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EXPLORING THE ALGEBRAIC AND TOPOLOGICAL PROPERTIES OF SEMIGROUPS THROUGH THEIR PRIME m-BI IDEALS

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Abstract. We introduce the concepts of the prime m-bi ideal and their associated types in the semigroups. Different characterizations of the semigroups using these m-bi ideals are presented. The forms of the topologies induced by the prime and strongly prime m-bi ideals in the semigroups are also explored. The result shows that either both the conditions of m-regularity and m-intraregularity or existence of pairwise comaximal m-bi ideals in a semigroup is necessary for strongly prime m-bi ideals to induce a topology; whereas the existence of pairwise comaximal m-bi ideals is necessary for the prime m-bi ideals to induce topology on the semigroups. We concluded that the prime m-bi ideals are as important to study the semigroups as the prime bi ideals.

1. Introduction and Preliminaries

A short introduction to our work and the important concepts are described in this section.

1.1. **Introduction.** A non-empty set M together with a given associative binary operation $\cdot$ is called a semigroup. A semigroup primarily need not to possess the additive identity 0 or absorbing *zero* [15], or the multiplicative identity e as against many other algebraic structures which do possess these two or one of these elements. Moreover, different powers of semigroups through a positive integer so-called *index* also produce sub-structures like subsemigroups and ideals which have different forms,

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and behave differently. In order to explore these properties, the author of this article generalized the bi ideals in the semirings and semigruops respectively in articles [9] and [10] through the *indices*. As a follow-up of these generalizations, it was felt that the major classes of the m-bi ideals like, the prime, strongly prime, semiprime, maximal and minimal, irreducible and strongly irreducible should also be studied in a similar way so that the hidden properties of the semigroups should also be uncovered. To meet this end, we started this work to explore the algebraic and consequently topological properties of these ideals in the semigroups. Our work is aimed at mainly semigroups without *zero*. We divide this article into six sections.

In Subsection 1.2, we present the preliminary concepts from the literature consisting of books on semigroups especially [5] and [2], and the research articles in the field of semigroup theory. We demonstrate the major results of our work on the prime and associated m-bi ideals in semigroups in Section 2. Section 3 deals with the characterization of semigroups through maximal m-bi ideals. In Section 4, we characterize the m-regular and m-intraregular semirings using the prime, semiprime and strongly m-bi ideals. We present the forms of the topologies formed by these m-bi ideals in Section 5. The conclusion of the whole work is given in Section 6.

1.2. **Preliminaries.** A nonempty subset K of M is called its *subsemirgoup* if K itself is a semigroup under the operation $\cdot$ of M. The subsemigroup K of M becomes a left (right) ideal of M if the condition $MK \subseteq K$ ($KM \subseteq K$) is imposed on M [13]. If K is a left as well as a right ideal, then it is called an ideal (or a two-sided ideal) of M. The intersection (if it is nonempty) and sum of two (ideals, left-ideals, right-rights) of a semgroup is (an ideal, a left-ideal, a right-ideal). Right and left ideals generalize to the quasi ideals. A quasi ideal Q of semigroup M is a subsemigroup if $MQ \cap QM \subseteq Q$. A further generalization of quasi ideals gives introduction to bi ideals. A subsemigroup B of M is called a *bi ideal* of M if $BMB \subseteq B$. Every bi ideals is a quasi ideal, however every quasi ideal may or may not be a bi ideal. An *m-bi ideal* B of M is a subsemigroup of M such that $BM^mB \subseteq B$, where $m \geq 1$ is a positive integer. Every *bi ideal* B of M is a 1-bi ideal of M, but every m-bi ideals is not a bi ideal. An m-bi ideal is called principal m-bi ideal if it is generated by a single element. If $a \in M$, the m-bi ideal generated by M is $\langle a \rangle_{m-b} = \{a\} \cup \{a^2\} \cup aM^m a$. A semigroup having no nontrivial *two-sided* ideals is called a simple semigroup [13].

2. Prime m-Bi Ideals

In this section, we develop the definitions of prime m-bi and their associated types, and characterize the semigroups using their properties.

Definition 2.1. An m-bi ideal B of a semigroup M is known as a prime m-bi ideal (strongly prime m-bi ideal) if the proposition "$B_1B_2 \subseteq B$" ("$B_1B_2 \cap B_2B_1 \subseteq B$") infers either "$B_1 \subseteq B$" or "$B_2 \subseteq B$" for any two m-bi ideals B_1 and B_2 of M.

Definition 2.2. An m-bi ideal B of a semigroup M is known as semiprime m-bi ideal if the proposition "$B_1^2 \subseteq B_1$" implies "$B_1 \subseteq B$" for any m-bi ideal B_1 of M.

This is very obvious that each strongly prime m-bi ideal of M is its prime m-bi ideal and its each prime m-bi-ideal is its semiprime m-bi ideal; but the converses do not hold. The prime (strongly prime) bi ideals as defined in [15] are the 1-bi ideals, but the prime (strongly prime) m-bi-ideal of M are not its prime (strongly prime) bi ideals. Similarly, semiprime bi ideals are the semiprime 1-bi ideals; but the converse does not follow.

Example 2.1. The semigroup M itself is always a prime, a semiprime and a strongly m-bi ideal of M. Moreover, M can have these type of ideals different from M.

Example 2.2. Consider the semigroup $M = \{\alpha, \beta, \gamma, \delta\}$ with the binary operation $\cdot$ given in the following table:

$\cdot$	α	β	γ	δ
α	α	α	α	α
β	α	β	α	α
γ	α	α	γ	α
δ	α	α	α	α

.

Taking $m = 2$, we get $M^2 = \{\alpha, \beta, \gamma\}$. Then the 2-bi ideals in M are $\{\alpha\}$, $\{\alpha, \beta\}$, $\{\alpha, \gamma\}$, $\{\alpha, \beta, \gamma\}$ and $\{\alpha, \beta, \gamma, \delta\}$. The prime 2-bi ideals are $\{\alpha\}$, $\{\alpha, \beta\}$, $\{\alpha, \gamma\}$ and $\{\alpha, \beta, \gamma, \delta\}$ and so are semiprime 2-bi-ideals of M. These are also strongly prime except the prime 2-bi ideal, $\{\alpha\}$, which is not strongly prime 2-bi ideal because $\{\alpha, \beta\}\{\alpha, \gamma\} \cap \{\alpha, \gamma\}\{\alpha, \beta\} = \{\alpha, \beta\} \cap \{\alpha, \gamma\} = \{\alpha\}$, but none of $\{\alpha, \beta\}$ and $\{\alpha, \gamma\}$ is contained in $\{\alpha\}$. More comments on the ideal viz., $\{\alpha, \beta, \gamma\}$ will be given in Section 3, Remark 3.1.

The following two successive examples demonstrate when the m-bi ideals and the subsets of a semigroup are also the prime and semiprime m-bi ideals.

Example 2.3. In a right zero semigroup M with the cardinality $|M| > 1$, we have $yx = x$ for all $x, y \in M$. So, for an arbitrary $x \in M$, $xx = x$, i.e., x is idempotent, $M^2 = M$. Consequently, $M^m = M$ for any integer $m \geq 1$. If S is a subset of M, then $SMS = MS = S$, i.e., S is bi ideal of M. Moreover, since $SM^mS = S$, every bi ideal is an m-bi ideal. That is, every subset is m-bi ideal of M.

In this case, all m-bi ideals of M coincide with the prime m-bi ideal, and so with the semiprime m-bi ideals. This is because for m-bi ideals B_1, B_2, we have $B_1B_2 = B_2$. On the other hand, if B is any m-bi ideal of M such that $|M - B| \geq 2$, then B is not strongly prime since for distinct $a, b \in M - B$, $(B \cup \{a\})(B \cup \{b\}) \cap (B \cup \{b\})(B \cup \{a\}) = (B \cup \{a\}) \cap (B \cup \{b\}) = B$, but none of $(B \cup \{a\})$, $(B \cup \{b\})$ is contained in B.

Example 2.4. Let M be a Kronecker delta semigroup, that is, M has a zero 0 and

$$xy = \begin{cases} x, & \text{if } x = y, \\ 0, & \text{otherwise,} \end{cases}$$

and assume that $|M| > 2$. Since $x^2 = x$ for all $x \in M$, so consequently, $M^m = M$. Clearly, every subset B of M is its m-bi ideals because $B^2 = B$ and $BM^mB = BMB = B$. Moreover, if $B_1^2 \subseteq B$, then since and $B_1^2 = B_1$ for any subsets B_1, B of M, so $B_1 \subseteq B$ imply that all subsets of M are all semiprime m-bi ideals of M. If B is an m-bi ideal of M such that $|M - B| > 2$, then B is not a prime m-bi ideal of M since for distinct $a, b \in M - B$, $(B \cup \{a\})(B \cup \{b\}) = (B \cup \{a\}) \cap (B \cup \{b\}) = B$, neither $(B \cup \{a\})$ nor $(B \cup \{b\})$ is contained in B. In particular case, $\{0\}$ is a semiprime m-bi ideal of M which is not a prime m-bi ideal.

Definition 2.3. An m-bi ideal B of a semigroup M is known as an irreducible (strongly irreducible) m-bi ideal if the proposition "$B_1 \cap B_2 = B$" ("$B_1 \cap B_2 \subseteq B$") infers either "$B_1 = B$" or "$B_2 = B$" (either "$B_1 \subseteq B$" or "$B_2 \subseteq B$"), for any two m-bi ideals B_1 and B_2 of M.

In a semigroup, strongly irreducible m-bi ideal irreducible m-bi ideal, but the converse is not true. This is evident by the following example.

Example 2.5. For the semigroup, $M = \{\pi, \rho, \sigma, \tau, \phi, \psi, \omega\}$ with binary operation $\cdot$ defined in the Table 1. We take $m = 2$, $M^2 = \{\pi, \rho, \sigma, \tau, \phi, \psi\}$. The m-bi ideals

TABLE 1.

.	π	ρ	σ	τ	ϕ	ψ	ω
π	π	π	π	π	π	π	π
ρ	π	ρ	ρ	ρ	ρ	ρ	ρ
σ	π	ρ	σ	τ	ρ	ρ	ρ
τ	π	ρ	ρ	ρ	σ	τ	ρ
ϕ	π	ρ	ϕ	ψ	ρ	ρ	ρ
ψ	π	ρ	ρ	ρ	ϕ	ψ	ρ
ω	π	ρ	ρ	ρ	ρ	ρ	ρ

in M are $\{\pi\}$, $\{\pi, \rho\}$, $\{\pi, \rho, \sigma\}$, $\{\pi, \rho, \tau\}$, $\{\pi, \rho, \phi\}$, $\{\pi, \rho, \psi\}$, $\{\pi, \rho, \sigma, \phi\}$, $\{\pi, \rho, \tau, \psi\}$, $\{\pi, \rho, \sigma, \tau\}$, $\{\pi, \rho, \phi, \psi\}$ and M itself. The irreducible m-bi ideals are $\{\pi\}$, $\{\pi, \rho, \sigma, \phi\}$, $\{\pi, \rho, \tau, \psi\}$, $\{\pi, \rho, \sigma, \tau\}$, $\{\pi, \rho, \phi, \psi\}$ and M. Strongly irreducible m-bi ideal is $\{\pi\}$.

Remark 2.1. (a) The intersection of any collection of *prime* (*strongly prime*) m-bi ideals in a semigroup M is generally not a *prime* (*strongly prime*) m-bi ideal. M be the semigroup of non-zero integers under ordinary multiplication. Let B_1 be the 2-bi ideal of M divisible by 2 and B_2 is the 3-bi ideal of M divisible by 3. Both of them are prime m-bi ideals of M as 2 and 3 are prime integers. Now $B_1 \cap B_2$ consists of non-zero integers divisible by 6, and $B_1B_2 = B_1 \cap B_2$, but neither $B_1 \cap B_1 \cap B_2$ nor $B_2 \cap B_1 \cap B_2$ implying that $B_1 \cap B_2 \subseteq B_1 \cap B_2$ is not prime.

(b) The intersection of any collection of *semiprime* m-bi ideals in M is a *semiprime* m-bi ideal.

This is to be reminded that each prime m-bi ideal of a semigroup is semiprime m-bi ideal. The following proposition describes the conditions when the semiprime m-bi ideal is a prime m-bi ideal in a semigroup.

Proposition 2.1. *If a semiprime m-bi ideal B of a semigroup M is strongly irreducible, then B is a strongly prime m-bi-ideal.*

Proof. We consider B_1, B_2 as two m-bi ideals of M with the additional assumption that

$$B_1B_2 \cap B_2B_1 \subseteq B. \tag{2.1}$$

Then, after a little simplification, we get,

$$(B_1 \cap B_2)^2 \subseteq B_1B_2 \cap B_2B_1. \tag{2.2}$$

Combining (2) and (2.2) by the transitive property of inclusion, we get, $(B_1 \cap B_2)^2 \subseteq B$, which gives $B_1 \cap B_2 \subseteq B$, because B is a semiprime. Moreover, since B is strongly irreducible m-bi ideal of M, so we obtain $B_1 \subseteq B$ or $B_2 \subseteq B$, making B a strongly prime m-bi ideal of M. □

Proposition 2.2. *For any m-bi B of a semigroup M, such that $c \in M$ and $c \notin B$, there exists an irreducible m-bi ideal I such that $B \subseteq I$ and $c \notin I$.*

Proof. Take $\mathcal{B} = \{B : B$ is an m-bi ideal of M so that $c \in M$ and $c \notin B\}$. Then $\mathcal{B} \neq \emptyset$, because $B \in \mathcal{B}$. $\mathcal{B}$ is clearly a partially ordered set under the binary operation of *inclusion* of m-bi ideals in $\mathcal{B}$. If $\mathcal{S}$ is any totally ordered subset of $\mathcal{B}$, then $S = \bigcup_{S_\alpha \in \mathcal{S}, \alpha \in \wedge} S_\alpha$ is an m-bi ideal of M containing B. So we can find a maximal m-bi ideal, J, in $\mathcal{B}$ [6]. To show that J is an irreducible, we suppose $J = J_1 \cap J_2$ for two m-bi ideals J_1 and J_2 of M. If, on contrary, both J_1 and J_2 contain J properly, then $c \in J_1$ and $c \in J_2$. Hence $c \in J_1 \cap J_2 = J$, which contradicts the hypothesis that $c \notin J$. Thus, $J = J_1$ or $J = J_2$; implying that J is an irreducible m-bi ideal. □

The last theorem in this section characterizes the semigroups in which each m-bi ideal is irreducible and strongly irreducible.

Theorem 2.1. *The following statements are equivalent for a given semigroup M.*

(a) *The set $\mathcal{B}$ of all m-bi ideals of M is a totally ordered set under binary operation of inclusion of sets.*
(b) *Each m-bi ideal of M is strongly irreducible m-bi ideal.*
(c) *Each m-bi ideal of M is irreducible m-bi ideal.*

Proof. (a) ⇒ (b) Suppose B is an m-bi ideal of M and for B_1, B_2 to be any two m-bi ideals of M, the statement $B_1 \cap B_2 \subseteq B$ holds. Since $\mathcal{B}$ is totally ordered set under inclusion, so $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. This implies, either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. So, from our previous articulation $B_1 \cap B_2 \subseteq B$, we derive either $B_1 \subseteq B$ or $B_2 \subseteq B$, making B a strongly irreducible m-bi ideal of M.

(b) $\Rightarrow$ (c) Straightforward as *strongly irreducible m-bi ideal of M is irreducible m-bi ideal.*

(c) $\Rightarrow$ (a) For any two m-bi ideals of M, namely B_1 and B_2, we can compose the statement $B_1 \cap B_2 = B_1 \cap B_2$. Since each m-bi ideal of M is irreducible m-bi, $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$, which further implies, $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. That is, B_1 and B_2 are comparable making the collection of m-bi ideals of M a totally ordered set. ☐

3. Maximal m-Bi Ideals

Maximal ideals of semigroups, like all other algebraic structures, are an important category of ideals used to characterize semigroups in a different way along with the prime, strong prime and semiprime ideals. In the following section, we define the maximal m-bi ideals and characterize semigroups through their properties.

Definition 3.1. An m-bi ideal S of a semigroup M is called its maximal m-bi ideal if $S \subset M$ (M contains S properly) and there exists no m-bi ideal S_1 of M to give $S \subset S_1 \subset M$ [14].

In Example 2.2 of Section 2, $\mathcal{M} = \{\alpha, \beta, \gamma\}$ is the maximal m-bi ideals of the semigroup, $M = \{\alpha, \beta, \gamma, \delta\}$.

Theorem 3.1. *Every maximal m-bi ideal S of a semigroup M is a prime m-bi ideal of M, if $M = M^2$.*

Proof. Suppose $M - S = P$, where P is the set complement of the ideal S with respect to the semigroup M. Then, $M = (S \cup P)^2 = S^2 \cup SP \cup PS \cup P^2 \subset S \cup P^2$. That is, $M \subset S \cup P^2$. This gives that $M \cap P \subset S \cap P \cup P^2$. But $S \cap P \neq \emptyset$, therefore, we get $P \subset P^2$. Assume $B_1 B_2 \subset S$ for two m-bi ideals B_1 and B_2 of M. Suppose on contrary that neither B_1 nor B_2 is contained in S. Since $B_1 \nsubseteq S$ and S is maximal, we have $B_1 \cup S = M$, hence $P \subset B_1$. Analogously, $P \subset B_2$. Thus, $P^2 \subset B_1 B_2$, hence $P \subset B_1 B_2$, which is contradiction to $B_1 B_2 \subset S$. ☐

Remark 3.1. If $M \neq M^2$, then every maximal m-bi ideal of M is not prime m-bi ideal. This is evident in Example 2.2. The maximal m-bi ideal $\{\alpha, \beta, \gamma\}$ is not prime because $\{\alpha, \beta, \gamma, \delta\}\{\alpha, \beta, \gamma, \delta\} \subseteq \{\alpha, \beta, \gamma\}$, but $\{\alpha, \beta, \gamma, \delta\} \nsubseteq \{\alpha, \beta, \gamma\}$, so $\{\alpha, \beta, \gamma\}$ is not prime.

Intersection of maximal ideals performs an important role in characterizing semigroups [3]. The following theorems deal with the sets of maximal m-bi ideals, their intersections and their complement sets in the semigroups. These help us in defining the topologies on the semigroups, and tell when the prime m-bi ideals are maximal m-bi ideals. Š. Schwarz proved these theorem in [14] for ideals in semigroups, we prove them for the maximal m-bi ideals in semigroups.

Theorem 3.2. *Let $\{S_\alpha : \alpha \in \wedge\}$ be the set of different maximal m-bi ideals of a semigroup M. Suppose $|\wedge| \geq 2$ and denote $Q_\alpha = M^m - S_\alpha$ and $S = \bigcap_{\alpha\in\wedge} S_\alpha$, we have the following.*

(a) $Q_\alpha \cap Q_\beta = \emptyset$ *for* $\alpha \neq \beta$.
(b) $M^m = \left(\bigcup_{\alpha\in\wedge} Q_\alpha\right) \cup S$.
(c) *For every* $\nu \neq \alpha$, *we have* $Q_\alpha \subset S_\nu$.
(d) *If J is an m-bi ideal of M and $J \cap Q_\alpha \neq \emptyset$, then $Q_\alpha \subset J$.*
(e) *For $\alpha \neq \beta$, we have $Q_\alpha Q_\beta^m Q_\alpha \subset S$, that is S is not empty.*

Proof. The case $|\wedge| = 1$ is obvious.
(a) We have $S_\alpha \cup S_\beta = M^m$ for $\alpha \neq \beta$. Thus, $Q_\alpha \cap Q_\beta = (M^m - S_\alpha) \cap (M^m - S_\beta) = M^m - (S_\alpha \cup S_\beta) = \emptyset$.
(b) Since $S = \bigcap_{\alpha\in\wedge} S_\alpha = \bigcap_{\alpha\in\wedge} (M^m - Q_\alpha) = M^m - \bigcup_{\alpha\in\wedge} Q_\alpha$. Thus, $M^m = (\bigcup_{\alpha\in\wedge} Q_\alpha) \cup S$.
(c) For $\nu \neq \alpha$, we have $Q_\alpha = M^m \cap Q_\alpha = (S_\nu \cup Q_\nu) \cap Q_\alpha = S_\nu \cap Q_\alpha$. Thus, $Q_\alpha \subset S_\nu$.
(d) Since $J \cap Q_\alpha \neq \emptyset$ and J is an m-bi ideal of M whereas S_α is the maximal m-bi ideal, therefore union set $S_\alpha \cup J$ is an m-bi ideal of M greater than S_α. So, $S_\alpha \cup J = M^m$. Since $S_\alpha \cap Q_\alpha = \emptyset$, we have $Q_\alpha \cap S_\alpha \cup J = Q_\alpha \cap M^m$, i.e., $Q_\alpha \cap (S_\alpha \cup J) = Q_\alpha \cap M^m$, which gives that $(Q_\alpha \cap S_\alpha) \cup (Q_\alpha \cap S_\alpha) = Q_\alpha$, and $\emptyset \cup (Q_\alpha \cap S_\alpha) = Q_\alpha$ i.e., $(Q_\alpha \cap J) = Q_\alpha$, which gives that $Q_\alpha \subset J$.
(e) Suppose on contrary that there exist $u_\alpha, u_\delta \in Q_\alpha$ and $u_\beta \in Q_\beta$ such that $u_\alpha u_\beta u_\delta = u_\gamma$ and $u_\gamma \notin S$. Using (ii), we can find Q_γ such that $u_\gamma \in Q_\gamma$. If $Q_\gamma \neq Q_\alpha$. Then $Q_\alpha \subset M^m - Q_\gamma = S_\gamma$. That is, $Q_\alpha \subset S_\gamma$ and similarly, $Q_\delta \subset S_\gamma$. This gives, $Q_\alpha Q_\beta^m Q_\alpha \subset S_\gamma Q_\beta^m S_\gamma \subset S_\gamma M^m S_\gamma \subset S_\gamma$, hence, $u_\gamma \in S_\gamma$, which is a contradiction to $u_\gamma \in Q_\gamma = M^m \backslash S_\gamma$. Suppose now, $Q_\gamma = Q_\beta$. Then, $Q_\beta \subset M^m - Q_\gamma = S_\gamma$ and $Q_\alpha Q_\beta^m Q_\alpha \subset S_\alpha M^m S_\alpha \subset S_\alpha$, hence $u_\gamma \in S_\alpha = M^m - Q_\alpha$, which is a contradiction to $u_\gamma \in Q_\gamma$. Thus, $Q_\alpha Q_\beta^m Q_\alpha \subset S$ and S is not empty. □

Theorem 3.3. *Let M be a semigroup containing maximal m-bi ideals and let S be the intersection of all maximal m-bi ideals of M. Then every prime m-bi ideal of M containing S and different from M is a maximal m-bi ideal of M.*

Proof. Let U be a prime m-bi ideal of M containing S and $U \neq M$. Then Theorem 3.2, part (iv),

$$U = M^m - \left(\bigcup_{\nu\in\Lambda} Q_\nu\right) = \bigcap_{\nu\in\Lambda} (M^m - Q_\nu) = \bigcap_{\nu\in\Lambda} S_\nu,$$

where $\Lambda \subseteq \wedge$ and $\Lambda \neq \emptyset$. If $|\Lambda| = 1$, we have $U = S_\nu$, i.e. U is a maximal m-bi ideal of M and the theorem is proved. We shall show that $|\Lambda| \geq 2$ is not possible. Suppose on contrary that $|\Lambda| \geq 2$. Let $\beta \in \Lambda$ and denote $H = \bigcup_{\nu\in\Lambda, \nu\neq\beta} S_\nu$. Then we have $U = H \cap S_\beta$. Since both H and S_β are m-bi ideals, their product is also m-bi ideal, and so $HS_\beta \subset H \cap S_\beta = U$. Since U is prime m-bi ideal, so either $H \subset U$ or $S_\beta \subset U$. We discuss these two possibilities separately.

(a) Let $H \subset U$. Since $U \subset H$ also, so $U = H$. Further $H = U = H \cap S_\beta$ implies $H \subseteq S_\beta$, by Theorem 3.2, part (iii), we have $Q_\beta \subseteq \bigcup_{\nu \in \Lambda, \nu \neq \beta} S_\nu = H$. Hence $Q_\beta \subset S_\beta$, a contradiction with $Q_\beta \cap S_\beta = \emptyset$.

(b) Let $S_\beta \subset U$. Since also $U \subset S_\beta$, so $U = S_\beta$. Now $U = S_\beta = H \cap S_\beta$ would imply $S_\beta \subset H$. Since S_β is maximal and H is a proper subset of M, so $H = S_\beta$. The relation $Q_\beta \subset H = S_\beta$ gives an another contradiction.

These two cases complete the proof of the theorem. □

Theorem 3.4. *If M is a semigroup containing at least one maximal m-bi ideal, then a prime m-bi ideal U different from M is a maximal m-bi ideal of M if and only if $S \subset U$, where $S = \bigcap_{\alpha \in \wedge} S_\alpha$.*

Proof. If U is a maximal ideal, then clearly $S \subset U$. Conversely, if $S \subset U$, then by Theorem 3.3, U is a maximal ideal of M. □

Definition 3.2. An m-bi ideal N different from $\{0\}$ (if $0 \in M$) of a semigroup M is known as its minimal m-bi ideal if there does exist any other proper m-bi ideal in M which is contained in N properly.

In Example 2.2, $\mathcal{N} = \{\alpha\}$ is a minimal m-bi ideal of M. Detailed studies of the maximal and minimal m-bi ideals of a semigroup will be given in a future article on chains of the m-bi ideals in a semigroup.

4. Charterizing m-Regular and m-Intraregular Semigroups

In this section, we describe the m-regular and m-intraregular semigroups using the properties of their m-bi ideals, prime, semiprime and strongly prime m-bi ideals.

Definition 4.1. An element a of a semigroup M is called m-regular if $axa = a$ for some $x \in M^m$. A semigroup M is called m-regular if every element of M is m-regular. M is m-regular if $a \in aM^m a$ for all $a \in M$ [11].

Definition 4.2. An element a of a semigroup M is called m-intraregular if $ya^2z = a$ for some elements $y, z \in M^m$. Semigroup M is called m-intraregular if every element of M is m-intraregular [11].

Theorem 4.1. *For a semigroup M, the given conditions are equivalent.*

(a) *M is m-regular & m-intraregular.*
(b) *$B^2 = B$ for all m-bi ideal B in M.*
(c) *$B_1 \cap B_2 = B_1B_2 \cap B_2B_1$ for all m-bi ideals B_1, B_2 in M.*
(d) *Every m-bi ideal of M is semiprime.*
(e) *For any proper m-bi ideal B of M, if $B = \bigcap_{\alpha \in \wedge} \{B_\alpha : B_\alpha$ is irreducible semiprime m-bi ideals of M containing $B\}$.*

Proof. (a) ⇒ (b) Let M be m-regular and m-intraregular. Trivially, $B^2 \subseteq B$. For the converse, let $b \in B$. So, $b \in M$, using (a), we have $b = bsb$ and $b = ub^2w$, for some $s \in M^m$, $y, w \in M^m$. So, $b = bsb = bsbsb = bs(sb^2u)sb = (bssb)(busb)$. As $b \in B$, therefore, $b(ss)b \in BM^mB \subseteq B$. Also, $b(us)b \in BM^mB \subseteq B$. So, $b = (bssb)(busb) \in BB = B^2$. We get $B \subseteq B^2$.

(b) $\Rightarrow$ (c) Since $(B_1 \cap B_2)$ being intersection of two m-bi ideal is again an m-bi ideals, by the truth of part (b), we have $B_1 \cap B_2 = (B_1 \cap B_2)^2$. After a short simplification, we

$$B_1 \cap B_2 \subseteq B_1B_2 \cap B_2B_1. \tag{4.1}$$

Since $B_1B_2 \cap B_2B_1$ being the intersection of the products B_1B_2 and B_2B_1 of m-bi ideals of M is again an m-bi ideal [10]. So, by (b), we obtain $B_1B_2 \cap B_2B_1 = (B_1B_2 \cap B_2B_1)^2 \subseteq B_1B_2B_2B_1 \subseteq B_1M^mB_1 \subseteq B_1$. Analogously, $B_1B_2 \cap B_2B_1 \subseteq B_2$. Thus,

$$B_1B_2 \cap B_2B_1 \subseteq B_1 \cap B_2. \tag{4.2}$$

Consequently, by (4.1) and (4.2), we get $B_1 \cap B_2 = B_1B_2 \cap B_2B_1$.

(c) $\Rightarrow$ (d) In order to show that each m-bi ideal B of M is semiprime, we take another arbitrary m-bi C of M and assume that $C^2 \subseteq B$. By the truth of (c), we get $C = C \cap C = CC \cap CC = C^2$. This gives that $C \subseteq B$. Thus, every m-bi ideal of M is semiprime.

(d) $\Rightarrow$ (e) For a proper m-bi ideal B of M, let $\mathcal{B} = \bigcap_{\alpha \in \wedge} \{B_\alpha :\ B_\alpha$ is irreducible semiprime m-bi ideals of M containing $B\}$. Clearly, $B \subseteq \mathcal{B}$. We claim that $\mathcal{B} \subseteq B$, because if not, let $c \in \mathcal{B}$ and $c \notin B$. Then, Proposition 2.2 says that there exists an irreducible m-bi ideal in M say B_γ, for some $\gamma \in \wedge$, such that $B_\gamma \supset B$ and $c \notin B_\gamma$. By our assumption, every m-bi ideal is semiprime, and so each B_γ is irreducible semiprime m-bi ideal. But $c \notin B_\gamma$ creates contradiction to the assumption that $c \in B_\gamma$ for all $\gamma \in \wedge$. Hence, our claim is valid that $\mathcal{B} \subseteq B$. This completes the proof of the theorem.

(e) $\Rightarrow$ (b) Assuming the validity of (v), we have to show that each m-bi ideals B of M is idempotent i.e., $B^2 = B$. Clearly, $B^2 \subseteq B$. We again claim that $B \subseteq B^2$, because if not, let $c \in B$ such that $c \notin B^2$. Two possibilities arise. Firstly, when B^2 is contained in M properly, then, by (e), $B^2 = \bigcap_{\alpha \in \wedge} \{B_\alpha :\ B_\alpha$ is irreducible semiprime m-bi ideals of M containing $B^2\}$. That is, $B^2 \subseteq B_\alpha$ for all α. But B_α is semiprime, so $B \subseteq \bigcap_{\alpha \in \wedge} B_\alpha = B^2$, i.e., $B \subseteq B^2$. Lastly, $B^2 = B$. Secondly, when B^2 is not a proper m-bi ideal of M, then $B^2 = M$, so B is idempotent, i.e., $B^2 = B$. ☐

Proposition 4.1. *An m-bi ideal B of an m-regular and m-intraregular semigroup M is strongly irreducible if and only if B is strongly prime.*

Proof. Suppose that B is strongly irreducible m-bi ideal of M, then from (4.1) of Theorem 4.1, for any two m-bi ideals B_1 and B_2 of M, $B_1 \cap B_2 \subseteq B_1B_2 \cap B_2B_1 \subseteq B(say)$. But by our hypothesis "B is strongly irreducible", we obtain $B_1 \subseteq B$ or $B_2 \subseteq B$, resulting B into a strongly prime m-bi ideal of M.

Conversely, suppose that B is strongly prime, then (4.2) of Theorem 4.1 leads to $B_1B_2 \cap B_2B_1 \subseteq B_1 \cap B_2 \subseteq B$ (say). This produces $B_1 \subseteq B$ or $B_2 \subseteq B$ because of our hypothesis. Thus, B becomes a strongly irreducible m-bi ideal. □

The following theorem characterizes the semigroups in which all m-bi ideals are strongly prime.

Theorem 4.2. *Each m-bi ideal of a semigroup M is strongly prime if and only if M is m-regular, m-intraregular and all the m-bi ideals of M become a totally ordered set with respect to inclusion.*

Proof. Assume that all m-bi ideal of M are strongly prime, so are also semiprime. By Theorem 4.1, M is m-regular and m-intraregular, and $B_1B_2 \cap B_2B_1 = B_1 \cap B_2$ for two m-bi ideals B_1 and B_2 of M. It stays to show that the collection of all m-bi ideals of M is totally ordered with regards to the inclusion of m-bi ideals. Since $B_1 \cap B_2$ being an m-bi ideal of M is strongly prime, so the result, $B_1B_2 \cap B_2B_1 = B_1 \cap B_2$ gives either $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$. Eventually, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus, the collection of m-bi ideals of M is totally ordered set.

Conversely, assume that M is m-regular, m-intraregular and the collection of m-bi ideals of M is totally ordered under the set inclusion. Let B be any m-bi ideal of M. We want to show B is strongly prime. Suppose B_1, B_2 be any two m-bi ideals of M with the property that $B_1B_2 \cap B_2B_1 \subseteq B$. By Theorem 4.1, $B_1B_2 \cap B_2B_1 = B_1 \cap B_2$. So, $B_1 \cap B_2 \subseteq B$. By our assumption, either $B_1 \subseteq B_2$ or $B_2 \subset B_1$, that is, either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Along these lines, either $B_1 \subseteq B$ or $B_2 \subseteq B$. Subsequently B is strongly prime. □

Theorem 4.3. *If the collection of all m-bi ideals in a semigroup M becomes a totally ordered set under the inclusion, then M is m-regular and m-intraregular if and only if each m-bi ideal of M is prime.*

Proof. Assume M is m-regular and m-intraregular and that B, B_1 and B_2 be any three m-bi ideals of M with the assumption that $B_1B_2 \subseteq B$. This to be noted, by theorem 4.1, B is semiprime. Since the collection of m-bi ideals of M is totally ordered, so by the definition of the total order on B_1 and B_2, one gets $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Without loss of generality, let $B_1 \subseteq B_2$, this produces $B_1^2 \subseteq B_1B_2 \subseteq B$. At this point, $B_1 \subseteq B$ because B is semiprime. Thus, B is a prime m-bi ideal.

For the converse, suppose that every m-bi ideal of M is prime. It is given that the collection of m-bi standards of M is totally ordered, therefore the prime m-bi ideals coincide with the semiprime m-bi. Eventually, by Theorem 4.2, M is both m-standard and m-intraregular. □

5. Topologies of m-Bi Ideals

Topology studies the set structures that are aimed to generalize the geometrical properties of the objects [1]. The ideals in all algebraic structures form topological spaces called the structure space [8]. Such structure spaces are compact and T_1. All algebraic structures have different families of ideals, so these families represent the topologies intrinsically [4]. A natural way in which to study this situation is the semigroups. Detailed procedures of defining the topologies in semigroups and other algebraic structures are given in [8] and [4]. In the beginning of this section, we define the *comaximal* m-bi ideals in semigroups, which will be used later in the section.

Definition 5.1. We say that two m-bi ideals A and B in a semigroup M intersect *transversally* or said to be *comaximal* m-bi ideals ([7, 12]), if $A \cap B = AB$.

We shall use the following notations in our onward work.

Notation 5.1. Let $\mathcal{B}$ is set of all the m-bi ideals of M, we define for each $B \in \mathcal{B}$, the collections.

(a) $\mathcal{P}$ be the set of all prime proper m-bi ideals of M, $\mathcal{K}_B = \{I \in \mathcal{P} : B \nsubseteq I\}$ and $\mathcal{T}(\mathcal{P}) = \{\mathcal{K}_B : B \text{ is an } m\text{-bi ideal of } M\}$.

(b) $\mathcal{S}$ be the set of all strongly prime proper m-bi ideals of M, $\mathcal{C}_B = \{I \in \mathcal{S} : B \nsubseteq I\}$ and $\mathcal{T}(\mathcal{S}) = \{\mathcal{C}_B : B \text{ is an } m\text{-bi ideal of } M\}$.

(c) $\mathcal{H}$ be the family of all properly containing semiprime m-bi ideals in M, $\mathcal{Y}_B = \{I \in \mathcal{H} : B \nsubseteq I\}$ and $\mathcal{T}(\mathcal{H}) = \{\mathcal{Y}_B : B \text{ is an } m\text{-bi ideal of } M\}$.

Notation 5.2. Let $\mathcal{L}$ be the set of all m-left ideals of M, we define for each $L \in \mathcal{L}$, the collections.

(a) $\mathcal{P}_L$ be the set of all prime proper m-bi ideals of M, $\mathcal{K}_L = \{I \in \mathcal{P} : L \nsubseteq I\}$ and $\mathcal{T}_L(\mathcal{P}_L) = \{\mathcal{K}_L : L \text{ is } m\text{-left ideal in } M\}$.

(b) $\mathcal{S}$ be the set of all strongly prime proper m-bi ideals of M, $\mathcal{C}_L = \{I \in \mathcal{S} : L \nsubseteq I\}$ and $\mathcal{T}_L(\mathcal{S}_L) = \{\mathcal{C}_L : L \text{ is } m\text{-left ideal in } M\}$.

(c) $\mathcal{H}$ be the family of all properly containing semiprime m-bi ideals of M, $\mathcal{Y}_L = \{I \in \mathcal{H} : L \nsubseteq I\}$ and $\mathcal{T}_L(\mathcal{H}_L) = \{\mathcal{Y}_L : L \text{ is } m\text{-left ideal in } M\}$.

Notation 5.3. Let $\mathcal{R}$ be the set of all m-right ideals of M, we define for each $R \in \mathcal{R}$, the collections.

(a) $\mathcal{P}_R$ be the set of all prime proper m-bi ideals of M, $\mathcal{K}_R = \{I \in \mathcal{P} : R \nsubseteq I\}$ and $\mathcal{T}_R(\mathcal{P}_R) = \{\mathcal{K}_R : R \text{ is } m\text{-right ideal in } M\}$.

(b) $\mathcal{S}_R$ be the set of all strongly prime proper m-bi ideals of M, $\mathcal{C}_R = \{I \in \mathcal{S} : R \nsubseteq I\}$ and $\mathcal{T}_R(\mathcal{S}_R) = \{\mathcal{C}_R : R \text{ is } m\text{-right ideal in } M\}$.

(c) $\mathcal{H}_R$ be the collection of all properly containing semiprime m-bi ideals of M, $\mathcal{Y}_R = \{I \in \mathcal{H} : R \nsubseteq I\}$ and $\mathcal{T}_R(\mathcal{H}_R) = \{\mathcal{Y}_R : R \text{ is } m\text{-right ideal in } M\}$.

Theorem 5.1. *If, in the semigroup M containing* 0*, the m-bi ideals are pairwise comaximal in the sense of Definition* 5.1*, then the $\mathcal{T}(\mathcal{P})$ forms a topology on the set $\mathcal{P}$.*

Proof. We show that $\mathcal{T}(\mathcal{P})$ satisfies all the three *axioms* of a topology.

(a) Since $\{0\}$ is an m-bi ideal of M, $\mathcal{K}_{\{0\}} = \{I \in \mathcal{P} : \{0\} \nsubseteq I\} = \emptyset$ because 0 belongs to every m-bi ideal of M. So, $\emptyset \in \mathcal{T}(\mathcal{P})$. Since M is also an m-bi ideal of M

whereas $\mathcal{P}$ is the family of all properly containing strongly prime m-bi ideals of M, so $\mathcal{K}_M = \{I \in \mathcal{P} : M \nsubseteq I\} = \mathcal{P}$. Therefore, $\mathcal{P} \in \mathcal{T}(\mathcal{P})$.

(b) Let $\{\mathcal{K}_{B_\alpha} : \alpha \in I\}$ be an arbitrary collection from $\mathcal{T}(\mathcal{P})$. Then

$$\underset{\alpha\in I}{\cup}\mathcal{K}_{B_\alpha} = \{I \in \mathcal{P} : B_\alpha \nsubseteq I \text{ for some } \alpha \in I\} = \left\{I \in \mathcal{P} : \overset{\wedge}{\bigcup_{\alpha\in I}} B_\alpha \nsubseteq I\right\} = \mathcal{K}_{\overset{\wedge}{\bigcup_{\alpha\in I}} B_\alpha},$$

where $\overset{\wedge}{\bigcup_{\alpha\in I}} B_\alpha$ is the m-bi ideal of M generated by $\underset{\alpha\in I}{\cup} B_\alpha$. Therefore, $\underset{\alpha\in I}{\cup}\mathcal{K}_{B_\alpha} \in \mathcal{T}(\mathcal{P})$.

(c) Let $\mathcal{K}_{B_1}$ and $\mathcal{K}_{B_2} \in \mathcal{T}(\mathcal{P})$. If $I \in \mathcal{K}_{B_1} \cap \mathcal{K}_{B_2}$, then $I \in \mathcal{P}$ and neither $B_1 \nsubseteq I$ nor $B_2 \nsubseteq I$. Let $B_1 \cap B_2 \subseteq I$. Since B_1 and B_2 are two *comaximal*, $B_1 \cap B_2 = B_1B_2$. So, $B_1B_2 \subseteq I$. But I is strongly prime m-bi ideal, so either $B_1 \subseteq I$ or $B_2 \subset I$. This is contradiction. Therefore, I does not contain $B_1 \cap B_2$, and so $I \in \mathcal{K}_{B_1\cap B_2}$. Thus $\mathcal{K}_{B_1} \cap \mathcal{K}_{B_2} \subseteq \mathcal{K}_{B_1\cap B_2}$. On the other hand, $I \in \mathcal{K}_{B_1\cap B_2}$ gives that $I \in \mathcal{P}$, but $B_1 \cap B_2 \nsubseteq I$. That is, $B_1 \nsubseteq I$ and $B_2 \nsubseteq I$. Thus $I \in \mathcal{K}_{B_1}$, $I \in \mathcal{K}_{B_2}$. Therefore, $I \in \mathcal{K}_{B_1} \cap \mathcal{K}_{B_2}$. Hence $\mathcal{K}_{B_1\cap B_2} \subseteq \mathcal{K}_{B_1} \cap \mathcal{K}_{B_2}$. Thus, $\mathcal{K}_{B_1} \cap \mathcal{K}_{B_2}\mathcal{K}_{B_1\cap B_2}$. So, $\mathcal{K}_{B_1} \cap \mathcal{K}_{B_2} \in \mathcal{T}(\mathcal{P})$. This shows that $\mathcal{T}(\mathcal{P})$ is a topology on $\mathcal{P}$. □

Corollary 5.1. *The collection $\mathcal{T}_L(\mathcal{P}_L)$ as defined in Notations* 5.2 *for the m-left ideals forms topology.*

Corollary 5.2. *The collection $\mathcal{T}_R(\mathcal{P}_R)$ as defined in Notations* 5.3 *for the m-right ideals forms a topology.*

Theorem 5.2. *If M is an m-regular and m-intraregular semigroup containing* 0*, then $\mathcal{T}(\mathcal{S})$ forms a topology on the set $\mathcal{S}$.*

Proof. We show that $\mathcal{T}(\mathcal{S})$ satisfies all the three *axioms* of a topology.

(a) Similar to (a) of proof of Theorem 5.1.

(b) Similar to (b) of proof of Theorem 5.1.

(c) Let $\mathcal{C}_{B_1}$ and $\mathcal{C}_{B_2} \in \mathcal{T}(\mathcal{S})$. If $I \in \mathcal{C}_{B_1} \cap \mathcal{C}_{B_2}$, then $I \in \mathcal{S}$ and $B_1 \nsubseteq I$, $B_2 \nsubseteq I$. Suppose $B_1 \cap B_2 \subseteq I$. Since M is both m-regular and m-intraregular, $B_1 \cap B_2 = B_1B_2 \cap B_2B_1$. Hence, $B_1B_2 \cap B_2B_1 \subseteq I$. This implies either $B_1 \subseteq I$ or $B_2 \subset I$; a contradiction. Therefore, $B_1 \cap B_2 \nsubseteq I$, and so $I \in \mathcal{C}_{B_1\cap B_2}$. Thus $\mathcal{C}_{B_1} \cap \mathcal{C}_{B_2} \subseteq \mathcal{C}_{B_1\cap B_2}$. If $I \in \mathcal{C}_{B_1\cap B_2}$, then we have $I \in \mathcal{S}$ and $B_1 \cap B_2 \nsubseteq I$. This implies that $B_1 \nsubseteq I$ and $B_2 \nsubseteq I$. Thus, $I \in \mathcal{C}_{B_1}$ and $I \in \mathcal{C}_{B_2}$, and therefore $I \in \mathcal{C}_{B_1} \cap \mathcal{C}_{B_2}$. Hence $\mathcal{C}_{B_1\cap B_2} \subseteq \mathcal{C}_{B_1} \cap \mathcal{C}_{B_2}$. Consequently, $\mathcal{C}_{B_1} \cap \mathcal{C}_{B_2}\mathcal{C}_{B_1\cap B_2}$. So, $\mathcal{C}_{B_1} \cap \mathcal{C}_{B_2} \in \mathcal{T}(\mathcal{S})$.

This shows that $\mathcal{T}(\mathcal{S})$ is a topology on $\mathcal{S}$. □

Corollary 5.3. *The collection $\mathcal{T}_L(\mathcal{S}_L)$ as defined in Notations* 5.2 *for the m-left ideals forms topology.*

Corollary 5.4. *The collection $\mathcal{T}_R(\mathcal{S}_R)$ as defined in Notations* 5.2 *for the m-right ideals forms a topology.*

Remark 5.1. (a) In Theorem 5.2 and Theorem 5.1, and their associated corollaries, we took semigroups with 0 in order to present more explicit form of topologies; however, semigroups without 0 also form the topologies.

(b) The m-regularity and m-intraregularity alone on the semigroup M are not enough to prove Theorem 5.1. Theorem 5.2 can also be proved if M possesses the pairwise *comaximal* m-bi ideals, even if in the absence of m-regularity and m-intraregularity property. The imposition of both these two conditions on M does not admit the collection $\mathcal{T}(\mathcal{Y})$ defined in Notation 5.1, (iii) to form a topology on $\mathcal{H}$.

(c) We can compare the strength of these two conditions with that of the pairwise *comaximal* m-bi ideals of M in the domain of the topological spaces. This signifies that the pairwise *comaximal* property is stronger than the m-regularity and m-intraregularity.

(d) The collection $\mathcal{T}_L(\mathcal{H}_L)$ as defined in Notations 5.2 for the m-left ideals does not admit topology.

(e) The collection $\mathcal{T}_R(\mathcal{H}_R)$ as defined in Notations 5.2 for the m-right ideals does not admit a topology.

6. Conclusions

The main conclusions of the article are summarized in the following lines.

(a) The concepts of the prime, semiprime and strongly prime m-bi in the semigroups were introduced. With the help of the examples, we showed that the m-bi ideals have different properties than the bi ideals in semigroups.

(b) We also presented the concept of the maximal, minimal, irreducible and strongly irreducible m-bi ideals and gave the important characterizations of m-regular and m-intraregular semigroups using these m-bi ideals.

(c) We showed that the prime m-bi ideals form topology when the pairwise *comaximal* property is satisfied. The strongly prime m-bi ideals form the topology separately when the semigroups is m-regular and m-intraregular. And also, when the pairwise comaximal property holds in it. However, the semiprime m-bi ideals do not admit the topology even if both the properties are satisfied by the semigroups.

(d) In the future, we can extend this work to explore more topologies of these m-bi ideals. We can explore metric topologies on these ideals. We can extend the work on the chains of the m-bi deals and characterize the semigroups through the maximal m-bi ideals, and other classes of prime m-bi ideals. The work can be extended to other algebraic structures, especially semirings. The idea of the m-bi ideal is also of importance to explore the properties of the simple semigroups.

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References

[1] K. Anitha, *Rough set theory on topological spaces*, in: *International Conference on Rough Sets and Knowledge Technology*, Springer, Cham., 2014, 69–74.

[2] J. S. Golan, *The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science*, Addison - Wesley Longman Ltd., 1992.

[3] P. Grillet, *Intersections of maximal ideals in semigroups*, Amer. Math. Monthly **76**(5) (1969), 503–509.

[4] J. Horne, *On the ideal structure of certain semirings and compactification of topological spaces*, Trans. Amer. Math. Soc. **90**(3) (1959), 408–430.

[5] J. Howie, *An Introduction to Semigroup Theory*, New York, London, 1976.

[6] K. Iseki, *Ideal theory of semiring*, Proc. Japan Acad. **32**(8) (1956), 554–559.

[7] C. U. Jensen, *Arithmetical rings*, Acta Math. Hungar. **17**(1–2) (1966), 115–123.

[8] K. Magill, *Topological spaces determined by left ideals of semigroups*, Pacific J. Math. **24**(2) (1968), 319–330.

[9] M. Munir and A. Shafiq, *A generalization of bi ideals in semirings*, Bull. Int. Math. Virtual Inst. **8**(1) (2018), 123–133. https://doi.org/10.7251/BIMVI1801123M

[10] M. Munir, *On m-bi ideals in semigroups*, Bull. Int. Math. Virtual Inst. **8**(3) (2018), 461–467. https://doi.org10.7251/BIMVI1803461M

[11] M. Munir and A. Ali, *On generalization of quasi ideals in semirings*, Bull. Int. Math. Virtual Inst. **10**(1) (2020), 83–94. https://doi.org10.7251/BIMVI2001083M

[12] J. Saha, I. Sengupta and G. Tripathi, *Transversal intersection of monomial ideals*, Proceedings - Mathematical Sciences **129**(5) (2019), Article ID 75. https://doi.org/10.1007/s12044-019-0509-5

[13] Š. Schwarz, *On the structure of simple semigroups without zero*, Czechoslovak Math. J. **1**(1) (1951), 41–53.

[14] Š. Schwarz, *Prime ideals and maximal ideals in semigroups*, Czechoslovak Math. J. **19**(1) (1969), 72–79.

[15] M. Shabir and N. Kanwal, *Prime bi-ideals of semigroups*, Southeast Asian Bull. Math. **31**(4) (2007), 757–764.

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