \frac{\gamma_n^+}{3-\gamma_n^+}.$$

Suppose $i \neq j$. Then equality in (5.6) holds if and only if $\frac{\gamma_i^+}{3-\gamma_i^+} = \frac{\gamma_j^+}{3-\gamma_j^+}$, that is if and only if $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+$. By Lemma 2.2, equality in (5.5) holds if and only if $G \cong K_n$. $\square$

Conisdering the same idea in Theorem 5.3 together with Lemma 2.1, 2.4 and 2.6, we have the following.

Theorem 5.4. *Let G be a connected bipartite graph with $n \geq 3$ vertices. Then*

$$ERNS(G) = ERN(G) \geq \frac{1}{3}\left(n+2+\frac{2(n-2)^2}{6K(G)-2n+1}\right).$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.

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