

## LOCAL $K$ -CONVOLUTED $C$ -GROUPS AND ABSTRACT CAUCHY PROBLEMS

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ABSTRACT. We first present a new form of a local  $K$ -convoluted  $C$ -group on a Banach space  $X$ , and then deduce some basic properties of a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$  and some generation theorems of local  $K$ -convoluted  $C$ -groups, which can be applied to obtain some equivalence relations between the generation of a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$  with subgenerator  $A$  and the unique existence of solutions of the abstract Cauchy problem  $\text{ACP}(A, f, x)$ .

### 1. INTRODUCTION

Let  $X$  be a Banach space over the field  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$  with norm  $\|\cdot\|$ , and let  $L(X)$  denote the family of all bounded linear operators from  $X$  into itself. For each  $0 < T_0 \leq \infty$ , we consider the following abstract Cauchy problem:

$$\text{ACP}(A, f, x) \begin{cases} u'(t) = Au(t) + f(t), & \text{for } t \in (-T_0, T_0), \\ u(0) = x, \end{cases}$$

where  $x \in X$ ,  $A$  is a closed linear operator in  $X$ , and  $f \in L_{loc}^1((-T_0, T_0), X)$  (the family of all locally integrable functions from  $(-T_0, T_0)$  into  $X$ ). A function  $u$  is called a solution of  $\text{ACP}(A, f, x)$  if  $u \in C((-T_0, T_0), X)$  satisfies  $\text{ACP}(A, f, x)$  (that is,  $u(0) = x$  and for a.e.  $t \in (-T_0, T_0)$ ,  $u(t)$  is differentiable and  $u(t) \in D(A)$ , and  $u'(t) = Au(t) + f(t)$  for a.e.  $t \in (-T_0, T_0)$ ). For each  $C \in L(X)$  and  $K \in L_{loc}^1([0, T_0], \mathbb{F})$ , a family  $S(\cdot) = \{S(t) \mid |t| < T_0\}$  in  $L(X)$  is called a local  $K$ -convoluted  $C$ -group on

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$X$  if  $S(\cdot)$  is strongly continuous,  $S(\cdot)C = CS(\cdot)$ , and satisfies

$$S(t)S(s)x = \left( \operatorname{sgn} t \operatorname{sgn} s \operatorname{sgn} (t+s) \int_0^{t+s} - \operatorname{sgn} s \int_0^t - \operatorname{sgn} t \int_0^s \right) K(|t+s-r|)S(r)Cx dr,$$

for all  $x \in X$  and  $|t|, |s|, |t+s| < T_0$ . In particular,  $S(\cdot)$  is called a local (0-times integrated)  $C$ -group on  $X$  if  $K = j_{-1}$  (the Dirac measure at 0) or equivalently,  $S(\cdot)$  is strongly continuous,  $S(\cdot)C = CS(\cdot)$ , and satisfies

$$S(t)S(s)x = S(t+s)Cx, \quad \text{for all } x \in X \text{ and } |t|, |s|, |t+s| < T_0,$$

(see [2]). Moreover, we say that  $S(\cdot)$  is nondegenerate, if  $x = 0$  whenever  $S(t)x = 0$  for all  $|t| < T_0$ . The nondegeneracy of a local  $K$ -convoluted  $C$ -group  $S(\cdot)$  on  $X$  implies that

$$S(0) = C \text{ if } K = j_{-1} \text{ and } S(0) = 0 \text{ (the zero operator on } X) \text{ otherwise,}$$

and the (integral) generator  $A : D(A) \subset X \rightarrow X$  of  $S(\cdot)$  is a closed linear operator in  $X$  defined by

$$D(A) = \{x \in X \mid S(\cdot)x - K_0(|\cdot|)Cx = \tilde{S}(\cdot)y_x \text{ on } (-T_0, T_0) \text{ for some } y_x \in X\}$$

and  $Ax = y_x$  for all  $x \in D(A)$ . Here  $\tilde{S}(t)z = \int_0^t S(s)z ds$ . In general, a local  $K$ -convoluted  $C$ -group on  $X$  is called a  $K$ -convoluted  $C$ -group on  $X$  if  $T_0 = \infty$ ; a (local)  $K$ -convoluted  $C$ -group on  $X$  is called a (local)  $K$ -convoluted group on  $X$  if  $C = I$  (the identity operator on  $X$ ) or a (local)  $\alpha$ -times integrated  $C$ -group on  $X$  if  $K$  is equal to the function  $j_{\alpha-1}$  for some  $\alpha \geq 0$ , defined by  $j_\alpha(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$  (see [4, 7, 21]). Here  $\Gamma(\cdot)$  denotes the Gamma function, a (local)  $\alpha$ -times integrated  $C$ -group on  $X$  is called a (local)  $\alpha$ -times integrated group on  $X$  if  $C = I$ ; and a (local)  $C$ -group on  $X$  is called a  $c_0$ -group on  $X$  if  $C = I$  (see [1, 5]). Some basic properties of a nondegenerate (local)  $\alpha$ -times integrated  $C$ -semigroup on  $X$  have been established by many authors (in [2, 3, 26–28] for  $\alpha = 0$ , and in [6, 10, 17–20, 22, 23, 25, 29, 30] for  $\alpha > 0$ ), which can be extended to the case of local  $K$ -convoluted  $C$ -semigroup just as results in [7–10, 13–16]. Some equivalence relations between the generation of a nondegenerate (local)  $K$ -convoluted  $C$ -semigroup on  $X$  with subgenerator  $A$  and the unique existence of solutions of the abstract Cauchy problem  $ACP(A, f, x)$  are also discussed in [2, 26, 27] for the case  $K = j_{\alpha-1}$  with  $\alpha = 0$  and in [11–13, 30, 31] with  $\alpha > 0$ , and in [8, 13, 16] for the general case. The purpose of this paper is to investigate the following basic properties of a nondegenerate local  $K$ -convoluted  $C$ -group  $S(\cdot) (= \{S(t) \mid |t| < T_0\})$  on  $X$  just as results in [13] concerning local  $K$ -convoluted  $C$ -semigroups on  $X$  when  $C$  is injective and some additional conditions are taken into consideration

$$(1.1) \quad C^{-1}AC = A,$$

$$(1.2) \quad \tilde{S}(t)x \in D(A) \text{ and } A\tilde{S}(t)x = S(t)x - K_0(|t|)Cx, \quad \text{for all } x \in X \text{ and } |t| < T_0,$$

$$(1.3) \quad S(t)x \in D(A) \text{ and } AS(t)x = S(t)Ax, \quad \text{for all } x \in D(A) \text{ and } |t| < T_0;$$

and

$$(1.4) \quad S(t)S(s) = S(s)S(t), \quad \text{on } X, \text{ for all } |t|, |s| < T_0,$$

(see Theorems 2.5, 2.6 and 2.7 below), which have been established partially in [8] by another method, and then deduce some equivalence relations between the generation of a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$  with subgenerator  $A$  and the unique existence of solutions of  $\text{ACP}(A, f, x)$ , which are similar to some results in [13] concerning equivalence relations between the generation of a nondegenerate local  $K$ -convoluted  $C$ -semigroup on  $X$  with subgenerator  $A$  and the unique existence of solutions of  $\text{ACP}(A, f, x)$ . To do these, we will first prove an important lemma which shows that a strongly continuous family  $S(\cdot)$  in  $L(X)$  is a local  $K$ -convoluted  $C$ -group on  $X$  is equivalent to  $\text{sgn}(\cdot)\tilde{S}(\cdot)$  is a local  $K_0$ -convoluted  $C$ -group on  $X$  (see Lemma 2.1 below), and then show that a strongly continuous family  $S(\cdot)$  in  $L(X)$  which commutes with  $C$  on  $X$  is a local  $K$ -convoluted  $C$ -group on  $X$  is equivalent to  $\tilde{S}(t)[S(s) - K_0(|s|)C] = [S(t) - K_0(|t|)C]\tilde{S}(s)$  for all  $|t|, |s|, |t+s| < T_0$  (see Theorem 2.1 below). In order to show that  $\text{sgn}(\cdot)b * S(\cdot)$  is a local  $a * K$ -convoluted  $C$ -group on  $X$  if  $S(\cdot)$  is a local  $K$ -convoluted  $C$ -group on  $X$  and  $b(\cdot) = a(|\cdot|)$  for some  $a \in L^1_{loc}([0, T_0], \mathbb{F})$ . In particular,  $\text{sgn}(\cdot)J_\beta * S(\cdot)$  is a local  $K_\beta$ -convoluted  $C$ -group on  $X$  if  $S(\cdot)$  is a local  $K$ -convoluted  $C$ -group on  $X$  and  $\beta > -1$ , which can be applied to show that its only if part is also true when  $\beta$  is a nonnegative integer (see Proposition 2.1 below). Here  $K_\beta(t) = K * j_\beta(t)$  for  $\beta > -1$ ,  $J_\beta(\cdot) = j_\beta(|\cdot|)$ ,  $f * S(t)x = \int_0^t f(t-s)S(s)x ds$  for all  $x \in X$  and  $f \in L^1_{loc}((-T_0, T_0), \mathbb{F})$ . We also show that a strongly continuous family  $S(\cdot)$  in  $L(X)$  which commutes with  $C$  on  $X$  is a local  $K$ -convoluted  $C$ -group on  $X$  when it has a subgenerator (see Theorem 2.4 below). Moreover,  $S(\cdot)$  is nondegenerate if  $C$  is injective and the generator of a nondegenerate local  $K$ -convoluted  $C$ -group  $S(\cdot)$  on  $X$  is the unique subgenerator of  $S(\cdot)$  which contains all its subgenerators, and each subgenerator of  $S(\cdot)$  is closable and its closure is also a subgenerator of  $S(\cdot)$  when  $S(\cdot)$  has a subgenerator (see Theorems 2.5, 2.6 and 2.7 below). This can be applied to show that  $CA \subset AC$  and  $S(\cdot)$  is a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$  with generator  $C^{-1}AC$  when  $C$  is injective,  $K_0$  a kernel on  $[0, T_0)$  (that is,  $f = 0$  on  $[0, T_0)$  whenever  $f \in C([0, T_0), \mathbb{F})$  with  $\int_0^t K_0(t-s)f(s)ds = 0$  for all  $0 \leq t < T_0$ ) and  $S(\cdot)$  a strongly continuous family in  $L(X)$  with closed subgenerator  $A$ . In this case,  $C^{-1}\overline{A_0}C$  is the generator of  $S(\cdot)$  for each subgenerator  $A_0$  of  $S(\cdot)$  (see Theorem 2.8 below). Some illustrative examples concerning these theorems are also presented in the final part of this paper.

## 2. BASIC PROPERTIES OF LOCAL $K$ -CONVOLUTED $C$ -GROUPS

In the following we will note some facts concerning local  $K$ -convoluted  $C$ -groups which can be expansively applied in this paper.

*Remark 2.1.* Let  $S(\cdot) (= \{S(t) \mid |t| < T_0\})$  be a strongly continuous family in  $L(X)$ . Then the following are equivalent.

- (i)  $S(\cdot)$  is a local  $K$ -convoluted  $C$ -group on  $X$ .
- (ii) (see [8])  $S_+(\cdot)$  and  $S_-(\cdot)$  are local  $K$ -convoluted  $C$ -semigroups on  $X$ ,  $S(t)S(s)x = S(s)S(t)x$  on  $X$  for all  $-T_0 < t \leq 0 \leq s < T_0$ ,

$$S(t)S(s)x = \int_{t+s}^s K(r-t-s)S(r)Cxdr + \int_t^0 K(t+s-r)S(r)Cxdr,$$

for all  $x \in X$  and  $-T_0 < t \leq 0 \leq s < T_0$  with  $t+s \geq 0$ , and

$$S(t)S(s)x = \int_t^{t+s} K(t+s-r)S(r)Cxdr + \int_0^s K(r-t-s)S(r)Cxdr,$$

for all  $x \in X$  and  $-T_0 < t \leq 0 \leq s < T_0$  with  $t+s \leq 0$ .

- (iii)

$$T(t)T(s)x = \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) K(|t+s-r|)T(r)Cxdr,$$

for all  $x \in X$  and  $|t|, |s|, |t+s| < T_0$ .

Here  $T(\cdot) = \text{sgn}(\cdot)S(\cdot)$  on  $(-T_0, T_0)$ ,  $S_+(\cdot) = S(\cdot)$  and  $S_-(\cdot) = S(-\cdot)$  on  $[0, T_0)$ .

Next we will deduce an important lemma which can be used to obtain a new equivalence relation between the generation of a local  $K$ -convoluted  $C$ -group  $S(\cdot)$  on  $X$  and the equality of

$$\tilde{S}(t)[S(s) - K_0(|s|)C] = [S(t) - K_0(|t|)C]\tilde{S}(s),$$

on  $X$  for all  $|t|, |s|, |t+s| < T_0$  when  $S(\cdot) (= \{S(t) \mid |t| < T_0\})$  is a strongly continuous family in  $L(X)$  commuting with  $C$  on  $X$  just as a result in [13] for the case of local  $K$ -convoluted  $C$ -semigroup and in [19] for the case of local  $\alpha$ -times integrated  $C$ -semigroup.

**Lemma 2.1.** *Let  $S(\cdot) (= \{S(t) \mid |t| < T_0\})$  be a strongly continuous family in  $L(X)$ . Then  $S(\cdot)$  is a local  $K$ -convoluted  $C$ -group on  $X$  if and only if  $\text{sgn}(\cdot)\tilde{S}(\cdot)$  is a local  $K_0$ -convoluted  $C$ -group on  $X$ . In this case,*

- (i)  $S(\cdot)$  is nondegenerate if and only if  $\tilde{S}(\cdot)$  is;
- (ii)  $A$  is the generator of  $S(\cdot)$  if and only if it is the generator of  $\text{sgn}(\cdot)\tilde{S}(\cdot)$ .

*Proof.* Let  $x \in X$  be given. We set  $T(\cdot) = \text{sgn}(\cdot)\tilde{S}(\cdot)$ . Then

$$(2.1) \quad \begin{aligned} & \frac{d}{dt} \left[ \int_{t+s}^s K_0(r-t-s)\tilde{S}(r)Cxdr - \int_t^0 K_0(t+s-r)\tilde{S}(r)Cxdr \right] \\ &= - \int_{t+s}^s K(r-t-s)\tilde{S}(r)Cxdr - \int_t^0 K(t+s-r)\tilde{S}(r)Cxdr + K_0(s)\tilde{S}(t)Cx \end{aligned}$$

and

$$(2.2) \quad \begin{aligned} & \frac{d}{ds} \left[ \int_{t+s}^s K_0(r-t-s)\tilde{S}(r)Cxdr - \int_t^0 K_0(t+s-r)\tilde{S}(r)Cxdr \right] \\ &= - \int_{t+s}^s K(r-t-s)\tilde{S}(r)Cxdr - \int_t^0 K(t+s-r)\tilde{S}(r)Cxdr + K_0(|t|)\tilde{S}(s)Cx, \end{aligned}$$

for  $-T_0 < t \leq 0 \leq s < T_0$  with  $t + s \geq 0$ . Using integration by parts to the right-hand sides of (2.1) and (2.2), we obtain

$$(2.3) \quad - \int_{t+s}^s K(r-t-s)\tilde{S}(r)Cxdr - \int_t^0 K(t+s-r)\tilde{S}(r)Cxdr + K_0(s)\tilde{S}(t)Cx \\ = \int_{t+s}^s K_0(r-t-s)S(r)Cxdr - \int_t^0 K_0(t+s-r)S(r)Cxdr - K_0(|t|)\tilde{S}(s)Cx$$

and

$$(2.4) \quad - \int_{t+s}^s K(r-t-s)\tilde{S}(r)Cxdr - \int_t^0 K(t+s-r)\tilde{S}(r)Cxdr + K_0(|t|)\tilde{S}(s)Cx \\ = \int_{t+s}^s K_0(r-t-s)S(r)Cxdr - \int_t^0 K_0(t+s-r)S(r)Cxdr - K_0(s)\tilde{S}(t)Cx,$$

for  $-T_0 < t \leq 0 \leq s < T_0$  with  $t + s \geq 0$ . Combining (2.1)–(2.4), we have

$$(2.5) \quad \frac{d}{dt} \left[ \int_{t+s}^s K_0(r-t-s)\tilde{S}(r)Cxdr - \int_t^0 K_0(t+s-r)\tilde{S}(r)Cxdr \right] \\ = \int_{t+s}^s K_0(r-t-s)S(r)Cxdr - \int_t^0 K_0(t+s-r)S(r)Cxdr - K_0(|t|)\tilde{S}(s)Cx$$

and

$$(2.6) \quad \frac{d}{ds} \left[ \int_{t+s}^s K_0(r-t-s)\tilde{S}(r)Cxdr - \int_t^0 K_0(t+s-r)\tilde{S}(r)Cxdr \right] \\ = \int_{t+s}^s K_0(r-t-s)S(r)Cxdr - \int_t^0 K_0(t+s-r)S(r)Cxdr - K_0(s)\tilde{S}(t)Cx,$$

for  $-T_0 < t \leq 0 \leq s < T_0$  with  $t + s \geq 0$ . Similarly, we can show that

$$(2.7) \quad \frac{d}{dt} \left[ - \int_t^{t+s} K_0(t+s-r)\tilde{S}(r)Cxdr + \int_0^s K_0(r-t-s)\tilde{S}(r)Cxdr \right] \\ = - \int_t^{t+s} K_0(t+s-r)S(r)Cxdr + \int_0^s K_0(r-t-s)S(r)Cxdr - K_0(|t|)\tilde{S}(s)Cx$$

and

$$(2.8) \quad \frac{d}{ds} \left[ - \int_t^{t+s} K_0(t+s-r)\tilde{S}(r)Cxdr + \int_0^s K_0(r-t-s)\tilde{S}(r)Cxdr \right] \\ = - \int_t^{t+s} K_0(t+s-r)S(r)Cxdr + \int_0^s K_0(r-t-s)S(r)Cxdr - K_0(s)\tilde{S}(t)Cx,$$

for  $-T_0 < t \leq 0 \leq s < T_0$  with  $t + s \leq 0$ . By (2.6) and (2.8), we have

$$(2.9) \quad \frac{d}{ds} \frac{d}{dt} \left[ \int_{t+s}^s K_0(r-t-s)\tilde{S}(r)Cxdr - \int_t^0 K_0(t+s-r)\tilde{S}(r)Cxdr \right] \\ = - \int_{t+s}^s K(r-t-s)S(r)Cxdr + \int_t^0 K(t+s-r)S(r)Cxdr,$$

for  $-T_0 < t \leq 0 \leq s < T_0$  with  $t + s \geq 0$  and

$$(2.10) \quad \frac{d}{dt} \frac{d}{ds} \left[ - \int_t^{t+s} K_0(t+s-r)\tilde{S}(r)Cxdr + \int_0^s K_0(r-t-s)\tilde{S}(r)Cxdr \right]$$

$$= - \int_t^{t+s} K(t+s-r)S(r)Cxd r + \int_0^s K(r-t-s)S(r)Cxd r,$$

for  $-T_0 < t \leq 0 \leq s < T_0$  with  $t+s \leq 0$ . Suppose that  $T(\cdot)$  is a local  $K_0$ -convoluted  $C$ -group on  $X$ . Then  $T_+(\cdot)$  and  $T_-(\cdot)$  both are local  $K_0$ -convoluted  $C$ -semigroups on  $X$ ,  $T_+(\cdot) = \widetilde{S}_+(\cdot)$  and  $T_-(\cdot) = \widetilde{S}_-(\cdot)$  on  $[0, T_0)$ ,  $T(t)T(s) = T(s)T(t)$  on  $X$  for all  $-T_0 < t \leq 0 \leq s < T_0$ ,

$$T(t)T(s)x = \int_{t+s}^s K_0(r-t-s)T(r)Cxd r + \int_t^0 K_0(t+s-r)T(r)Cxd r,$$

for all  $x \in X$  and  $-T_0 < t \leq 0 \leq s < T_0$  with  $t+s \geq 0$  and

$$T(t)T(s)x = \int_t^{t+s} K_0(t+s-r)T(r)Cxd r + \int_0^s K_0(r-t-s)T(r)Cxd r,$$

for all  $x \in X$  and  $-T_0 < t \leq 0 \leq s < T_0$  with  $t+s \leq 0$  or equivalently,  $S_+(\cdot)$  and  $S_-(\cdot)$  both are local  $K$ -convoluted  $C$ -semigroups on  $X$ ,  $S(t)S(s) = S(s)S(t)$  on  $X$  for all  $-T_0 < t \leq 0 \leq s < T_0$

$$(2.11) \quad -\widetilde{S}(t)\widetilde{S}(s)x = \int_{t+s}^s K_0(r-t-s)\widetilde{S}(r)Cxd r - \int_t^0 K_0(t+s-r)\widetilde{S}(r)Cxd r,$$

for all  $x \in X$  and  $-T_0 < t \leq 0 \leq s < T_0$  with  $t+s \geq 0$ , and

$$(2.12) \quad -\widetilde{S}(t)\widetilde{S}(s)x = - \int_t^{t+s} K_0(t+s-r)\widetilde{S}(r)Cxd r + \int_0^s K_0(r-t-s)\widetilde{S}(r)Cxd r,$$

for all  $x \in X$  and  $-T_0 < t \leq 0 \leq s < T_0$  with  $t+s \leq 0$ . Combining (2.7)–(2.10), we have

$$(2.13) \quad S(t)S(s)x = \int_{t+s}^s K(r-t-s)S(r)Cxd r + \int_t^0 K(t+s-r)S(r)Cxd r,$$

for all  $x \in X$  and  $-T_0 < t \leq 0 \leq s < T_0$  with  $t+s \geq 0$  and

$$(2.14) \quad S(t)S(s)x = \int_t^{t+s} K(t+s-r)S(r)Cxd r + \int_0^s K(r-t-s)S(r)Cxd r,$$

for all  $x \in X$  and  $-T_0 < t \leq 0 \leq s < T_0$  with  $t+s \leq 0$ . Consequently,  $S(\cdot)$  is a local  $K$ -convoluted  $C$ -group on  $X$ . Conversely, suppose that  $S(\cdot)$  is a local  $K$ -convoluted  $C$ -group on  $X$ . Then  $T_+(\cdot)$  and  $T_-(\cdot)$  both are local  $K_0$ -convoluted  $C$ -semigroups on  $X$ ,  $T(t)T(s) = T(s)T(t)$  on  $X$  for all  $-T_0 < t \leq 0 \leq s < T_0$ , and (2.13)–(2.14) both hold. By (2.9) and (2.10), we have (2.11) and (2.12) both hold. Consequently,  $T(\cdot)$  is a local  $K_0$ -convoluted  $C$ -group on  $X$ . □

**Theorem 2.1.** *Let  $S(\cdot) (= \{S(t) \mid |t| < T_0\})$  be a strongly continuous family in  $L(X)$  which commutes with  $C$  on  $X$ . Then  $S(\cdot)$  is a local  $K$ -convoluted  $C$ -group on  $X$  if and only if*

$$(2.15) \quad \widetilde{S}(t)[S(s) - K_0(|s|)C] = [S(t) - K_0(|t|)C]\widetilde{S}(s), \quad \text{on } X,$$

for all  $|t|, |s|, |t+s| < T_0$ .

*Proof.* We set  $T(\cdot) = \text{sgn}(\cdot)\tilde{S}(\cdot)$ . Suppose that  $S(\cdot)$  is a local  $K$ -convoluted  $C$ -group on  $X$ . Then  $S_+(\cdot)$  and  $S_-(\cdot)$  both are local  $K$ -convoluted  $C$ -semigroups on  $X$ . To show that (2.15) holds for all  $|t|, |s|, |t + s| < T_0$ , we observe from [13, Theorem 2.2] that we need only to show that  $\tilde{S}(t)[S(s) - K_0(|s|)C]x = [S(t) - K_0(|t|)C]\tilde{S}(s)x$  for all  $x \in X$  and  $|t|, |s| < T_0$  with  $ts \leq 0$ . Let  $x \in X$  and  $-T_0 < t \leq 0 \leq s < T_0$  be given with  $t + s \geq 0$ . By Lemma 2.1, (2.1) and (2.2), we have

$$\begin{aligned} -S(t)\tilde{S}(s)x - K_0(|s|)\tilde{S}(t)Cx &= \frac{d}{dt}T(t)T(s)x - K_0(|s|)\tilde{S}(t)Cx \\ &= \frac{d}{ds}T(t)T(s)x - K_0(|t|)\tilde{S}(s)Cx \\ &= -\tilde{S}(t)S(s)x - K_0(|t|)\tilde{S}(s)Cx, \end{aligned}$$

or equivalently,  $\tilde{S}(t)[S(s) - K_0(|s|)C]x = [S(t) - K_0(|t|)C]\tilde{S}(s)x$ . Similarly, we can show that  $\tilde{S}(t)[S(s) - K_0(|s|)C]x = [S(t) - K_0(|t|)C]\tilde{S}(s)x$  for all  $x \in X$  and  $-T_0 < t \leq 0 \leq s < T_0$  with  $t + s \leq 0$ . Since  $S(t)S(s) = S(s)S(t)$  on  $X$  for all  $|t|, |s|, |t + s| < T_0$ , we also have  $\tilde{S}(t)[S(s) - K_0(|s|)C]x = [S(t) - K_0(|t|)C]\tilde{S}(s)x$  for all  $x \in X$  and  $-T_0 < s \leq 0 \leq t < T_0$ . Consequently, (2.15) holds for all  $|t|, |s|, |t + s| < T_0$ . Conversely, suppose that (2.15) holds for all  $|t|, |s|, |t + s| < T_0$ . Then  $T_+(\cdot)$  and  $T_-(\cdot)$  both are local  $K_0$ -convoluted  $C$ -semigroups on  $X$  and  $\tilde{S}(t)S(s)x - S(t)\tilde{S}(s)x = K_0(|s|)\tilde{S}(t)Cx - K_0(|t|)\tilde{S}(s)Cx$  for all  $x \in X$  and  $|t|, |s|, |t + s| < T_0$  with  $t + s \geq 0$ . Fix  $x \in X$  and  $-T_0 < t < 0 \leq s < T_0$  with  $t + s \geq 0$ , we have

$$\begin{aligned} (2.16) \quad &\tilde{S}(t + s - r)S(r)x - S(t + s - r)\tilde{S}(r)x \\ &= K_0(|r|)\tilde{S}(t + s - r)Cx - K_0(|t + s - r|)\tilde{S}(r)Cx, \end{aligned}$$

for all  $t \leq r \leq 0$ . Using integration by parts to the left-hand side of (2.16) over  $[t, 0]$  and change of variables to the right-hand side of (2.16) over  $[t, 0]$ , we obtain

$$\begin{aligned} (2.17) \quad T(t)T(s)x &= -\tilde{S}(t)\tilde{S}(s)x \\ &= \int_t^0 [\tilde{S}(t + s - r)S(r)x - S(t + s - r)\tilde{S}(r)x]dr \\ &= \int_t^0 [K_0(|r|)\tilde{S}(t + s - r)Cx - K_0(|t + s - r|)\tilde{S}(r)Cx]dr \\ &= \int_s^{t+s} K_0(|t + s - r|)\tilde{S}(r)Cxdx - \int_t^0 K_0(|t + s - r|)\tilde{S}(r)Cxdx \\ &= \int_{t+s}^s K_0(|t + s - r|)T(r)Cxdx + \int_t^0 K_0(|t + s - r|)T(r)Cxdx. \end{aligned}$$

Using change of variables to the left-hand side of (2.16) over  $[t, 0]$ , we also have

$$(2.18) \quad T(s)T(t)x = -\tilde{S}(s)\tilde{S}(t)x = \int_t^0 [\tilde{S}(t + s - r)S(r)x - S(t + s - r)\tilde{S}(r)x]dr.$$

Combining (2.17) with (2.18), we have  $T(t)T(s) = T(s)T(t)$  on  $X$  for all  $|t|, |s|, |t+s| < T_0$  with  $ts \leq 0$  and

$$T(t)T(s)x = \int_{t+s}^s K_0(|t+s-r|)T(r)Cxdr + \int_t^0 K_0(|t+s-r|)T(r)Cxdr,$$

for all  $x \in X$  and  $-T_0 < t \leq 0 \leq s < T_0$  with  $t+s \geq 0$ . Similarly, we can show that

$$T(t)T(s)x = \int_t^{t+s} K_0(|t+s-r|)T(r)Cxdr + \int_0^s K_0(|t+s-r|)T(r)Cxdr,$$

for all  $x \in X$  and  $-T_0 < t \leq 0 \leq s < T_0$  with  $t+s \leq 0$  when the interval  $[t, 0]$  of the integration of (2.16) is replaced by  $[t, t+s]$ . Consequently,  $T(\cdot)$  is a local  $K_0$ -convoluted  $C$ -group on  $X$ . Combining this with Lemma 2.1, we get that  $S(\cdot)$  is a local  $K$ -convoluted  $C$ -group on  $X$ . □

**Proposition 2.1.** *Let  $S(\cdot)$  be a local  $K$ -convoluted  $C$ -group on  $X$ ,  $a \in L_{loc}^1([0, T_0], \mathbb{F})$ , and  $b(\cdot) = a(|\cdot|)$ . Then  $\text{sgn}(\cdot)b * S(\cdot)$  is a local  $a * K$ -convoluted  $C$ -group on  $X$ . In particular, for each  $\beta > -1$ ,  $\text{sgn}(\cdot)J_\beta * S(\cdot)$  is a local  $K_\beta$ -convoluted  $C$ -group on  $X$ . Here  $J_\beta(\cdot) = j_\beta(|\cdot|)$ . Moreover,  $S(\cdot)$  is a local  $K$ -convoluted  $C$ -group on  $X$  if it is a strongly continuous family in  $L(X)$  such that  $\text{sgn}^k(\cdot)j_{k-1} * S(\cdot) = \text{sgn}(\cdot)J_{k-1} * S(\cdot)$  is a local  $K_{k-1}$ -convoluted  $C$ -group on  $X$  for some nonnegative integer  $k$ .*

*Proof.* Clearly,  $\text{sgn}(\cdot)b * S(\cdot)$  is strongly continuous family in  $L(X)$  which commutes with  $C$  on  $X$ . To show that  $\text{sgn}(\cdot)b * S(\cdot)$  is a local  $a * K$ -convoluted  $C$ -group on  $X$ , we remain only to show that

$$\begin{aligned} & [(\text{sgn } t)b * S(t) - \widetilde{a * K}(|t|)C]j_0 * [\text{sgn}(\cdot)b * S(\cdot)](s) \\ &= j_0 * [\text{sgn}(\cdot)b * S(\cdot)](t)[(\text{sgn } s)b * S(s) - \widetilde{a * K}(|s|)C], \end{aligned}$$

on  $X$  for all  $|t|, |s|, |t+s| < T_0$ . Here  $\widetilde{a * K} = j_0 * (a * K)$ . Clearly,

$$b * K_0(|\cdot|)(t) = (\text{sgn } t)j_0 * (b * K)(|t|),$$

on  $X$  for all  $0 \leq t < T_0$ . Next we will show that  $b * K_0(|\cdot|)(t) = (\text{sgn } t)j_0 * b * K(|t|)$  on  $X$  for all  $-T_0 < t \leq 0$ . Let  $-T_0 < t \leq 0$  be given, then

$$\begin{aligned} b * K_0(|\cdot|)(t) &= \int_0^t b(s)K_0(|t-s|)ds = \int_0^t b(s)K_0(s-t)ds \\ &= - \int_0^t a(-s) \int_s^t K(s-r)drds = - \int_t^0 \int_s^t a(-s)K(s-r)drds \\ &= - \int_0^t \int_r^t a(-r)K(r-s)dsdr = \int_t^0 \int_r^t a(-r)K(r-s)dsdr \end{aligned}$$

and

$$\begin{aligned} \int_t^0 \int_r^t a(-r)K(r-s)dsdr &= - \int_t^0 \int_s^0 a(-r)K(r-s)drds \\ &= - \int_0^t \int_0^s a(|r|)K(r-s)drds \end{aligned}$$

$$\begin{aligned} &= \int_0^t \int_0^{-s} a(|r|)K(-r-s)drds \\ &= \int_0^t b * K(-s)ds = - \int_0^{-t} b * K(s)ds \\ &= (\operatorname{sgn} t)j_0 * (b * K)(|t|). \end{aligned}$$

Since  $b * K(|t|) = a * K(|t|)$  for all  $|t| < T_0$ , we have  $b * K_0(| \cdot |)(t) = (\operatorname{sgn} t)\widetilde{a * K}(|t|)$  for all  $|t| < T_0$ . Clearly,  $b * \widetilde{S}(t) = j_0 * (b * S)(t)$  on  $X$  for all  $|t| < T_0$ . Since  $j_0 * [\operatorname{sgn}(\cdot)b * S(\cdot)](t) = (\operatorname{sgn} t)j_0 * (b * S)(t) = (\operatorname{sgn} t)b * \widetilde{S}(t)$  on  $X$  for all  $|t| < T_0$ , we also have

$$\begin{aligned} & [(\operatorname{sgn} t)(b * S)(t) - \widetilde{a * K}(|t|)C](\operatorname{sgn} s)\widetilde{b * S}(s)x \\ &= [(\operatorname{sgn} t)(b * S)(t) - (\operatorname{sgn} t)b * K_0(| \cdot |)(t)C](\operatorname{sgn} s)b * \widetilde{S}(s)x \\ &= (\operatorname{sgn} t)[(b * S)(t) - b * K_0(| \cdot |)(t)C](\operatorname{sgn} s)b * \widetilde{S}(s)x \\ &= (\operatorname{sgn} t) \int_0^t b(t-s)[S(r) - K_0(|r|)C](\operatorname{sgn} s)b * \widetilde{S}(s)xdr \\ &= (\operatorname{sgn} t)b * \left[ \int_0^t b(t-r)(S(r) - K_0(|r|)C)\widetilde{S} \right] (s)(\operatorname{sgn} s)xdr \end{aligned}$$

and

$$\begin{aligned} & (\operatorname{sgn} t)b * \left[ \int_0^t b(t-r)(S(r) - K_0(|r|)C)\widetilde{S} \right] (s)(\operatorname{sgn} s)xdr \\ &= (\operatorname{sgn} t)b * \left[ \int_0^t b(t-r)\widetilde{S}(r)(S(\cdot) - K_0(| \cdot |)C) \right] (s)(\operatorname{sgn} s)xdr \\ &= (\operatorname{sgn} t)b * \widetilde{S}(t)b * [S(\cdot) - K_0(| \cdot |)C](s)(\operatorname{sgn} s)x \\ &= (\operatorname{sgn} t)b * \widetilde{S}(t)[b * S(s) - b * K_0(| \cdot |)(s)C](\operatorname{sgn} s)x \\ &= (\operatorname{sgn} t)\widetilde{b * S}(t)[(\operatorname{sgn} s)b * S(s) - (\operatorname{sgn} s)b * K_0(| \cdot |)(s)C]x \\ &= (\operatorname{sgn} t)\widetilde{b * S}(t)[(\operatorname{sgn} s)b * S(s) - \widetilde{a * K}(|s|)C]x, \end{aligned}$$

for all  $x \in X$  and  $|t|, |s|, |t + s| < T_0$ . □

**Definition 2.1.** Let  $S(\cdot) = \{S(t) \mid |t| < T_0\}$  be a strongly continuous family in  $L(X)$ . A linear operator  $A$  in  $X$  is called a subgenerator of  $S(\cdot)$  if

$$S(t)x - K_0(|t|)Cx = \int_0^t S(r)Axdr,$$

for all  $x \in D(A)$  and  $|t| < T_0$ , and

$$(2.19) \quad \int_0^t S(r)xdr \in D(A) \quad \text{and} \quad A \int_0^t S(r)xdr = S(t)x - K_0(|t|)Cx,$$

for all  $x \in X$  and  $|t| < T_0$ . A subgenerator  $A$  of  $S(\cdot)$  is called the maximal subgenerator of  $S(\cdot)$  if it is an extension of each subgenerator of  $S(\cdot)$  to  $D(A)$ .

*Remark 2.2.* Let  $S(\cdot)(= \{S(t) \mid |t| < T_0\})$  be a strongly continuous family in  $L(X)$ , and  $A$  a linear operator in  $X$ . Then  $A$  is a subgenerator of  $S(\cdot)$  if and only if  $A$  is a subgenerator of  $S_+(\cdot)$  and  $-A$  a subgenerator of  $S_-(\cdot)$ .

*Remark 2.3.* Let  $S(\cdot)(= \{S(t) \mid |t| < T_0\})$  be a strongly continuous family in  $L(X)$ , and  $A$  a (closed) linear operator in  $X$ . Then  $A$  is the maximal subgenerator of  $S(\cdot)$  if  $A$  is the maximal subgenerator of  $S_+(\cdot)$  and  $-A$  the maximal subgenerator of  $S_-(\cdot)$ .

**Theorem 2.2.** *Let  $S(\cdot)$  be a local  $K$ -convoluted  $C$ -group on  $X$  and  $K_0$  not the zero function on  $[0, T_0)$ , or a  $K$ -convoluted  $C$ -group on  $X$ . Assume that  $C$  is injective. Then  $S(\cdot)$  is nondegenerate if and only if  $S_+(\cdot)$  and  $S_-(\cdot)$  both are nondegenerate if and only if  $S_+(\cdot)$  or  $S_-(\cdot)$  is nondegenerate.*

*Proof.* Clearly,  $S(\cdot)$  is nondegenerate if either  $S_+(\cdot)$  or  $S_-(\cdot)$  is nondegenerate. Conversely, suppose that  $S(\cdot)$  is nondegenerate and  $S_+(\cdot)x = 0$  on  $[0, T_0)$  for some  $x \in X$ . By Theorem 2.1, we have  $\tilde{S}(t)[S(s) - K_0(|s|)C]x = [S(t) - K_0(|t|)C]\tilde{S}(s)x = 0$  for all  $-T_0 < t \leq 0 \leq s < T_0$ , and so  $\tilde{S}(t)K_0(|s|)Cx = 0$ . Hence,  $\tilde{S}(t)x = 0$ . Since  $-T_0 < t \leq 0$  is arbitrary, we have  $S(\cdot)x = 0$  on  $(-T_0, 0]$ , which together with the nondegeneracy of  $S(\cdot)$  implies that  $x = 0$ . Consequently,  $S_+(\cdot)$  is nondegenerate. Similarly, we can show that  $S_-(\cdot)$  is nondegenerate when  $S(\cdot)$  is nondegenerate.  $\square$

**Theorem 2.3.** *Let  $S(\cdot)$  be a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$  and  $K_0$  not the zero function on  $[0, T_0)$ , or a  $K$ -convoluted  $C$ -group on  $X$ . Assume that  $C$  is injective. Then  $A$  is the generator of  $S(\cdot)$  if and only if  $A$  is the generator of  $S_+(\cdot)$  and  $-A$  the generator of  $S_-(\cdot)$  if and only if  $A$  is the generator of  $S_+(\cdot)$  or  $-A$  the generator of  $S_-(\cdot)$ .*

*Proof.* Suppose that  $A$  is the generator of  $S_+(\cdot)$  and  $-A$  is the generator of  $S_-(\cdot)$ . We set  $B$  to denote the generator of  $S(\cdot)$ . Then  $S(\cdot)x - K_0(|\cdot|)Cx = \tilde{S}(\cdot)Ax$  on  $(-T_0, T_0)$  for all  $x \in D(A)$  or equivalently,  $A \subset B$ . Since  $S(\cdot)x - K_0(|\cdot|)Cx = \tilde{S}(\cdot)Bx$  on  $(-T_0, T_0)$  for all  $x \in D(B)$ , we have  $B \subset A$ . Consequently,  $A = B$  is the generator of  $S(\cdot)$ . Suppose that  $A$  is the generator of  $S(\cdot)$ . We set  $B_+$  and  $B_-$  to denote the generators of  $S_+(\cdot)$  and  $S_-(\cdot)$ , respectively. To show that  $B_+ = A$  and  $B_- = -A$ , we observe from the preceding argument, we need only to show that  $B_+ = -B_-$ . Let  $x \in D(B_-)$  be given, then

$$\begin{aligned} \tilde{S}(t)[S(s) - K_0(|s|)C]x &= [S(t) - K_0(|t|)C]\tilde{S}(s)x = \tilde{S}(s)[S(t) - K_0(|t|)C]x \\ &= \tilde{S}(s)[- \tilde{S}(t)B_-x] = \tilde{S}(s)[\tilde{S}(t)(-B_-)x] \\ &= \tilde{S}(t)[\tilde{S}(s)(-B_-)x], \end{aligned}$$

for all  $-T_0 < t \leq 0 \leq s < T_0$ . By the nondegeneracy of  $S_-(\cdot)$ , we have  $[S(s) - K_0(|s|)C] = \tilde{S}(s)[-B_-x]$  for all  $0 \leq s < T_0$ , and so  $x \in D(B_+)$  and  $B_+x = -B_-x$ . Hence,  $-B_- \subset B_+$ . By symmetry, we also have  $B_+ \subset -B_-$ . Consequently,  $B_+ = -B_-$ .  $\square$

**Theorem 2.4.** *Let  $S(\cdot) (= \{S(t) \mid |t| < T_0\})$  be a strongly continuous family in  $L(X)$  which commutes with  $C$  on  $X$ . Assume that  $S(\cdot)$  has a subgenerator. Then  $S(\cdot)$  is a local  $K$ -convoluted  $C$ -group on  $X$ . Moreover,  $S(\cdot)$  is nondegenerate if the injectivity of  $C$  is added and  $K_0$  is a nonzero function on  $[0, T_0)$ .*

Combining Remark 2.2 with [13, Lemma 2.8], the next lemma is also obtained.

**Lemma 2.2.** *Let  $A$  be a closed subgenerator of a strongly continuous family  $S(\cdot) (= \{S(t) \mid |t| < T_0\})$  in  $L(X)$ , and  $K_0$  a kernel on  $[0, t_0)$  (or equivalently,  $K$  is a kernel on  $[0, t_0)$ ) for some  $0 < t_0 \leq T_0$ . Assume that  $C$  is injective, and  $u \in C((-t_0, t_0), X)$  satisfies  $u(\cdot) = Aj_0 * u(\cdot)$  on  $(-t_0, t_0)$ . Then  $u = 0$  on  $(-t_0, t_0)$ .*

By slightly modifying the proof of [13, Theorem 2.7], we can apply Lemma 2.2 to deduce the next theorem concerning nondegenerate  $K$ -convoluted  $C$ -groups, and so its proof is omitted.

**Theorem 2.5.** *Let  $S(\cdot)$  be a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$  with generator  $A$ . Assume that  $S(\cdot)$  has a subgenerator. Then  $A$  is the maximal subgenerator of  $S(\cdot)$ , and each subgenerator of  $S(\cdot)$  is closable and its closure is also a subgenerator of  $S(\cdot)$ . Moreover, if  $C$  is injective. Then (1.1)–(1.3) hold, and (1.4) also holds when  $K_0$  is a kernel on  $[0, T_0)$  or  $T_0 = \infty$ .*

**Lemma 2.3.** *Let  $S(\cdot)$  be a local  $K$ -convoluted  $C$ -group on  $X$  and  $0 \in \text{supp } K_0$  (the support of  $K_0$ ), or a  $K$ -convoluted  $C$ -group on  $X$  and  $K_0$  not the zero function on  $[0, \infty)$ . Assume that  $S(\cdot)x = 0$  on  $[0, t_0)$  or on  $(-t_0, 0]$  for some  $x \in X$  and  $0 < t_0 \leq T_0$ . Then  $CS(\cdot)x = 0$  on  $(-T_0, T_0)$ . In particular,  $S(t)x = 0$  for all  $|t| < T_0$  if the injectivity of  $C$  is added.*

*Proof.* Let  $S(\cdot)x = 0$  on  $[0, t_0)$  and  $|t| < T_0$  be given, then  $|t|+s < T_0$  and  $K_0(s) \neq 0$  for some  $0 < s < t_0$ , so that  $\tilde{S}(s)S(t)x = S(t)\tilde{S}(s)x = 0$ ,  $S(s)\tilde{S}(t)x = \tilde{S}(t)S(s)x = 0$ , and  $\tilde{S}(s)K_0(|t|)Cx = K_0(|t|)C\tilde{S}(s)x = 0$ . By Theorem 2.3, we have  $\tilde{S}(s)[S(t) - K_0(|t|)C]x = [S(s) - K_0(s)C]\tilde{S}(t)x$ . Hence,  $K_0(s)\tilde{S}(t)Cx = K_0(s)C\tilde{S}(t)x = 0$ , which implies that  $\tilde{S}(t)Cx = 0$ . Since  $|t| < T_0$  is arbitrary, we have  $CS(t)x = S(t)Cx = 0$  for all  $|t| < T_0$ . In particular,  $S(t)x = 0$  for all  $|t| < T_0$  if the injectivity of  $C$  is added. □

**Lemma 2.4.** *Let  $S(\cdot)$  be a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$  with generator  $A$  and  $0 \in \text{supp } K_0$ . Assume that  $C$  is injective. Then  $A$  is a subgenerator of  $S(\cdot)$ .*

*Proof.* By Theorems 2.2 and 2.3,  $A$  is the generator of  $S_+(\cdot)$  and  $-A$  is the generator of  $S_-(\cdot)$ . It follows from [13, Theorem 2.9] that  $A$  is a subgenerator of  $S_+(\cdot)$  and  $-A$  is a subgenerator of  $S_-(\cdot)$ , which together with Remark 2.2 implies that  $A$  is a subgenerator of  $S(\cdot)$ . □

By slightly modifying the proof of Lemma 2.4, the next lemma concerning nondegenerate  $K$ -convoluted  $C$ -groups is also attained.

**Lemma 2.5.** *Let  $S(\cdot)$  be a nondegenerate  $K$ -convoluted  $C$ -group on  $X$  with generator  $A$ . Then  $C$  is injective, and  $A$  is a subgenerator of  $S(\cdot)$ .*

Combining Theorem 2.5 with Lemma 2.5, the next theorem concerning nondegenerate  $K$ -convoluted  $C$ -groups is also obtained.

**Theorem 2.6.** *Let  $S(\cdot)$  be a nondegenerate  $K$ -convoluted  $C$ -group on  $X$  with generator  $A$ . Then  $A$  is the maximal subgenerator of  $S(\cdot)$ , and each subgenerator of  $S(\cdot)$  is closable and its closure is also a subgenerator of  $S(\cdot)$ . Moreover, (1.1)–(1.4) hold.*

Since  $0 \in \text{supp}K_0$  implies that  $K_0$  is a kernel on  $[0, T_0)$ , we can apply Theorem 2.5 and Lemma 2.4 to obtain the next theorem.

**Theorem 2.7.** *Let  $S(\cdot)$  be a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$  with generator  $A$  and  $0 \in \text{supp}K_0$ . Assume that  $C$  is injective. Then  $A$  is the maximal subgenerator of  $S(\cdot)$ , and each subgenerator of  $S(\cdot)$  is closable and its closure is also a subgenerator of  $S(\cdot)$ . Moreover, (1.1)–(1.4) hold.*

**Theorem 2.8.** *Let  $S(\cdot) (= \{S(t) \mid |t| < T_0\})$  be a strongly continuous family in  $L(X)$  which has a subgenerator and  $K_0$  a kernel on  $[0, T_0)$ . Assume that  $C$  is injective. Then  $S(\cdot)$  is a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$ ,  $CA \subset AC$  and  $C^{-1}AC$  is the generator of  $S(\cdot)$  for each closed subgenerator  $A$  of  $S(\cdot)$ . In particular,  $C^{-1}\overline{A_0}C$  is the generator of  $S(\cdot)$  for each subgenerator  $A_0$  of  $S(\cdot)$ .*

*Proof.* Suppose that  $A$  is a closed subgenerator of  $S(\cdot)$ . By Remark 2.2,  $A$  is a closed subgenerator of  $S_+(\cdot)$ . By [13], Theorem 2.13, we have  $CA \subset AC$  and  $C^{-1}AC$  is the generator of  $S_+(\cdot)$ . By Theorem 2.3,  $C^{-1}AC$  is the generator of  $S(\cdot)$ . Similarly, we can show that  $C^{-1}\overline{A_0}C$  is the generator of  $S(\cdot)$  for each subgenerator  $A_0$  of  $S(\cdot)$ .  $\square$

**Corollary 2.1.** *Let  $S(\cdot)$  be a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$  which has a subgenerator and  $K_0$  a kernel on  $[0, T_0)$ . Assume that  $C$  is injective and  $R(C)$  is dense in  $X$ . Then  $A$  is a closed subgenerator of  $S_+(\cdot)$  if and only if  $-A$  is a closed subgenerator of  $S_-(\cdot)$ .*

*Proof.* By Remark 2.2, we need only to show that  $A$  is a closed subgenerator of  $S(\cdot)$  when  $A$  is a closed subgenerator of  $S_+(\cdot)$ . Since  $\int_0^t S(r)Axdr = \int_0^t S(r)C^{-1}ACxdr = S(t)x - K_0(|t|)Cx$  for all  $x \in D(A)$  and  $|t| < T_0$ , we remain only to show that (2.19) holds for all  $x \in X$  and  $|t| < T_0$ . Suppose that  $x \in X$  and  $|t| < T_0$  are given. By [13], Theorem 2.13,  $C^{-1}AC$  is the generator of  $S_+(\cdot)$ . By Theorem 2.3,  $C^{-1}AC$  is the generator of  $S(\cdot)$ . By Theorems 2.5 and 2.8,  $C^{-1}AC$  is the maximal subgenerator of  $S(\cdot)$ , and so  $C^{-1}AC \int_0^t S(r)xdr = S(t)x - K_0(|t|)Cx$ . Hence,  $AC \int_0^t S(r)xdr = A \int_0^t S(r)Cxdr = S(t)Cx - K_0(|t|)CCx$ , which together with the denseness of  $R(C)$  implies that  $A \int_0^t S(r)xdr = S(t)x - K_0(|t|)Cx$  for all  $x \in X$  and  $|t| < T_0$ .  $\square$

*Remark 2.4.* Let  $S(\cdot) (= \{S(t) \mid |t| < T_0\})$  be a strongly continuous family in  $L(X)$ . Then  $S(\cdot)$  is a local  $K$ -convoluted  $C$ -group on  $X$  with closed subgenerator  $A$  if and only if  $\text{sgn}(\cdot)\tilde{S}(\cdot)$  is a local  $K_0$ -convoluted  $C$ -group on  $X$  with closed subgenerator  $A$ .

### 3. ABSTRACT CAUCHY PROBLEMS

In the following, we always assume that  $C \in L(X)$  is injective,  $K_0$  a kernel on  $[0, T_0)$ , and  $A$  a closed linear operator in  $X$  such that  $CA \subset AC$ . We first note some basic properties concerning the solutions of  $ACP(A, f, x)$  just as results in [13] for the case of  $A$  is the generator of a nondegenerate local  $K_0$ -convoluted  $C$ -semigroup on  $X$ .

**Proposition 3.1.** *Let  $A$  be a subgenerator of a nondegenerate local  $K_0$ -convoluted  $C$ -group  $S(\cdot)$  on  $X$ . Then for each  $x \in D(A)$ ,  $\text{sgn}(\cdot)S(\cdot)x$  is the unique solution of  $ACP(A, K_0(|\cdot|)Cx, 0)$  in  $C((-T_0, T_0), [D(A)])$ . Here  $[D(A)]$  denotes the Banach space  $D(A)$  equipped with the graph norm  $\|x\|_A = \|x\| + \|Ax\|$  for  $x \in D(A)$ .*

**Proposition 3.2.** *Let  $A$  be a subgenerator of a nondegenerate local  $K$ -convoluted  $C$ -group  $S(\cdot)$  on  $X$  and  $C^1 = \{x \in X \mid S(\cdot)x \text{ is continuously differentiable on } (-T_0, T_0)\}$ . Then*

- (i) *for each  $x \in C^1$ ,  $S(t)x \in D(A)$  for a.e.  $t \in (-T_0, T_0)$ ;*
- (ii) *for each  $x \in C^1$ ,  $S(\cdot)x$  is the unique solution of  $ACP(A, \text{sgn}(\cdot)K(|\cdot|)Cx, 0)$ ;*
- (iii) *for each  $x \in D(A)$ ,  $S(\cdot)x$  is the unique solution of  $ACP(A, \text{sgn}(\cdot)K(|\cdot|)Cx, 0)$  in  $C((-T_0, T_0), [D(A)])$ .*

**Proposition 3.3.** *Let  $A$  be the generator of a nondegenerate local  $K$ -convoluted  $C$ -group  $S(\cdot)$  on  $X$  and  $x \in X$ . Assume that  $S(t)x \in R(C)$  for all  $|t| < T_0$  and  $C^{-1}S(\cdot)x \in C((-T_0, T_0), X)$  is differentiable a.e. on  $(-T_0, T_0)$ . Then  $C^{-1}S(t)x \in D(A)$  for a.e.  $t \in (-T_0, T_0)$  and  $C^{-1}S(\cdot)x$  is the unique solution of*

$$ACP(A, \text{sgn}(\cdot)K(|\cdot|)x, 0).$$

*Proof.* Clearly,  $S(\cdot)x = CC^{-1}S(\cdot)x$  is differentiable a.e. on  $(-T_0, T_0)$ . By (1.1)–(1.4), we have

$$\begin{aligned} C \frac{d}{dt} C^{-1} S(t)x &= \frac{d}{dt} S(t)x \\ &= AS(t)x + (\text{sgn } t)K(|t|)Cx = ACC^{-1}S(t)x + (\text{sgn } t)K(|t|)Cx, \end{aligned}$$

for a.e.  $t \in (-T_0, T_0)$ . Hence, for a.e.  $t \in (-T_0, T_0)$ ,  $C^{-1}S(t)x \in D(C^{-1}AC) = D(A)$  and

$$\frac{d}{dt} C^{-1} S(t)x = (C^{-1}AC)C^{-1}S(t)x + (\text{sgn } t)K(|t|)x = AC^{-1}S(t)x + (\text{sgn } t)K(|t|)x,$$

which implies that  $C^{-1}S(\cdot)x$  is a solution of  $ACP(A, \text{sgn}(\cdot)K(|\cdot|)x, 0)$ . □

Applying Theorem 2.8, we can investigate an important result concerning the relation between the generation of a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$  with subgenerator  $A$  and the unique existence of solutions of  $ACP(A, f, x)$ , which extends some results in [13] for the case of local  $K$ -convoluted  $C$ -semigroup

**Theorem 3.1.** *The following statements are equivalent.*

- (i)  *$A$  is a subgenerator of a nondegenerate local  $K$ -convoluted  $C$ -group  $S(\cdot)$  on  $X$ .*

- (ii) For each  $x \in X$  and  $g \in L^1_{loc}((-T_0, T_0), X)$ ,  $ACP(A, K_0(|\cdot|)Cx + K_0(|\cdot|) * Cg(\cdot), 0)$  has a unique solution in  $C^1((-T_0, T_0), X) \cap C((-T_0, T_0), [D(A)])$ .
- (iii) For each  $x \in X$  the problem  $ACP(A, K_0(|\cdot|)Cx, 0)$  has a unique solution in  $C^1((-T_0, T_0), X) \cap C((-T_0, T_0), [D(A)])$ .
- (iv) For each  $x \in X$  the integral equation  $v(\cdot) = Aj_0 * v(\cdot) + K_0(|\cdot|)Cx$  has a unique solution  $v(\cdot; x)$  in  $C((-T_0, T_0), X)$ .

In this case,  $\tilde{S}(\cdot)x + \tilde{S} * g(\cdot)$  is the unique solution of  $ACP(A, K_0(|\cdot|)Cx + K_0(|\cdot|) * Cg(\cdot), 0)$  and  $v(\cdot; x) = S(\cdot)x$ .

*Proof.* We will first prove that (i) $\Rightarrow$ (ii) holds. Let  $x \in X$  and  $g \in L^1_{loc}([0, T_0), X)$  be given. We set  $u(\cdot) = \tilde{S}(\cdot)x + \tilde{S} * g(\cdot)$ , then  $u \in C^1((-T_0, T_0), X) \cap C((-T_0, T_0), [D(A)])$ ,  $u(0) = 0$ , and

$$\begin{aligned} Au(t) &= A\tilde{S}(t)x + A \int_0^t \tilde{S}(t-s)g(s)ds \\ &= S(t)x - K_0(|t|)Cx + \int_0^t [S(t-s) - K_0(|t-s|)C]g(s)ds \\ &= S(t)x + \int_0^t S(t-s)g(s)ds - [K_0(|t|)Cx + K_0(|\cdot|) * Cg(t)] \\ &= u'(t) - [K_0(|t|)Cx + K_0(|\cdot|) * Cg(t)], \end{aligned}$$

for all  $0 \leq t < T_0$ . Hence,  $u$  is a solution of  $ACP(A, K_0(|\cdot|)Cx + K_0(|\cdot|) * Cg(\cdot), 0)$  in  $C^1((-T_0, T_0), X) \cap C((-T_0, T_0), [D(A)])$ . The uniqueness of solutions for  $ACP(A, K_0(|\cdot|)Cx + K_0(|\cdot|) * Cg(\cdot), 0)$  follows directly from the uniqueness of solutions for  $ACP(A, 0, 0)$ .

Clearly, (ii) $\Rightarrow$ (iii) holds, and (iii) and (iv) both are equivalent. We remain only to show that (iv) $\Rightarrow$ (i) holds. The assumption of (iv) implies that for each  $x \in X$ ,  $v_+(\cdot) = v(\cdot; x)$  on  $[0, T_0)$  is a unique solution of the integral equation  $v(\cdot) = Aj_0 * v(\cdot) + K_0(|\cdot|)Cx$  on  $[0, T_0)$ , which together with [13, Theorem 3.4] implies that  $A$  is a subgenerator of a nondegenerate local  $K$ -convoluted  $C$ -semigroup on  $X$ . Similarly, we can show that  $-A$  is a subgenerator of a nondegenerate local  $K$ -convoluted  $C$ -semigroup on  $X$ . It follows from Remark 2.2 and Theorem 2.2 that  $A$  is a subgenerator of a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$ . □

Just as in the proof of Theorem 3.1, we can apply Remark 2.2 with [13, Theorem 3.5] to obtain the next result, and so its proof is omitted.

**Theorem 3.2.** Assume that  $R(C) \subset R(\lambda - A)$  for some  $\lambda \in \mathbb{F}$  and

$$ACP(A, \text{sgn}(\cdot)K(|\cdot|)x, 0)$$

has a unique solution in  $C((-T_0, T_0), [D(A)])$  for each  $x \in D(A)$  with  $(\lambda - A)x \in R(C)$ . Then  $A$  is a subgenerator of a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$ .

Since  $C^{-1}AC = A$  and  $R((\lambda - A)^{-1}C) = C(D(A))$  if  $\rho(A) \neq \emptyset$ , we can apply Theorem 3.2 to obtain the next corollary.

**Corollary 3.1.** *Assume that the resolvent set of  $A : D(A) \rightarrow X$  is nonempty. Then  $A$  is the generator of a nondegenerate local  $K$ -convoluted  $C$ -group on  $X$  if and only if for each  $x \in D(A)$   $ACP(A, \text{sgn}(\cdot)K(|\cdot|)Cx, 0)$  has a unique solution in  $C((-T_0, T_0), [D(A)])$ .*

Just as in the proof of Theorem 3.1, we can apply Remark 2.2 with [13, Theorem 3.7] to obtain the next result, and so its proof is omitted.

**Theorem 3.3.** *Assume that  $A$  is densely defined. Then the following are equivalent.*

- (i)  $A$  is a subgenerator of a nondegenerate local  $K$ -convoluted  $C$ -group  $S(\cdot)$  on  $X$ .
- (ii) For each  $x \in D(A)$   $ACP(A, \text{sgn}(\cdot)K(|\cdot|)Cx, 0)$  has a unique solution  $u(\cdot; Cx)$  in  $C((-T_0, T_0), [D(A)])$  which depends continuously on  $x$ . That is, if  $\{x_n\}_{n=1}^\infty$  is a Cauchy sequence in  $(D(A), \|\cdot\|)$ , then  $\{u(\cdot; Cx_n)\}_{n=1}^\infty$  converges uniformly on compact subsets of  $(-T_0, T_0)$ .

We end this paper with several illustrative examples.

*Example 3.1.* Let  $X = C_b(\mathbb{R})$ , and  $S(t)$  for  $t \in \mathbb{R}$  be bounded linear operators on  $X$  defined by  $S(t)f(x) = f(x + t)$  for all  $x \in \mathbb{R}$ . Then for each  $K \in L^1_{loc}([0, T_0], \mathbb{F})$  and  $\beta > -1$ ,  $\text{sgn}(\cdot)K_\beta(|\cdot|) * S(\cdot) = \{\text{sgn}(t)K_\beta(|\cdot|) * S(t) \mid |t| < T_0\}$  is local a  $K_\beta$ -convoluted group on  $X$  which is also nondegenerate with a closed subgenerator  $\frac{d}{dx}$  when  $K_0$  is not the zero function on  $[0, T_0)$  (or equivalently,  $K$  is not the zero in  $L^1_{loc}([0, T_0], \mathbb{F})$ ), but  $\text{sgn}(\cdot)K(|\cdot|) * S(\cdot)$  may not be a local  $K$ -convoluted group on  $X$  except for  $K \in L^1_{loc}([0, T_0], \mathbb{F})$  so that  $K * S(\cdot)$  is a strongly continuous family in  $L(X)$  for which  $\frac{d}{dx}$  is a closed subgenerator of  $\text{sgn}(\cdot)K(|\cdot|) * S(\cdot)$  when  $K_0$  is not the zero function on  $[0, T_0)$ . Moreover, (1.1)–(1.4) hold and  $\frac{d}{dx}$  is its generator and maximal subgenerator when  $K_0$  is a kernel on  $[0, T_0)$ . In this case,  $\frac{d}{dx} = \overline{A_0}$  for each subgenerator  $A_0$  of  $\text{sgn}(\cdot)K(|\cdot|) * S(\cdot)$ .

*Example 3.2.* Let  $X = C_b(\mathbb{R})$  (or  $L^\infty(\mathbb{R})$ ), and  $A$  be the maximal differential operator in  $X$  defined by  $Au = \sum_{j=0}^k a_j D^j u$  on  $\mathbb{R}$  for all  $u \in D(A)$ , then  $UC_b(\mathbb{R})$  (or  $C_0(\mathbb{R})$ ) =  $\overline{D(A)}$ .

Here  $a_0, a_1, \dots, a_k \in \mathbb{C}$  and  $D^j u(x) = u^{(j)}(x)$  for all  $x \in \mathbb{R}$ . It is shown in [2, 19] that  $\{S(t) \mid |t| < T_0\}$  defined by

$$(S(t)f)(x) = \frac{1}{\sqrt{2\pi}} \text{sgn}(t) \int_0^t \int_{-\infty}^\infty K(|t-s|) \widetilde{\phi}_s(x-y) f(y) dy ds,$$

for all  $f \in X$  and  $|t| < T_0$ , is a norm continuous local  $K_0$ -convoluted group on  $X$  with closed subgenerator  $A$  if the real-valued polynomial  $p(x) = \sum_{j=0}^k a_j (ix)^j$  satisfies  $\sup_{x \in \mathbb{R}} p(x) < \infty$ , and  $K \in L^1_{loc}([0, T_0], \mathbb{F})$  is not the zero function on  $[0, T_0)$ . Here  $\widetilde{\phi}_t$  denotes the inverse Fourier transform of  $\phi_t$  with  $\phi_t(x) = \int_0^t e^{p(x)s} ds$  for all  $t \geq 0$ . Now if  $K_0$  is a kernel on  $[0, T_0)$ , then  $A$  is its generator and maximal subgenerator. Applying Theorem 3.1, we get that for each  $f \in X$  and continuous function  $g$  on  $(-T_0, T_0) \times \mathbb{R}$  with  $\int_{-t}^t \sup_{x \in \mathbb{R}} |g(s, x)| ds < \infty$  for all  $0 \leq t < T_0$ , the function  $u$  on

$(-T_0, T_0) \times \mathbb{R}$  defined by

$$u(t, x) = \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} K_0(|t-s|) \widetilde{\phi}_s(x-y) f(y) dy ds \\ + \frac{1}{\sqrt{2\pi}} \int_0^t \int_0^{t-r} \int_{-\infty}^{\infty} K_0(|t-r-s|) \widetilde{\phi}_s(x-y) g(r, y) dy ds dr,$$

for all  $|t| < T_0$  and  $x \in \mathbb{R}$ , is the unique solution of

$$\frac{\partial u(t, x)}{\partial t} = \sum_{j=0}^k a_j \left( \frac{\partial}{\partial x} \right)^j u(t, x) + K_1(|t|) f(x) + \int_0^t K_1(|t-s|) g(s, x) ds, \\ u(0, x) = 0, \quad \text{for } t \in (-T_0, T_0) \text{ and a.e. } x \in \mathbb{R},$$

in  $C^1((-T_0, T_0), X) \cap C((-T_0, T_0), [D(A)])$ .

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