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ITERATIVE ALGORITHM OF SPLIT MONOTONE VARIATIONAL INCLUSION PROBLEM FOR NEW MAPPINGS

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Abstract. In this paper, we developed a new type iterative scheme to approximate a common solution of split monotone variational inclusion, variational inequality and fixed point problems for an infinite family of nonexpansive mappings in the framework of Hilbert spaces. Further, we proved that the sequence generated by the proposed iterative method converges strongly to a common solution of split monotone variational inclusion, variational inequality and fixed point problems. Furthermore, we give some consequences of the main result. Finally, we discuss a numerical example to demonstrate the applicability of the iterative algorithm. The result presented in this paper unifies and extends some known results in this area.

1. Introduction

Throughout the paper, let C_1 and C_2 be nonempty subsets of real Hilbert spaces H_1 and H_2, respectively.

A mapping $S_1 : C_1 \to C_1$ is said to be nonexpansive if

$$\|S_1x_1 - S_1x_2\| \leq \|x_1 - x_2\|, \quad \text{for all } x_1, x_2 \in C_1.$$

Let $\mathrm{Fix}(S_1)$ denotes the fixed points of S_1 that is $\mathrm{Fix}(S_1) = \{x_1 \in C_1 : S_1x_1 = x_1\}$.

The classical scalar nonlinear variational inequality problem (in brief, VIP) is: Find $x_1 \in C_1$ such that

$$\langle Bx_1, x_2 - x_1\rangle \geq 0, \quad \text{for all } x_2 \in C_1, \tag{1.1}$$

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where $B : C_1 \to H_1$ is a nonlinear mapping. It was introduced by Hartman and Stampacchia [10].

A mapping $T : H_1 \to H_1$ is said to be

(i) monotone, if

$$\langle Tx_1 - Tx_2, x_1 - x_2\rangle \geq 0, \quad \text{for all } x_1, x_2 \in H_1;$$

(ii) γ-inverse strongly monotone (in brief, ism), if

$$\langle Tx_1 - Tx_2, x_1 - x_2\rangle \ \geq \gamma\|Tx_1 - Tx_2\|^2, \quad \text{for all } x_1, x_2 \in H_1 \text{ and } \gamma > 0;$$

(iii) firmly nonexpansive, if

$$\langle Tx_1 - Tx_2, x_1 - x_2\rangle \ \geq \|Tx_1 - Tx_2\|^2, \quad \text{for all } x_1, x_2 \in H_1;$$

(iv) L-Lipschitz continuous, if

$$\|Tx_1 - Tx_2\| \leq L\|x_1 - x_2\|, \quad \text{for all } x_1, x_2 \in H_1 \text{ and } L > 0.$$

A set valued mapping $M_1 : H_1 \to 2^{H_1}$ is called monotone if for every $x_1, x_2 \in H_1$, $u_1 \in M_1x_1$ and $u_2 \in M_1x_2$ such that

$$\langle x_1 - x_2, u_1 - u_2\rangle \ \geq 0.$$

And it is maximal if $G(M_1)$, graph of M_1 defined as $G(M_1) = \{(x_1, u_1) : u_1 \in M_1x_1\}$ does not contain properly in the graph of other. Note that, M_1 is maximal if and only if for $(x_1, u_1) \in H_1 \times H_1$, $\langle x_1 - x_2, u_1 - u_2\rangle \ \geq 0$, for all $(x_2, u_2) \in G(M_1)$ implies $u_1 \in M_1x_1$.

An operator $J^{M_1}_{\rho_1} : H_1 \to H_1$ is defined as

$$J^{M_1}_{\rho_1}x_1 = (I + \rho_1M_1)^{-1}x_1, \quad \text{for all } x_1 \in H_1,$$

known as resolvent operator, where $\rho_1 > 0$ and I stands for identity mapping on H_1.

In this paper, we consider the split monotone variational inclusion problem (in brief, SpMVIP). Find $\tilde{x} \in H_1$ such that

$$0 \in g_1(\tilde{x}) + M_1(\tilde{x}) \tag{1.2}$$

and

$$\tilde{y} = D\tilde{x} \in H_2 \text{ solves } 0 \in g_2(\tilde{y}) + M_2(\tilde{y}), \tag{1.3}$$

where $g_1 : H_1 \to H_1$, $g_2 : H_2 \to H_2$ be inverse strongly monotone mappings, $D : H_1 \to H_2$ be a bounded linear mapping and $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings, which is introduced by Moudafi [17]. Let $\Lambda = \{\tilde{x} \in H_1 : \tilde{x} \in \text{Sol(MVIP(1.2))} \text{ and } D\tilde{x} \in \text{Sol(MVIP(1.3))}\}$ denote the solution of SpMVIP (1.2)–(1.3).

The split feasibility, split zero and the split fixed point problems include as a special cases. It studied broadly by various authors and solved real life problems essentially in modelling of inverse problems, sensor networks in computerised tomography and radiation therapy; for details [3, 5, 7].

If $g_1 \equiv 0$ and $g_2 \equiv 0$ then we find a split null point problem (in brief, $\mathrm{S_P}$NPP):Find $\tilde{x} \in H_1$ such that

$$0 \in M_1(\tilde{x}) \tag{1.4}$$

and

$$\tilde{y} = D\tilde{x} \in H_2 \text{ solves } 0 \in M_2(\tilde{y}). \tag{1.5}$$

The iterative algorithm for $\mathrm{S_P}$MVIP (1.2)–(1.3) was introduced and studied by Moudafi [17]:

$$x_0 \in H_1, \quad x_{n+1} = P(x_n + \eta D^*(Q - I)Dx_n), \quad \text{for } \rho > 0,$$

where $P := J_\rho^{M_1}(I - \rho g_1)$, $Q := J_\rho^{M_2}(I - \rho g_2)$, D^* be the adjoint operator of D and $0 < \eta < \frac{1}{\varsigma}$, ς be the spectral radius of D^*D.

The convergence analysis was studied by Byrne et al. [4] of some iterative algorithm for $\mathrm{S_P}$NPP (1.4)–(1.5). Moreover, Kazmi et al. [15] established an iterative method to find a common solution of $\mathrm{S_P}$NPP (1.4)–(1.5) and fixed point problem. For instance, see [1, 12–14, 20–22].

Recently, Qin et al. [19] proposed an algorithm for infinite family of nonexpansive mappings as:

$$x_0 \in C_1,\ x_{n+1} = \mu_n \theta g(x_n) + \eta_n x_n + ((1 - \eta_n)I - \mu_n D)\mathbb{W}_n u_n,$$

where g be a contraction mapping on H_1, D be a strongly positive bounded linear operator, $\mathbb{W}_n$ generated by $S_1, S_2, \ldots$ as:

$$\begin{aligned}
\mathbb{V}_{n,n+1} &:= I,\\
\mathbb{V}_{n,n} &:= \lambda_n S_n \mathbb{V}_{n,n+1} + (1 - \lambda_n)I,\\
\mathbb{V}_{n,n-1} &:= \lambda_{n-1} S_{n-1} \mathbb{V}_{n,n} + (1 - \lambda_{n-1})I,\\
&\ \ \vdots\\
\mathbb{V}_{n,m} &:= \lambda_m S_m \mathbb{V}_{n,m+1} + (1 - \lambda_m)I,\\
\mathbb{V}_{n,m-1} &:= \lambda_{m-1} S_{m-1} \mathbb{V}_{n,m} + (1 - \lambda_{m-1})I,\\
&\ \ \vdots\\
\mathbb{V}_{n,2} &:= \lambda_2 S_2 \mathbb{V}_{n,3} + (1 - \lambda_3)I,\\
\mathbb{W}_n \equiv \mathbb{V}_{n,1} &:= \lambda_1 S_1 \mathbb{V}_{n,2} + (1 - \lambda_1)I,
\end{aligned} \tag{1.6}$$

where $S_1, S_2, \ldots, \mathbb{W}_n$ are nonexpansive mappings, $\{\lambda_n\} \subset (0, 1]$, for $n \geq 1$. For further work see [8, 11].

Inspirited by Moudafi [17], Byrne et al. [4], Kazmi et al. [14, 15], Qin et al. [19] and by continuing work, we propose and analyze a new type iterative algorithm to find a common solution of split monotone variational inclusion, variational inequality and fixed point problems for an infinite family of nonexpansive mappings in the framework of Hilbert spaces. Further, we endowed that the sequence generated by the algorithm converges strongly to common solution. Furthermore, we listed some consequences of our established theorem. Finally, we provide a numerical example to demonstrate the applicability of algorithm. We emphasize that the result accounted in manuscript is unifies and extends of various results in this field of study.

2. Preliminaries

This section is devoted to recall few definitions, entailing mathematical tools and helpful results that are required in the sequel.

To each $x_1 \in H_1$, there exists a unique nearest point $P_{C_1}x_1$ to x_1 in C_1 such that

$$\|x_1 - P_{C_1}x_1\| \leq \|x_1 - x_2\|, \quad \text{for all } x_2 \in C_1,$$

where P_{C_1} is a metric projection of H_1 onto C_1. Also, P_{C_1} is nonexpansive and holds

$$\langle x_1 - x_2, P_{C_1}x_1 - P_{C_1}x_2\rangle \geq \|P_{C_1}x_1 - P_{C_1}x_2\|^2, \quad \text{for all } x_1, x_2 \in H_1.$$

Moreover, $P_{C_1}x_1$ is characterized by the fact that $P_{C_1}x_1 \in C_1$ and

$$\langle x_1 - P_{C_1}x_1, x_2 - P_{C_1}x_1\rangle \leq 0, \quad \text{for all } x_2 \in C_1.$$

This implies that

$$\|x_1 - x_2\|^2 \geq \|x_1 - P_{C_1}x_1\|^2 + \|x_2 - P_{C_1}x_1\|^2, \quad \text{for all } x_1 \in H_1, \text{for all } x_2 \in C_1,$$

and

$$\|\mu x_1 + (1-\mu)x_2\|^2 = \mu\|x_1\|^2 + (1-\mu)\|x_2\|^2 - \mu(1-\mu)\|x_1 - x_2\|^2,$$

for all $x_1, x_2 \in H_1$ and $\mu \in [0,1]$.

Also, on H_1 holds following inequalities.

(a) Opial's condition [18], that is for any $\{x_n\}$ with $x_n \rightharpoonup x_1$ and

$$\liminf_{n\to\infty} \|x_n - x_1\| < \liminf_{n\to\infty} \|x_n - x_2\|,$$

holds, for all $x_2 \in H_1$ with $x_2 \neq x_1$.

(b)

$$\|x_1 + x_2\|^2 \leq \|x_1\|^2 + 2\langle x_2, x_1 + x_2\rangle, \quad \text{for all } x_1, x_2 \in H_1. \tag{2.1}$$

Definition 2.1. ([2]) A mapping $T_1 : H_1 \to H_1$ is called averaged if and only if

$$T_1 = (1-\lambda)I + \lambda S_1,$$

where $\lambda \in (0,1)$, I be the identity mapping on H_1 and $S_1 : H_1 \to H_1$ be nonexpansive mapping.

Lemma 2.1. ([17])

(i) *If $T_2 = (1-\lambda)T_1 + \lambda S_1$, where $T_1 : H_1 \to H_1$ be averaged, $S_1 : H_1 \to H_1$ be nonexpansive and $0 < \lambda < 1$, then T_2 is averaged.*

(ii) *If T_1 is γ-ism, then βT_1 is $\frac{\gamma}{\beta}$-ism, for $\beta > 0$.*

(iii) *T_1 is averaged if and only if $I - T$ is γ-ism for some $\gamma > \frac{1}{2}$.*

Lemma 2.2. ([17]) *Let $\rho > 0$, f be a γ-ism and M be a maximal monotone mapping. If $\rho \in (0, 2\gamma)$, then $J_\rho^M(I - \rho f)$ is averaged.*

Lemma 2.3. ([17]) *Let $\rho_1, \rho_2 > 0$ and M_1, M_2 be maximal monotone mapping. Then*

$$\tilde{x} \text{ solves (1.2)–(1.3)} \Leftrightarrow \tilde{x} = J_{\rho_1}^{M_1}(I - \rho_1 f_1)\tilde{x} \text{ and } B\tilde{x} = J_{\rho_2}^{M_2}(I - \rho_2 f_2)B\tilde{x}.$$

Lemma 2.4. ([24]) *Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in E, a Banach space and let $0 < \mu_n < 1$ with $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$. Consider $v_{n+1} = (1 - \mu_n)v_n + \mu_n u_n$, $n \geq 0$, and $\limsup_{n\to\infty}(\|v_{n+1} - v_n\| - \|u_{n+1} - u_n\|) \leq 0$. Then*

$$\lim_{n\to\infty} \|v_n - u_n\| = 0.$$

Lemma 2.5. ([16]) *Assume that B is a strongly positive self-adjoint bounded linear operator on H_1 with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.6. ([25]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers with*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \alpha_n, \quad n \geq 0,$$

where $\lambda_n \in (0, 1)$ and $\{\alpha_n\}$ in $\mathbb{R}$ with

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\alpha_n}{\lambda_n} \leq 0$ or $\sum_{n=1}^{\infty} |\alpha_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. ([9]) *Let $S_1 : C_1 \to H_1$ be a nonexpansive mapping. If S_1 has a fixed point, then $(I - S_1)$, where I be the identity mapping, be demiclosed that is if $x_n \rightharpoonup x_1 \in H_1$ and $x_n - S_1 x_n \to x_2$, then $(I - S_1)x_1 = x_2$.*

Lemma 2.8. ([23]) *Let $C_1 \neq \emptyset$ be closed convex subset of a strictly convex Banach space E. Let $S_1, S_2, \ldots$ be nonexpansive mappings of C_1 to C_1 such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers satisfying $0 < \lambda_i < 1$ for all $i \geq 1$. Then $\lim_{i\to\infty} \mathbb{V}_{i,j}\tilde{x}$ exists for all $\tilde{x} \in C_1$ and $j \in \mathbb{N}$.*

Remark 2.1. By Lemma 2.8, define a mapping $\mathbb{W} : C_1 \to C_1$ such that $\mathbb{W}\tilde{x} = \lim_{i\to\infty} \mathbb{W}_i\tilde{x} = \lim_{i\to\infty} \mathbb{V}_{i,1}\tilde{x}$ for all $\tilde{x} \in C_1$, which is called the $\mathbb{W}$-mapping generated by $S_1, S_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$ In the whole paper, we consider $0 < \lambda_i < 1$ for all $i \geq 1$.

Lemma 2.9. ([23]) *Let $C_1 \neq \emptyset$ be closed convex subset of a strictly convex Banach space E. Let $S_1, S_2, \ldots$ be nonexpansive mappings of C_1 to C_1 such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers satisfying $0 < \lambda_i < 1$ for all $i \geq 1$. Then $\mathrm{Fix}(\mathbb{W}) = \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$.*

Lemma 2.10. ([6]) *Let $C_1 \neq \emptyset$ be closed convex subset of H_1. Let $S_1, S_2, \dots$ be nonexpansive mappings of C_1 to C_1 such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \dots$ be real numbers satisfying $0 < \lambda_i < 1$ for all $i \geq 1$. If K be any bounded bounded subset of C_1, then $\lim_{i\to\infty} \sup_{\tilde{x}\in K} \|\mathbb{W}_i\tilde{x} - \mathbb{W}\tilde{x}\| = 0$.*

3. Main Result

We study the following convergence result for a new type iterative method to find a common solution of $\mathrm{S_P}$MVIP (1.2)–(1.3), VIP (1.1) and fixed point problem.

Theorem 3.1. *Let H_1 and H_2 denote the Hilbert spaces and $C_1 \subset H_1$ be nonempty closed convex subset of H_1. Let $B : H_1 \to H_1$ be a γ-inverse strongly monotone mapping, $D : H_1 \to H_2$ be a bounded linear operator with its adjoint operator D^*, $M_1 : H_1 \to 2^{H_1}$,$M_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone operators and $g_1 : H_1 \to H_1$, $g_2 : H_2 \to H_2$ be α_1, α_2-inverse strongly monotone mappings, respectively. Let $f : C_1 \to C_1$ be a contraction mapping with constant $\tau \in (0, 1)$, A be a strongly positive bounded linear self adjoint operator on C_1 with constant $\bar{\theta} > 0$ such that $0 < \theta < \frac{\bar{\theta}}{\tau} < \theta + \frac{1}{\tau}$ and $\{S_i\}_{i=1}^{\infty} : C_1 \to C_1$ be an infinite family of nonexpansive mappings such that $\Gamma := \Lambda \cap \mathrm{Sol(VIP(1.1))} \cap (\cap_{i=1}^{\infty}\mathrm{Fix}(S_i)) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated as:*

$$\left.\begin{aligned}
&x_1 \in C_1,\\
&z_n = R(I + \xi D^*(S - I)D)x_n,\\
&u_n = P_{C_1}(z_n - \sigma_n B z_n),\\
&v_n = \delta_n u_n + (1 - \delta_n)\mathbb{W}_n u_n,\\
&x_{n+1} = \mu_n \theta f(x_n) + \eta_n x_n + ((1 - \eta_n)I - \mu_n A)v_n, \quad n \geq 1,
\end{aligned}\right\}$$

where $R = J_{\rho_1}^{(g_1,M_1)}(I - \rho_1 g_1)$, $S = J_{\rho_2}^{(g_2,M_2)}(I - \rho_2 g_2)$, $\mathbb{W}_n$ defined in (1.6), $\{\mu_n\}$, $\{\eta_n\}$, $\{\delta_n\} \subset (0, 1)$ and $\xi \in (0, \frac{1}{\epsilon})$, ϵ be the spectral radius of D^*D. Let the control sequences satisfying conditions:

(i) $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n=0}^{\infty} \mu_n = \infty$;
(ii) $0 < \rho_1 < 2\alpha_1$, $0 < \rho_2 < 2\alpha_2$;
(iii) $0 < \liminf_{n\to\infty} \eta_n \leq \limsup_{n\to\infty} \eta_n < 1$;
(iv) $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 2\gamma$;
(v) $\lim_{n\to\infty} \delta_n = 0$.

Then, the sequence $\{x_n\}$ converges strongly to some $\tilde{x} \in \Gamma$, where $\tilde{x} = P_\Gamma(\theta f + (I - A))\tilde{x}$ which solves:

$$\langle (A - \theta f)\tilde{x}, v - \tilde{x}\rangle \geq 0, \quad \text{for all } v \in \Gamma.$$

Proof. For sake of simplicity, we divide the proof into several steps.

Step 1. We prove that $\{x_n\}$ is bounded.

Let $\tilde{x} \in \Gamma$ then $\tilde{x} \in \Lambda$ and thus $R\tilde{x} = \tilde{x}$, $S(D\tilde{x}) = D\tilde{x}$ and $P\tilde{x} = \tilde{x}$, where $P = I + \eta D^*(S - I)D$. By Lemma 2.2 and firmly nonexpansive, R and S are averaged.

Also, P is averaged since it is $\frac{\nu}{\epsilon}$-ism for some $\nu > \frac{1}{2}$. From Lemma 2.1 (iii), $I-S$ is ν-ism. Thus, we obtain

$$\begin{aligned}\langle D^*(I-S)Dx_1 - D^*(I-S)Dx_2, x_1 - x_2\rangle =&\langle (I-S)Dx_1 - (I-S)Dx_2, Dx_1 - Dx_2\rangle \\ \geq&\nu\|(I-S)Dx_1 - (I-S)Dx_2\|^2 \\ \geq&\frac{\nu}{\epsilon}\|D^*(I-S)Dx_1 - D^*(I-S)Dx_2\|^2.\end{aligned}$$

This implies that $\eta D^*(I-S)D$ is $\frac{\nu}{\xi\epsilon}$-ism. Since $0 < \xi < \frac{1}{\epsilon}$ therefore its complement $(I - \xi D^*(I-S)D)$ is averaged and hence $R(I + \xi D^*(S-I)D) = \mathbb{Z}$(say). Thus, $I + \xi D^*(S-I)D$, R, S and $\mathbb{Z}$ are nonexpansive mappings.

Next, we calculate

$$\begin{aligned}\|z_n - \tilde{x}\|^2 &= \|J_{\rho_1}^{g_1,M_1}(I-\rho_1 g_1)(x_n + \xi D^*(S-I)Dx_n) - J_{\rho_1}^{g_1,M_1}(I-\rho_1 g_1)\tilde{x}\|^2 \\ &\leq \|x_n + \xi D^*(S-I)Dx_n - \tilde{x}\|^2 \\ &= \|x_n - \tilde{x}\|^2 + \xi^2\|D^*(S-I)Dx_n\|^2 + 2\xi\langle x_n - \tilde{x}, D^*(S-I)Dx_n\rangle. \end{aligned} \tag{3.1}$$

Now,

$$\begin{aligned}\xi^2\|D^*(S-I)Dx_n\|^2 &= \xi^2\langle (S-I)Dx_n, DD^*(S-I)Dx_n\rangle \\ &\leq \epsilon\xi^2\langle (S-I)Dx_n, (S-I)Dx_n\rangle \\ &= \epsilon\xi^2\|(S-I)Dx_n\|^2. \end{aligned} \tag{3.2}$$

Consider $\Upsilon_n := 2\xi\langle x_n - \tilde{x}, D^*(S-I)Dx_n\rangle$ and we estimate

$$\begin{aligned}\Upsilon_n &= 2\xi\langle x_n - \tilde{x}, D^*(S-I)Dx_n\rangle \\ &= 2\xi\langle D(x_n - \tilde{x}) + (S-I)Dx_n - (S-I)Dx_n, (S-I)Dx_n\rangle \\ &= 2\xi[\langle S(D(x_n) - D\tilde{x}, (S-I)Dx_n\rangle - \|(S-I)Dx_n\|^2] \\ &\leq 2\xi\left[\frac{1}{2}\|(S-I)Dx_n\|^2 - \|(S-I)Dx_n\|^2\right] \\ &= -\xi\|(S-I)Dx_n\|^2. \end{aligned} \tag{3.3}$$

From (3.1), (3.2), (3.3), we obtain

$$\|z_n - \tilde{x}\|^2 \leq \|x_n - \tilde{x}\|^2 + \xi(\epsilon\xi - 1)\|(S-I)Dx_n\|^2. \tag{3.4}$$

Since $0 < \xi < \frac{1}{\epsilon}$, therefore

$$\|z_n - \tilde{x}\| \leq \|x_n - \tilde{x}\|. \tag{3.5}$$

Using γ-ism and $0 < \sigma_n < 2\gamma$, we have

$$\begin{aligned}\|u_n - \tilde{x}\|^2 &= \|P_{C_1}(z_n - \sigma_n Bz_n) - P_{C_1}(z_n - \sigma_n B\tilde{x})\|^2 \\ &\leq \|z_n - \sigma_n Bz_n - (z_n - \sigma_n B\tilde{x})\|^2 \\ &= \|(z_n - \tilde{x}) - \sigma_n(Bz_n - B\tilde{x})\|^2 \\ &= \|z_n - \tilde{x}\|^2 - 2\sigma_n\langle Bz_n - B\tilde{x}, z_n - \tilde{x}\rangle + \sigma_n^2\|Bz_n - B\tilde{x}\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq \|z_n - \tilde{x}\|^2 - 2\sigma_n\gamma\|Bz_n - B\tilde{x}\|^2 + \sigma_n^2\|Bz_n - B\tilde{x}\|^2 \\
&= \|z_n - \tilde{x}\|^2 + \sigma_n(\sigma_n - 2\gamma)\|Bz_n - B\tilde{x}\|^2 \\
&\leq \|z_n - \tilde{x}\|^2, \qquad (3.6)
\end{aligned}
$$

this implies

$$\|u_n - \tilde{x}\| \leq \|z_n - \tilde{x}\|. \qquad (3.7)$$

By using (3.5) and (3.6), we calculate

$$
\begin{aligned}
\|v_n - \tilde{x}\| &\leq \delta_n\|u_n - \tilde{x}\| + (1-\delta_n)\|\mathbb{W}_n u_n - \tilde{x}\| \\
&= \delta_n\|u_n - \tilde{x}\| + (1-\delta_n)\|u_n - \tilde{x}\| \\
&\leq \|u_n - \tilde{x}\| \\
&= \|z_n - \tilde{x}\| \\
&= \|x_n - \tilde{x}\|. \qquad (3.8)
\end{aligned}
$$

By using (3.7) and (3.8), we calculate

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\| &= \|\mu_n\theta f(x_n) + \eta_n x_n + ((1-\eta_n)I - \mu_n A)v_n - \tilde{x}\| \\
&= \|\mu_n(\theta f(x_n) - A\tilde{x}) + \eta_n(x_n - \tilde{x}) \\
&\quad + ((1-\eta_n)I - \mu_n A)(v_n - \tilde{x})\| \\
&\leq \mu_n\|\theta f(x_n) - A\tilde{x}\| + \eta_n\|x_n - \tilde{x}\| \\
&\quad + ((1-\eta_n)I - \mu_n\overline{\theta})\|v_n - \tilde{x}\| \\
&\leq \mu_n\|\theta f(x_n) - \theta f(\tilde{x}) + \theta f(\tilde{x}) - A\tilde{x}\| \\
&\quad + \eta_n\|x_n - \tilde{x}\| + ((1-\eta_n)I - \mu_n\overline{\theta})\|u_n - \tilde{x}\| \\
&\quad \leq \mu_n\theta\|f(x_n) - f(\tilde{x})\| + \mu_n\|\theta f(\tilde{x}) - A\tilde{x}\| \\
&\quad + \eta_n\|x_n - \tilde{x}\| + ((1-\eta_n)I - \mu_n\overline{\theta})\|x_n - \tilde{x}\| \\
&\leq \mu_n\theta\tau\|x_n - \tilde{x}\| + \mu_n\|\theta f(\tilde{x}) - A\tilde{x}\| \\
&\quad + (1 - \mu_n\overline{\theta})\|x_n - \tilde{x}\| \\
&\leq (1 - \mu_n(\overline{\theta} - \theta\tau))\|x_n - \tilde{x}\| + \mu_n\|\theta f(\tilde{x}) - A\tilde{x}\| \\
&\leq \max\left\{\|x_n - \tilde{x}\|, \frac{\|\theta f(\tilde{x}) - A\tilde{x}\|}{\overline{\theta} - \theta\tau}\right\}, \quad n \geq 1.
\end{aligned}
$$

Using induction, we get

$$\|x_{n+1} - \tilde{x}\| \leq \max\left\{\|x_1 - \tilde{x}\|, \frac{\|\theta f(\tilde{x}) - A\tilde{x}\|}{\overline{\theta} - \theta\tau}\right\}.$$

Thus, $\{x_n\}$ is bounded. Also, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$, $\{f(x_n)\}$ and $\{\mathbb{W}(x_n)\}$ are bounded due to (3.4), (3.7) and (3.8).

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty}\|x_n - \mathbb{W}_n u_n\| = 0$, $\lim_{n\to\infty}\|v_n - x_n\| = 0$ and $\lim_{n\to\infty}\|v_n - u_n\| = 0$.

Since $R(I + \xi D^*(S - I)D)$ is nonexpansive therefore

$$\begin{aligned}\|z_{n+1} - z_n\| &= \|R(I + \xi D^*(S - I)D)x_{n+1} - R(I + \xi D^*(S - I)D)x_n\| \\ &\leq \|x_{n+1} - x_n\|. \end{aligned} \tag{3.9}$$

Using (3.9), we estimate

$$\begin{aligned}\|u_{n+1} - u_n\| &= \|P_C(I - \sigma_{n+1}B)z_{n+1} - P_C(I - \sigma_n B)z_n\| \\ &\leq \|(I - \sigma_{n+1}B)z_{n+1} - (I - \sigma_n B)z_n\| \\ &= \|(I - \sigma_{n+1}B)z_{n+1} - (I - \sigma_{n+1}B)z_n + (\sigma_n - \sigma_{n+1})Av_n\| \\ &\leq \|z_{n+1} - z_n\| + |\sigma_n - \sigma_{n+1}|\|Bz_n\| \\ &\leq \|x_{n+1} - x_n\| + |\sigma_n - \sigma_{n+1}|\|Bz_n\| \\ &\leq \|x_{n+1} - x_n\| + \mathbb{N}_1|\sigma_n - \sigma_{n+1}|, \end{aligned} \tag{3.10}$$

where $\mathbb{N}_1 = \sup_{n\geq 1} \|Bz_n\|$.

For $i \in 1, 2, \ldots, n$, S_i and $\mathbb{V}_{n,i}$, are nonexpansive therefore from (1.6), we obtain

$$\begin{aligned}\|\mathbb{W}_{n+1}u_n - \mathbb{W}_n u_n\| &= \|\lambda_1 S_1 \mathbb{V}_{n+1,2}u_n - \lambda_1 S_1 \mathbb{V}_{n,2}u_n\| \\ &\leq \lambda_1\|\mathbb{V}_{n+1,2}u_n - \mathbb{V}_{n,2}u_n\| \\ &\leq \lambda_1\|\lambda_2 S_2 \mathbb{V}_{n+1,3}u_n - \lambda_2 S_2 \mathbb{V}_{n,3}u_n\| \\ &\leq \lambda_1\lambda_2\|\mathbb{V}_{n+1,3}u_n - \mathbb{V}_{n,3}u_n\| \\ &\vdots \\ &\leq \lambda_1\lambda_2 \cdots \lambda_n\|\mathbb{V}_{n+1,n+1}u_n - \mathbb{V}_{n,n+1}u_n\| \\ &\leq \mathbb{N}_2 \prod_{i=1}^{n} \lambda_i, \end{aligned} \tag{3.11}$$

where $\mathbb{N}_2 \geq 0$ with $\|\mathbb{V}_{n+1,n+1}u_n - \mathbb{V}_{n,n+1}u_n\| \leq \mathbb{N}_2$ for all $n \geq 1$.

Using (3.10) and (3.11), we estimate

$$\begin{aligned}\|v_{n+1} - v_n\| \leq& \|\delta_{n+1}u_{n+1} + (1 - \delta_{n+1})\mathbb{W}_{n+1}u_{n+1} - \delta_n u_n - (1 - \delta_n)\mathbb{W}_n u_n\| \\ \leq& \|\delta_{n+1}u_{n+1} + (1 - \delta_{n+1})\mathbb{W}_{n+1}u_{n+1} - \delta_n u_n - (1 - \delta_n)\mathbb{W}_n u_n \\ &+ (1 - \delta_n)\mathbb{W}_{n+1}u_n - (1 - \delta_n)\mathbb{W}_{n+1}u_n\| \\ \leq& (1 - \delta_n)\|\mathbb{W}_{n+1}u_n - \mathbb{W}_n u_n\| + \|\mathbb{W}_{n+1}u_{n+1} - \mathbb{W}_{n+1}u_n\| \\ &+ \delta_{n+1}\|\mathbb{W}_{n+1}u_{n+1} - u_{n+1}\| + \delta_n\|\mathbb{W}_{n+1}u_n - u_n\| \\ \leq& (1 - \delta_n)\mathbb{N}_2 \prod_{i=1}^{n} \lambda_i + \|u_{n+1} - u_n\| + \delta_{n+1}\mathbb{N}_3 + \delta_n\mathbb{N}_4 \\ \leq& (1 - \delta_n)\mathbb{N}_2 \prod_{i=1}^{n} \lambda_i + \|x_{n+1} - x_n\| + \mathbb{N}_1|\sigma_n - \sigma_{n+1}| + \delta_{n+1}\mathbb{N}_3 + \delta_n\mathbb{N}_4, \end{aligned} \tag{3.12}$$

where $\mathbb{N}_3 = \sup_{n\geq 1}\|\mathbb{W}_{n+1}u_{n+1} - u_{n+1}\|$ and $\mathbb{N}_4 = \sup_{n\geq 1}\|\mathbb{W}_{n+1}u_n - u_n\|$. Setting $x_{n+1} = (1-\eta_n)s_n + \eta_n x_n$, then we have $s_n = \frac{x_{n+1}-\eta_n x_n}{1-\eta_n}$ and

$$\begin{aligned} s_{n+1} - s_n =& \frac{\mu_{n+1}\theta f(x_{n+1}) + ((1-\eta_{n+1})I - \mu_{n+1}A)v_{n+1}}{1-\eta_{n+1}} \\ &- \frac{\mu_n\theta f(x_n) + ((1-\eta_n)I - \mu_n A)v_n}{1-\eta_n} \\ =& \frac{\mu_{n+1}}{1-\eta_{n+1}}(\theta f(x_{n+1}) - Av_{n+1}) + \frac{\mu_n}{1-\eta_n}(Av_n - \theta f(x_n)) + v_{n+1} - v_n. \end{aligned}$$

Hence,

$$\begin{aligned} \|s_{n+1} - s_n\| \leq& \frac{\mu_{n+1}}{1-\eta_{n+1}}(\|\theta f(x_{n+1})\| + \|Av_{n+1}\|) \\ &+ \frac{\mu_n}{1-\eta_n}(\|Av_n\| + \|\theta f(x_n)\|) + \|v_{n+1} - v_n\| \\ \leq& \frac{\mu_{n+1}}{1-\eta_{n+1}}\mathbb{N}_5 + \frac{\mu_n}{1-\eta_n}\mathbb{N}_6 + \|v_{n+1} - v_n\|, \end{aligned}$$

where $\mathbb{N}_5 = \sup_{n\geq 1}(\|\theta f(x_{n+1})\| + \|Av_{n+1}\|)$ and $\mathbb{N}_6 = \sup_{n\geq 1}(\|Av_n\| + \|\theta f(x_n)\|)$.

Using (3.12) in above inequality

$$\begin{aligned} \|s_{n+1} - s_n\| \leq& \frac{\mu_{n+1}}{1-\eta_{n+1}}\mathbb{N}_5 + \frac{\mu_n}{1-\eta_n}\mathbb{N}_6 + (1-\delta_n)\mathbb{N}_2\prod_{i=1}^{n}\lambda_i \\ &+ \|x_{n+1} - x_n\| + \mathbb{N}_1|\sigma_n - \sigma_{n+1}| + \delta_{n+1}\mathbb{N}_3 + \delta_n\mathbb{N}_4, \end{aligned}$$

and thus

$$\begin{aligned} \|s_{n+1} - s_n\| - \|x_{n+1} - x_n\| \leq& \frac{\mu_{n+1}}{1-\eta_{n+1}}\mathbb{N}_5 + \frac{\mu_n}{1-\eta_n}\mathbb{N}_6 \\ &+ \mathbb{N}_1|\sigma_n - \sigma_{n+1}| + (1-\delta_n)\mathbb{N}_2\prod_{i=1}^{n}\lambda_i + \delta_{n+1}\mathbb{N}_3 + \delta_n\mathbb{N}_4. \end{aligned}$$

Using the given conditions in above inequality, we have

$$\limsup_{n\to\infty}(\|s_{n+1} - s_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

By Lemma 2.4, we get

$$\lim_{n\to\infty}\|s_n - x_n\| = 0.$$

As $x_{n+1} = (1-\eta_n)s_n + \eta_n x_n$, therefore

$$\|x_{n+1} - x_n\| = \|(1-\eta_n)(s_n - x_n)\|,$$

which yields

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.13}$$

Now,

$$\begin{aligned} \|x_n - \mathbb{W}_n u_n\| =& \|x_n - x_{n+1} + x_{n+1} - \mathbb{W}_n u_n\| \\ \leq& \|x_{n+1} - x_n\| + \|\mu_n\theta f(x_n) + \eta_n x_n \end{aligned}$$

$$
\begin{aligned}
&+ ((1-\eta_n)I - \mu_n A)v_n - \mathbb{W}_n u_n\| \\
=&\|x_{n+1} - x_n\| + \|\mu_n(\theta f(x_n) - Av_n)\| \\
&+ ((1-\eta_n)I - \mu_n A)(v_n - \mathbb{W}_n u_n) + \eta_n(x_n - \mathbb{W}_n u_n) \\
\le&\|x_{n+1} - x_n\| + \mu_n\|\theta f(x_n) - Av_n\| + \eta_n\|x_n - \mathbb{W}_n u_n\|. \qquad (3.14)
\end{aligned}
$$

Hence,

$$(1-\eta_n)\|x_n - \mathbb{W}_n u_n\| \le \|x_{n+1} - x_n\| + \mu_n\|\theta f(x_n) - Av_n\|.$$

Using the given conditions and (3.13) in (3.14), we get

$$\lim_{n\to\infty} \|x_n - \mathbb{W}_n u_n\| = 0. \qquad (3.15)$$

By (3.5) and (3.7), we compute

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 =&\|\mu_n\theta f(x_n) + \eta_n x_n + ((1-\eta_n)I - \mu_n A)v_n - \tilde{x}\|^2 \\
\le&\|(1-\eta_n)(v_n - \tilde{x}) + \eta_n(x_n - \tilde{x})\|^2 \\
&+ 2\langle \mu_n\theta f(x_n) - \mu_n Av_n, x_{n+1} - \tilde{x}\rangle \\
\le&(1-\eta_n)\|v_n - \tilde{x}\|^2 + \eta_n\|x_n - \tilde{x}\|^2 \\
&- \eta_n(1-\eta_n)\|x_n - v_n\|^2 \\
&+ 2\mu_n\|\theta f(x_n) - Av_n\|\|x_{n+1} - \tilde{x}\| \\
\le&\eta_n\|x_n - \tilde{x}\|^2 + (1-\eta_n)\|v_n - \tilde{x}\|^2 + 2\mu_n\mathbb{N}_7, \qquad (3.16)
\end{aligned}
$$

where $\mathbb{N}_7 = \max\{\sup_{n\ge1}\|\theta f(x_n) - Av_n\|, \sup_{n\ge1}\|x_{n+1} - \tilde{x}\|\}$. From (3.4) and (3.8), we get

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 \le&\eta_n\|x_n - \tilde{x}\|^2 + (1-\eta_n)\|x_n - \tilde{x}\|^2 \\
&+ (1-\eta_n)\xi(\epsilon\xi - 1)\|(S-I)Dx_n\|^2 + 2\mu_n\mathbb{N}_7 \\
\le&\|x_n - \tilde{x}\|^2 + (1-\eta_n)\xi(\epsilon\xi - 1)\|(S-I)Dx_n\|^2 + 2\mu_n\mathbb{N}_7,
\end{aligned}
$$

which yields

$$(1-\eta_n)\xi(1-\epsilon\xi)\|(S-I)Dx_n\|^2 \le \|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 + 2\mu_n\mathbb{N}_7.$$

Since $\epsilon(1-\epsilon\xi) > 0$, $\lim_{n\to\infty}\mu_n = 0$ and $\{x_n\}$, $\{u_n\}$ are bounded, and using (3.13), we have

$$\lim_{n\to\infty} \|(S-I)Dx_n\| = 0. \qquad (3.17)$$

Next, prove that $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

By using firmly nonexpansive of $J_{\rho_1}^{(g_1,M_1)}$, we compute

$$
\begin{aligned}
\|z_n - \tilde{x}\|^2 =&\|J_{\rho_1}^{(g_1,M_1)}(x_n + \xi D^*(S-I)Dx_n) - J_{\rho_1}^{(g_1,M_1)}\tilde{x}\|^2 \\
\le&\langle z_n - \tilde{x}, x_n + \xi D^*(S-I)Dx_n - \tilde{x}\rangle \\
=&\frac{1}{2}\Big\{\|z_n - \tilde{x}\|^2 + \|x_n + \xi D^*(S-I)Dx_n - \tilde{x}\|^2 - \|(z_n - \tilde{x})
\end{aligned}
$$

$$
\begin{aligned}
&- [x_n + \xi D^*(S - I)Dx_n - \tilde{x}] \|^2 \Big\} \\
=&\frac{1}{2}\Big\{ \|z_n - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - \|z_n - x_n - \xi D^*(S - I)Dx_n\|^2 \Big\} \\
=&\frac{1}{2}\Big\{ \|z_n - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - \Big[\|z_n - x_n\|^2 + \xi^2\|D^*(S - I)Dx_n\|^2 \\
&- 2\xi\langle z_n - x_n, D^*(S - I)Dx_n\rangle\Big]\Big\}.
\end{aligned}
$$

Hence, we obtain

$$
\|z_n - \tilde{x}\|^2 \leq \|x_n - \tilde{x}\|^2 - \|z_n - x_n\|^2 + 2\xi\|D(z_n - x_n)\|\|(S - I)Dx_n\|.
$$

Using (3.7), (3.8) and (3.16) in above inequality, we obtain

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 \leq&\eta_n\|x_n - \tilde{x}\|^2 + (1 - \eta_n)\|x_n - \tilde{x}\|^2 \\
&- (1 - \eta_n)\|z_n - x_n\|^2 \\
&+ 2(1 - \eta_n)\xi\|D(z_n - x_n)\|\|(S - I)Dx_n\| + 2\mu_n\mathbb{N}_7 \\
(1 - \eta_n)\|z_n - x_n\|^2 \leq&\|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 \\
&+ 2(1 - \eta_n)\xi\|D(z_n - x_n)\|\|(S - I)Dx_n\| + 2\mu_n\mathbb{N}_7 \\
\leq&(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|)\|x_n - x_{n+1}\| \\
&+ 2(1 - \eta_n)\xi\|D(z_n - x_n)\|\|(S - I)Dx_n\| + 2\mu_n\mathbb{N}_7.
\end{aligned}
$$

By (3.13), (3.17) and the given conditions, we have

$$
\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.18}
$$

Next, prove that $\lim_{n\to\infty} \|z_n - u_n\| = 0$.

We estimate

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 =&\|(1 - \eta_n)(v_n - \tilde{x}) + \eta_n(x_n - \tilde{x}) \\
&+ \mu_n(\theta f(x_n) - Av_n)\|^2 \\
\leq&(1 - \eta_n)\|v_n - \tilde{x}\|^2 + \eta_n\|x_n - \tilde{x}\|^2 \\
&+ 2\mu_n\langle \kappa_n, x_{n+1} - \tilde{x}\rangle \\
\leq&(1 - \eta_n)\|v_n - \tilde{x}\|^2 + \eta_n\|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n \\
\leq&(1 - \eta_n)\|u_n - \tilde{x}\|^2 + \eta_n\|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n.
\end{aligned}
$$

In the above inequality we set $\kappa_n = \theta f(x_n) - Av_n$ and let $\omega > 0$ be a suitable constant with $\omega \geq \sup_n\{\|\kappa_n\|, \|x_n - \tilde{x}\|\}$. Thus,

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 \leq&(1 - \eta_n)\Big\{ \|P_{C_1}(z_n - \sigma_n Bz_n) - P_{C_1}(\tilde{x} - \sigma_n B\tilde{x})\|^2 \Big\} \\
&+ \eta_n\|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n \\
\leq&(1 - \eta_n)\Big\{\|z_n - \tilde{x}\|^2 + \sigma_n(\sigma_n - 2\gamma)\|Bz_n - B\tilde{x}\|^2\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \eta_n\|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n \\
\leq &(1-\eta_n)\Big\{\|x_n - \tilde{x}\|^2 + \sigma_n(\sigma_n - 2\gamma)\|Bz_n - B\tilde{x}\|^2\Big\} \\
&+ \eta_n\|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n \\
\leq &(1-\eta_n)\sigma_n(\sigma_n - 2\omega)\|Bz_n - B\tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n, \qquad (3.19)
\end{aligned}
$$

which implies

$$
\begin{aligned}
(1-\eta_n)\sigma_n(2\omega - \sigma_n)\|Bz_n - B\tilde{x}\|^2 \leq &\|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 + 2\omega^2\mu_n \\
\leq &(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|)\|x_n - x_{n+1}\| + 2\omega^2\mu_n.
\end{aligned}
$$

By (3.13) and the given conditions, we get

$$
\lim_{n\to\infty} \|Bz_n - B\tilde{x}\| = 0. \qquad (3.20)
$$

From (2.1), we compute

$$
\begin{aligned}
\|u_n - \tilde{x}\|^2 = &\|P_{C_1}(z_n - \sigma_n Bz_n) - P_{C_1}(\tilde{x} - \sigma_n B\tilde{x})\|^2 \\
\leq &\langle u_n - \tilde{x}, (z_n - \sigma_n Bz_n) - (\tilde{x} - \sigma_n B\tilde{x})\rangle \\
\leq &\frac{1}{2}\Big\{\|u_n - \tilde{x}\|^2 + \|(z_n - \sigma_n Bz_n) \\
&- (\tilde{x} - \sigma_n B\tilde{x})\|^2 - \|(u_n - z_n) + \sigma_n(Bz_n - B\tilde{x})\|^2\Big\} \\
\leq &\frac{1}{2}\Big\{\|u_n - \tilde{x}\|^2 + \|z_n - \tilde{x}\|^2 - \|(u_n - z_n) + \sigma_n(Bz_n - B\tilde{x})\|^2\Big\} \\
\leq &\|z_n - \tilde{x}\|^2 - \|u_n - z_n\|^2 - \sigma_n^2\|Bz_n - B\tilde{x}\|^2 \\
&+ 2\sigma_n\langle u_n - z_n, Bu_n - B\tilde{x}\rangle \\
\leq &\|z_n - \tilde{x}\|^2 - \|u_n - z_n\|^2 + 2\sigma_n\|u_n - z_n\|\|Bz_n - B\tilde{x}\| \\
\leq &\|x_n - \tilde{x}\|^2 - \|u_n - z_n\|^2 + 2\sigma_n\|u_n - z_n\|\|Bz_n - B\tilde{x}\|.
\end{aligned}
$$

By (3.19), we obtained

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 \leq &(1-\eta_n)\|u_n - \tilde{x}\|^2 + \eta_n\|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n \\
\leq &(1-\eta_n)\Big\{\|x_n - \tilde{x}\|^2 - \|u_n - z_n\|^2 \\
&+ 2\sigma_n\|u_n - z_n\|\|Bz_n - B\tilde{x}\|\Big\} + \eta_n\|x_n - \tilde{x}\|^2 + 2\omega^2\mu_n,
\end{aligned}
$$

which implies

$$
\begin{aligned}
(1-\eta_n)\|u_n - z_n\|^2 \leq &\|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 \\
&+ 2(1-\eta_n)\sigma_n\|u_n - z_n\|\|Bz_n - B\tilde{x}\| + 2\omega^2\mu_n \\
\leq &(\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|)\|x_n - x_{n+1}\| \\
&+ 2(1-\eta_n)\sigma_n\|u_n - z_n\|\|Bz_n - B\tilde{x}\| + 2\omega^2\mu_n.
\end{aligned}
$$

Using (3.13), (3.20) and the given conditions, we get

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.21}$$

From (3.18) and (3.21), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.22}$$

By (3.15) and (3.22), we get

$$\lim_{n\to\infty} \|\mathbb{W}_n u_n - u_n\| = 0. \tag{3.23}$$

Further, using (3.22) and (3.23)

$$\begin{aligned} \|v_n - x_n\|^2 &\le \|\delta_n u_n + (1-\delta_n)\mathbb{W}_n u_n - x_n\| \\ &\le \delta_n \|u_n - x_n\| + (1-\delta_n)\|\mathbb{W}_n u_n - x_n\| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.24}$$

Therefore, by (3.22) and (3.24), we get

$$\lim_{n\to\infty} \|u_n - v_n\| = 0. \tag{3.25}$$

Step 3. We claim that $\tilde{x} \in \Gamma$.

Since $\{x_n\}$ is bounded therefore consider $\tilde{x} \in H_1$ be any weak cluster point of $\{x_n\}$. Hence, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ with $x_{n_j} \rightharpoonup \tilde{x}$. By Lemma 2.7 and (3.23), we have $\tilde{x} \in \cap_{i=1}^{\infty} \mathrm{Fix}(S_i)$.

And $z_{n_j} = R[x_{n_j} + \eta D^*(S-I)Dx_{n_j}]$ can write as

$$\frac{(x_{n_j} - z_{n_j}) + D^*(S-I)Dx_{n_j}}{\rho_1} \in M_1 z_{n_j}. \tag{3.26}$$

Taking $j \to \infty$ in (3.26) and by (3.17), (3.18) and the concept of the graph of a maximal monotone mapping and $\frac{1}{\alpha_1}$-Lipschitz continuity of g_1, we get $0 \in M_1\tilde{x} + g_1\tilde{x}$ that is $\tilde{x} \in \mathrm{Sol}(\mathrm{MVIP}(1.2))$. Furthermore, since $\{x_n\}$ and $\{z_n\}$ have the same asymptotical behaviour, $Dx_{n_j} \rightharpoonup D\tilde{x}$. As S is nonexpansive, by (3.17) and Lemma 2.7, we get $(I-S)D\tilde{x} = 0$. Hence, by Lemma 2.3, $0 \in g_2(D\tilde{x}) + M_2 D\tilde{x}$ that is $D\tilde{x} \in \mathrm{Sol}(\mathrm{MVIP}(1.3))$. Thus, $\tilde{x} \in \Lambda$.

Next, we prove $\tilde{x} \in \mathrm{Sol}(\mathrm{VIP}(1.1))$. Since $\lim_{n\to\infty} \|z_n - u_n\| = 0$ and $\lim_{n\to\infty} \|z_n - x_n\| = 0$, there exist subsequences $\{z_{n_i}\}$ and $\{u_{n_i}\}$ of $\{z_n\}$ and $\{u_n\}$, respectively such that $z_{n_i} \rightharpoonup \tilde{x}$ and $u_{n_i} \rightharpoonup \tilde{x}$.

Define the mapping $\mathbb{M}$ as

$$\mathbb{M}(p_1) = \begin{cases} D(p_1) + \mathbb{N}_{C_1}(p_1), & \text{if } p_1 \in C_1, \\ \emptyset, & \text{if } p_1 \notin C_1, \end{cases}$$

where $\mathbb{N}_{C_1}(p_1) := \{p_2 \in H_1 : \langle p_1 - y, p_2 \rangle \ge 0 \text{ for all } y \in C_1\}$ is the normal cone to C_1 at $p_1 \in H_1$. Thus, $\mathbb{M}$ is a maximal monotone and hence $0 \in \mathbb{M}p_1$ mapping if and only if $p_1 \in \mathrm{Sol}(\mathrm{VIP}(1.1))$. Let $(p_1, p_2) \in \mathrm{graph}(\mathbb{M})$. Then, we have $p_2 \in \mathbb{M}p_1 =$

$Bp_1 + \mathbb{N}_{C_1}(p_1)$ and hence $p_2 - Bp_1 \in \mathbb{N}_{C_1}(p_1)$. So, we have $\langle p_1 - y, p_2 - Bp_1\rangle \geq 0$, for all $y \in C_1$. On the other hand, from $u_n = P_{C_1}(z_n - \sigma_n Bz_n)$ and $z_1 \in C_1$, we have

$$\langle (z_n - \sigma_n Bz_n) - u_n, u_n - p_1\rangle \geq 0.$$

This implies that

$$\left\langle p_1 - u_n, \frac{u_n - z_n}{\sigma_n} + Bz_n \right\rangle \geq 0.$$

Since $\langle p_1 - y, p_2 - Bp_1\rangle \geq 0$ for all $y \in C_1$ and $u_{n_i} \in C_1$, using monotonicity of B, we have

$$\begin{aligned}\langle p_1 - u_{n_i}, p_2\rangle \geq& \langle p_1 - u_{n_i}, Bp_1\rangle \\ \geq& \langle p_1 - u_{n_i}, Bp_1\rangle - \left\langle p_1 - u_{n_i}, \frac{u_{n_i} - z_{n_i}}{\sigma_{n_i}} + Bu_{n_i}\right\rangle \\ =& \langle p_1 - u_{n_i}, Bp_1 - Bu_{n_i}\rangle + \langle p_1 - u_{n_i}, Bu_{n_i} - Bz_{n_i}\rangle \\ & - \left\langle p_1 - u_{n_i}, \frac{u_{n_i} - z_{n_i}}{\sigma_{n_i}}\right\rangle \\ \geq& \langle p_1 - u_{n_i}, Bu_{n_i} - Bz_{n_i}\rangle - \left\langle p_1 - u_{n_i}, \frac{u_{n_i} - z_{n_i}}{\sigma_{n_i}}\right\rangle.\end{aligned}$$

Since B is continuous therefore on taking limit $i \to \infty$, we have $\langle p_1 - \tilde{x}, p_2\rangle \geq 0$. Since $\mathbb{M}$ is maximal monotone, we have $\tilde{x} \in \mathbb{M}^{-1}(0)$ and hence $\tilde{x} \in \mathrm{Sol}(\mathrm{VIP}(1.1))$. Thus, $\tilde{x} \in \Gamma$.

Step 4. Finally, we prove that $\limsup_{n\to\infty} \langle (\theta f - A)z, x_n - z\rangle \leq 0$, where $z = P_\Gamma(I - A + \theta f)z$ and $x_n \to \tilde{x}$.

By (3.24), we obtain

$$\begin{aligned}\limsup_{n\to\infty} \langle (\theta f - A)z, x_n - z\rangle &= \limsup_{n\to\infty} \langle (\theta f - A)z, v_n - z\rangle \\ &\leq \limsup_{i\to\infty} \langle (\theta f - A)z, v_{n_i} - z\rangle \\ &= \langle (\theta f - A)z, \tilde{x} - z\rangle \\ &\leq 0. \end{aligned} \tag{3.27}$$

Using (3.5) and (3.7), we calculate

$$\begin{aligned}\|x_{n+1} - \tilde{x}\|^2 =& \langle \mu_n(\theta f(x_n) - A\tilde{x}) + \eta_n(x_n - \tilde{x}) \\ & + ((1-\eta_n)I - \mu_n A)(v_n - \tilde{x}), x_{n+1} - \tilde{x}\rangle \\ =& \mu_n\langle \theta f(x_n) - A\tilde{x}, x_{n+1} - \tilde{x}\rangle + \eta_n\langle x_n - \tilde{x}, x_{n+1} - \tilde{x}\rangle \\ & + \langle ((1-\eta_n)I - \mu_n A)(v_n - \tilde{x}), x_{n+1} - \tilde{x}\rangle \\ \leq& \mu_n\left(\theta\langle f(x_n) - f(\tilde{x}), x_{n+1} - \tilde{x}\rangle + \langle \theta f(\tilde{x}) - A\tilde{x}, x_{n+1} - \tilde{x}\rangle\right) \\ & + \eta_n\|x_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\| + \|(1-\eta_n)I - \mu_n A\|\|v_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\| \\ \leq& \mu_n\tau\theta\|x_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\| + \mu_n\langle \theta f(\tilde{x}) - A\tilde{x}, x_{n+1} - \tilde{x}\rangle\end{aligned}$$

$$
\begin{aligned}
&+\eta_n\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\|+(1-\eta_n-\mu_n\bar{\theta})\|v_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\| \\
=&[1-\mu_n(\bar{\theta}-\theta\tau)]\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\|+\mu_n\langle\theta f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle, \\
\|x_{n+1}-\tilde{x}\|^2\leq&\frac{1-\mu_n(\bar{\theta}-\theta\tau)}{2}\left(\|x_n-\tilde{x}\|^2+\|x_{n+1}-\tilde{x}\|^2\right) \\
&+\mu_n\langle\theta f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle \\
\leq&\frac{1-\mu_n(\bar{\theta}-\theta\tau)}{2}\|x_n-\tilde{x}\|^2+\frac{1}{2}\|x_{n+1}-\tilde{x}\|^2+\mu_n\langle\theta f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2&\leq[1-\mu_n(\bar{\theta}-\theta\tau)]\|x_n-\tilde{x}\|^2+2\mu_n(\langle\theta f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&=[1-\mu_n(\bar{\theta}-\theta\tau)]\|x_n-\tilde{x}\|^2+2\mu_n\langle\theta f(\tilde{x})-A\tilde{x},x_{n+1}-\tilde{x}\rangle. \qquad (3.28)
\end{aligned}
$$

Thus, by (3.27), (3.28), Lemma 2.6 and using $\lim_{n\to\infty}\mu_n=0$, we get $x_n\to\tilde{x}$, where $\tilde{x}=P_\Gamma(I+\theta f-A)$. □

Now, we listed following consequences from Theorem 3.1.

Corollary 3.1. *Let H_1 and H_2 denote the Hilbert spaces and $C_1\subset H_1$ be nonempty closed convex subset of H_1. Let $B:H_1\to H_1$ be a γ-inverse strongly monotone mapping, $D:H_1\to H_2$ be a bounded linear operator with its adjoint operator D^*, $M_1:H_1\to 2^{H_1}$, $M_2:H_2\to 2^{H_2}$ be multi-valued maximal monotone operators and $g_1:H_1\to H_1$, $g_2:H_2\to H_2$ be α_1, α_2-inverse strongly monotone mappings, respectively. Let $f:C_1\to C_1$ be a contraction mapping with constant $\tau\in(0,1)$, A be a strongly positive bounded linear self adjoint operator on C_1 with constant $\bar{\theta}>0$ such that $0<\theta<\frac{\bar{\theta}}{\tau}<\theta+\frac{1}{\tau}$ and $S:C_1\to C_1$ be a nonexpansive mapping such that $\Gamma:=\Lambda\cap\mathrm{Sol(VIP(1.1))}\cap\mathrm{Fix}(S))\neq\emptyset$. Let $\{x_n\}$ be the sequence generated as:*

$$
\left.\begin{aligned}
&x_1\in C_1,\\
&z_n=R(I+\xi D^*(S-I)D)x_n,\\
&u_n=P_{C_1}(z_n-\sigma_nBz_n),\\
&v_n=\delta_nu_n+(1-\delta_n)Su_n,\\
&x_{n+1}=\mu_n\theta f(x_n)+\eta_nx_n+((1-\eta_n)I-\mu_nA)v_n,\quad n\geq1,
\end{aligned}\right\}
$$

where $R=J_{\rho_1}^{(g_1,M_1)}(I-\rho_1g_1)$, $S=J_{\rho_2}^{(g_2,M_2)}(I-\rho_2g_2)$, $\{\mu_n\},\{\eta_n\},\{\delta_n\}\subset(0,1)$ and $\xi\in(0,\frac{1}{\epsilon})$, ϵ be the spectral radius of D^*D. Let the control sequences satisfying conditions:

(i) $\lim_{n\to\infty}\mu_n=0$, $\sum_{n=0}^{\infty}\mu_n=\infty$;
(ii) $0<\rho_1<2\alpha_1$, $0<\rho_2<2\alpha_2$;
(iii) $0<\liminf_{n\to\infty}\eta_n\leq\limsup_{n\to\infty}\eta_n<1$;
(iv) $0<\liminf_{n\to\infty}\sigma_n\leq\limsup_{n\to\infty}\sigma_n<2\gamma$;
(v) $\lim_{n\to\infty}\delta_n=0$.

Then, the sequence $\{x_n\}$ *converges strongly to some* $\tilde{x} \in \Gamma$*, where* $\tilde{x} = P_\Gamma(\theta f + (I - A))\tilde{x}$ *which solves:*

$$\langle (A - \theta f)\tilde{x}, v - \tilde{x} \rangle \geq 0, \quad \text{for all } v \in \Gamma.$$

If we consider $\rho_1 = \rho_2$, $g_1 = g_2 = B \equiv 0$ and $\eta_n = 0$ in Theorem 3.1 then we have following corollary.

Corollary 3.2. *Let* H_1 *and* H_2 *denote the Hilbert spaces. Let* $D : H_1 \to H_2$ *be a bounded linear operator with its adjoint operator* D^**,* $M_1 : H_1 \to 2^{H_1}$*,* $M_2 : H_2 \to 2^{H_2}$ *be multi-valued maximal monotone operators, respectively. Let* $f : H_1 \to H_1$ *be a contraction mapping with constant* $\tau \in (0, 1)$*,* A *be a strongly positive bounded linear self adjoint operator on* H_1 *with constant* $\bar{\theta} > 0$ *such that* $0 < \theta < \frac{\bar{\theta}}{\tau} < \theta + \frac{1}{\tau}$ *and* $\{S_i\}_{i=1}^{\infty} : H_1 \to H_1$ *be an infinite family of nonexpansive mappings such that* $\Gamma := \Lambda \cap (\cap_{i=1}^{\infty} \mathrm{Fix}(S_i)) \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence generated as:*

$$\left.\begin{array}{l} x_1 \in H_1, \\ z_n = J_\rho^{M_1}(I + \xi D^*(J_\rho^{M_2} - I)D)x_n, \\ v_n = \delta_n z_n + (1 - \delta_n)\mathbb{W}_{\ltimes} z_n, \\ x_{n+1} = \mu_n \theta f(x_n) + (I - \mu_n A)v_n, \quad n \geq 1, \end{array}\right\}$$

where $\mathbb{W}_n$ *defined in* (1.6)*,* $\{\mu_n\}$*,* $\{\delta_n\} \subset (0, 1)$ *and* $\xi \in (0, \frac{1}{\epsilon})$*,* ϵ *be the spectral radius of* D^*D*. Let the control sequences satisfying conditions:*

(i) $\lim\limits_{n\to\infty} \mu_n = 0$, $\sum\limits_{n=0}^{\infty} \mu_n = \infty$;

(ii) $\lim\limits_{n\to\infty} \delta_n = 0$.

Then, the sequence $\{x_n\}$ *converges strongly to some* $\tilde{x} \in \Gamma$*, where* $\tilde{x} = P_\Gamma(\theta f + (I - A))\tilde{x}$ *which solves:*

$$\langle (A - \theta f)\tilde{x}, v - \tilde{x} \rangle \geq 0, \quad \text{for all } v \in \Gamma.$$

4. Numerical Example

Example 4.1. Let $H_1 = H_2 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle u, v \rangle = uv$ for all $u, v \in \mathbb{R}$, and induced usual norm $|\cdot|$. Let $C_1 = [0, \infty)$; let the mappings $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ be defined by $g_1(u) = \frac{3}{2}u$ for all $u \in H_1$ and $g_2(v) = v + 3$ for all $v \in H_2$. Let $M_1, M_2 : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by $M_1(u) = \{u - 1\}$ for all $u \in \mathbb{R}$ and $M_2(v) = \{4v\}$ for all $v \in \mathbb{R}$. Let the mapping $D : \mathbb{R} \to \mathbb{R}$ be defined by $D(u) = -\frac{3}{2}u$ for all $u \in \mathbb{R}$. Let the mappings $\{S_i\}_{i=1}^{\infty} : C_1 \to C_1$ be defined by $S_i u = \frac{u+2i}{1+5i}$ for each $i \in \mathbb{N}$, let the mapping $B : H_1 \to \mathbb{R}$ be defined by $Bu = 5u - 2$ for all $u \in H_1$. Let the mapping $f : C_1 \to C_1$ be defined by $f(u) = \frac{u}{5}$ for all $u \in C_1$ and $Au = \frac{u}{2}$ with $\theta = \frac{1}{10}$. Setting $\{\mu_n\} = \{\frac{1}{10n}\}$, $\{\eta_n\} = \{\frac{1}{2n^2}\}$, $\{\sigma_n\} = \frac{1}{4}$, $\{\delta_n\} = \frac{1}{n}$ and $\{\lambda_n\} = \{\frac{1}{3n^2}\}$ for all $n \geq 1$. Let $\mathbb{W}_n$ be the $\mathbb{W}$-mapping generated by $S_1, S_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$, which is defined by (1.6). Then, there are sequences $\{x_n\}$, $\{z_n\}$, $\{u_n\}$

and $\{v_n\}$ as: Given x_1,

$$\left.\begin{array}{l} t_n = SDx_n = J_{\rho_2}^{(g_2,M_2)}(I-\rho_2 g_2)Dx_n \\ r_n = x_n + \xi D^*(t_n - Dx_n) \\ z_n = J_{\rho_1}^{(g_1,M_1)} r_n \\ u_n = P_{C_1}(z_n - \sigma_n B z_n), \\ v_n = \delta_n z_n + (1-\delta_n)\mathbb{W}_{\ltimes} z_n, \\ x_{n+1} = \mu_n \theta f(x_n) + \eta_n x_n + ((1-\eta_n)I - \mu_n A)v_n, \end{array}\right\}$$

Then, $\{x_n\}$ converges to $\tilde{x} = \{\frac{2}{5}\} \in \Gamma$.

Proof. Obviously, D is a bounded linear operator on $\mathbb{R}$ with adjoint D^* and $\|D\| = \|D^*\| = \frac{3}{2}$, and hence $\xi \in (0, \frac{4}{9})$. Therefore, we choose $\xi = 0.1$. Further, g_1 and g_2 are 3 and 1-ism, therefore $\rho_1 \subset (0, \frac{4}{3})$ and $\rho_2 \subset (0, 2)$, thus choose $\rho_1 = \frac{1}{3} > 0$ and $\rho_2 = \frac{1}{3} > 0$. For each i, S_i is nonexpansive with $\mathrm{Fix}(S_i) = \left\{\frac{2}{5}\right\}$. Further, B is 5-ism and $\mathrm{Sol(VIP(1.1))} = \{\frac{2}{5}\}$. Furthermore, $\mathrm{Sol(MVIP(1.2))} = \left\{\frac{2}{5}\right\}$ and $\mathrm{Sol(MVIP(1.3))} = \left\{-\frac{3}{5}\right\}$, and thus $\Lambda = \{\frac{2}{5} \in C : \frac{2}{5} \in \mathrm{Sol(MVIP(1.2))} : D(\frac{2}{5}) \in \mathrm{Sol(MVIP(1.3))}\} = \left\{\frac{2}{5}\right\}$. Therefore, $\Gamma := \Lambda \cap \mathrm{Sol(VIP(1.1))} \cap (\cap_{i=1}^{\infty} \mathrm{Fix}(S_i)) \neq \emptyset$. Thus,

$$t_n = \frac{-6x_n + 9}{14}; \quad r_n = \frac{31x_n - 6t_n}{40}; \quad z_n = \frac{3r_n + 1}{4};$$

$$u_n = P_{C_1}(z_n - \sigma_n B z_n) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x > 1, \\ \frac{-z_n+2}{4} & \text{otherwise;} \end{cases}$$

$\mathbb{W}_n = u_n;$

Step 1:

$i = 1;$

$$\mathbb{W}_n = \frac{1}{3n^2}\frac{(\mathbb{W}_n + 2i)}{1+5i} + \left(1 - \frac{1}{3n^2}\right) u_n;$$

$i = i + 1;$

if $(i \leq N)$ go to Step 1;

$$v_n = \frac{1}{n} u_n + \left(1 - \frac{1}{n}\right) \mathbb{W}_n u_n,$$

$$x_{n+1} = \frac{1}{100n}\frac{x_n}{5} + \frac{1}{2n^2} x_n + \left(1 - \frac{1}{2n^2}\right) v_n - \frac{1}{10n}\frac{v_n}{2},$$

which show that $\{x_n\}$ converges to $\tilde{x} = \frac{2}{5}$ as $n \to +\infty$ and $\lim_{n\to\infty} \|\mathbb{W}_n x - \mathbb{W}x\| = 0$ for each $x \in C_1$. □

5. Figures

Finally, by the software Matlab 7.8.0, we obtain following figures which show that $\{x_n\}$ converges to $\tilde{x} = \frac{2}{5}$ as $n \to +\infty$, and $\lim_{n\to\infty} \|\mathbb{W}_n x - \mathbb{W}x\| = 0$ for each $x \in C_1$.

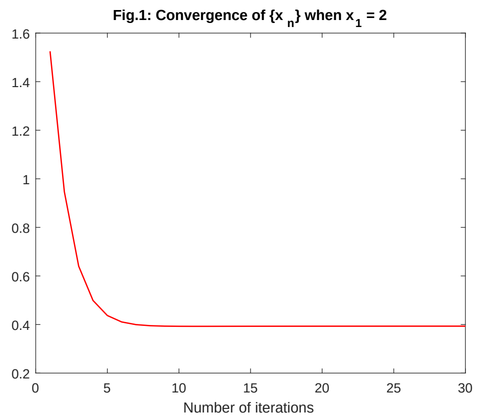

FIGURE 1. Convergence of $\{x_n\}$ when $x_1 = 2$.

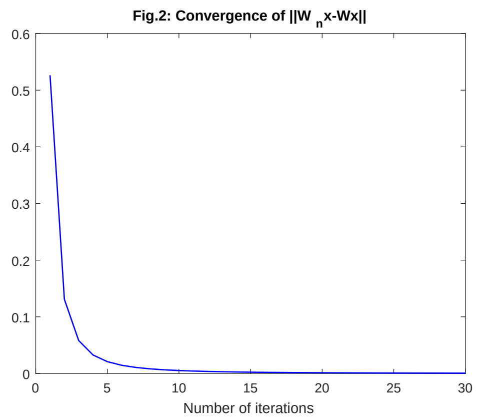

FIGURE 2. Convergence of $\|\mathbb{W}_n x - \mathbb{W}x\|$.

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