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GEOMETRIC INEQUALITIES FOR STATISTICAL SUBMANIFOLDS IN COSYMPLECTIC STATISTICAL MANIFOLDS

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Abstract. In this paper, we obtain two important geometric inequalities namely, Euler's inequality and Chen's inequality for statistical submanifolds in cosymplectic statistical manifolds with constant curvature, and discuss the equality case of the inequalities. We also give some applications of the inequalities obtained.

1. Introduction

Since Lauritzen introduced the notion of statistical manifolds in 1987 [18], the geometry of statistical manifolds has been developed in close relations with affine differential geometry and Hessian geometry as well as information geometry [2,17,28].

The notion of statistical submanifold was introduced in 1989 by Vos [27]. Though, it has made very little progress due to the hardness to find classical differential geometric approaches for study of statistical submanifolds. However, in the recent years many research has been published in the area and it remains a hot topic for the researchers [4,6,13,14,23,25,26].

On the other hand, in 1993, B.-Y. Chen [8] established the simple relationships between the main intrinsic invariants and the main extrinsic invariants of the submanifolds know as the theory of Chen invariants, which is one of the most interesting research area of differential geometry.

Since then different geometers obtained the similar inequalities for different submanifolds and ambient spaces due to its rich geometric importance [5,9,10,12,16,20,21,24]. In [22], A. Mihai and I. Mihai established a Chen-Ricci inequality with respect to a

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sectional curvature of the ambient Hessian manifold. In [3], Aquib obtained Chen's inequality for totally real statistical submanifolds of quaternion Kaehler-like statistical space forms. Recently, Chen et al. [11] obtained Chen first inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature. In [19], F. Malek and H. Akbari, obtained bounds for Casorati curvatures of submanifolds in Cosymplectic statistical space forms.

In the present article, motivated by the problems proposed in [7] we derive Chen's inequality for statistical manifolds in Cosymplectic statistical manifold with constant curvature and investigate the equality case of the inequality. We also give some applications of the inequalities we derived.

2. Preliminaries

Let $(\overline{M}, g)$ be a Riemannian manifold and $\overline{\nabla}$ and $\overline{\nabla}^*$ be torsion-free affine connections on $\overline{M}$ such that

$$Zg(X,Y) = g(\overline{\nabla}_Z X, Y) + g(X, \overline{\nabla}^*_Z Y),$$

for $X, Y, Z \in \Gamma(T\overline{M})$. Then Riemannian manifold $(\overline{M}, g)$ is called a statistical manifold. It is denoted by $(\overline{M}, g, \overline{\nabla}, \overline{\nabla}^*)$. The connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are called dual connections. The pair $(\overline{\nabla}, g)$ is said to be a statistical structure.

If $(\overline{\nabla}, g)$ is a statistical structure on $\overline{M}$, then $(\overline{\nabla}^*, g)$ is also statistical structure on $\overline{M}$.

For the dual connections $\overline{\nabla}$ and $\overline{\nabla}^*$ we have

$$2\overline{\nabla}^\circ = \overline{\nabla} + \overline{\nabla}^*, \tag{2.1}$$

where $\overline{\nabla}^\circ$ is Levi-Civita connection for g.

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold and let M be an $(m+1)$-dimensional submanifolds of $\overline{M}$. Then, the Gauss formulae are [27]

$$\begin{cases} \overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \\ \overline{\nabla}^*_X Y = \nabla^*_X Y + h^*(X,Y), \end{cases}$$

where h and h^* are symmetric, bilinear, imbedding curvature tensors of M in $\overline{M}$ for $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively.

Let $\overline{R}$ and $\overline{R}^*$ be Riemannian curvature tensor fields of $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively. Then [27]

$$\begin{aligned} g(\overline{R}(X,Y)Z,W) =& g(R(X,Y)Z,W) + g(h(X,Z), h^*(Y,W)) \\ & - g(h^*(X,W), h(Y,Z)) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} g(\overline{R}^*(X,Y)Z,W) =& g(R^*(X,Y)Z,W) + g(h^*(X,Z), h(Y,W)) \\ & - g(h(X,W), h^*(Y,Z)), \end{aligned} \tag{2.3}$$

where

$$g(\overline{R}^*(X,Y)Z,W) = -g(Z,\overline{R}(X,Y)W).$$

Let us denote the normal bundle of M by $TM^\perp$. The linear transformations A_N and A_N^* are defined by

$$\begin{cases} g(A_N X, Y) = g(h(X,Y), N), \\ g(A_N^* X, Y) = g(h^*(X,Y), N), \end{cases}$$

for any $N \in \Gamma(TM^\perp)$ and $X, Y \in \Gamma(TM)$. The corresponding Weingarten formulae are [27]

$$\begin{cases} \overline{\nabla}_X N = -A_N^* X + \nabla_X^\perp N, \\ \overline{\nabla}_X^* N = -A_N X + \nabla_X^{*\perp} N, \end{cases}$$

where $N \in \Gamma(TM^\perp)$, $X \in \Gamma(TM)$ and $\nabla_X^\perp$ and $\nabla_X^{*\perp}$ are Riemannian dual connections with respect to the induced metric on $\Gamma(TM^\perp)$.

For a statistical manifod $(\overline{M}, \overline{g}, \overline{\nabla})$, the difference (1,2)-tensor K of the torsion free affine connection $\overline{\nabla}$ and levi-Civita connection $\overline{\nabla}^\circ$ is defined as (see [15])

$$K_X Y = K(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_X^\circ Y.$$

K is a difference tensor field on $\overline{M}$, that is, $K_X Y = K_Y X$ and

$$\overline{g}(K_X Y, Z) = \overline{g}(Y, K_X Z).$$

Now, we consider a cosymplectic statistical structure on a cosymplectic manifold and define cosymplectic statistical manifold and cosymplectic statistical space form.

Definition 2.1 ([19])**.** $(\overline{\nabla}, \overline{g}, \phi, \xi, \eta)$ is called cosymplectic statistical structure on $\overline{M}$ if $(\overline{\nabla}, \overline{g})$ is a statistical structure and $\phi^2 X = -X + \eta(X)\xi$, $\eta(\xi) = 1$, $\phi(\xi) = 0$, $\overline{g}(\phi X, Y) = -\overline{g}(X, \phi Y)$, $\overline{\nabla}_X^\circ \phi = 0$. That means $(\overline{g}, \phi, \xi, \eta)$ is a cosymplectic structure on $\overline{M}$, and the formula $K_X \phi Y + \phi K_X Y = 0$ holds for any $X, Y \in \Gamma(T\overline{M})$.

Let $(\overline{M}, \overline{\nabla}, \overline{g})$ be a statistical manifold. The tensor field $\overline{R}(X,Y,Z,W)$ is not skew-symmetric relative to Z and W. Then, the sectional curvature on $\overline{M}$ can not be defined by the standard definition. In [15] Furuhata and Hasegawa have defined the statistical curvature tensor field $\overline{S}$ for a statistical manifold $(\overline{M}, \overline{\nabla}, \overline{g})$ as follows:

$$\overline{S}(X,Y)Z = \frac{1}{2}\{\overline{R}(X,Y)Z + \overline{R}^*(X,Y)Z\}. \tag{2.4}$$

Definition 2.2 ([19])**.** $(\overline{M}, \overline{\nabla}, \overline{g}, \phi, \xi)$ be cosymplectic statistical manifold and c a real constant. The cosymplectic statistical structure is said to be of constant ϕ-sectional curvature c if

$$\begin{aligned} \overline{S}(X,Y)Z = &\frac{c}{4}\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y + \overline{g}(X,\phi Z)\phi Y \\ &- \overline{g}(Y,\phi Z)\phi X + 2\overline{g}(X,\phi Y)\phi Z + \eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + \overline{g}(X,Z)\eta(Y)\xi - \overline{g}(Y,Z)\eta(X)\xi\} \end{aligned} \tag{2.5}$$

holds for any $X, Y, Z \in \Gamma(T\overline{M})$.

Let ξ be tangent to the submanifolds M and let $\{e_1, \ldots, e_{m+1} = \xi\}$ and $\{e_{m+2}, \ldots, e_{2n+1}\}$ be tangent orthonormal frame and normal orthonormal frame, respectively, on M. Then, the mean curvature vector fields H, H^*, H° are given by

$$H = \frac{1}{m+1} \sum_{i=1}^{m+1} h(e_i, e_i),$$
$$H^* = \frac{1}{m+1} \sum_{i=1}^{m+1} h^*(e_i, e_i)$$

and

$$H^\circ = \frac{1}{m+1} \sum_{i=1}^{m+1} h^\circ(e_i, e_i).$$

We also set

$$\|h\|^2 = \sum_{i,j=1}^{m+1} g(h(e_i, e_j), h(e_i, e_j)),$$
$$\|h^*\|^2 = \sum_{i,j=1}^{m+1} g(h^*(e_i, e_j), h^*(e_i, e_j))$$

and

$$\|h^\circ\|^2 = \sum_{i,j=1}^{m+1} g(h^\circ(e_i, e_j), h^\circ(e_i, e_j)).$$

The second fundamental form h°(resp. h, or h^*) has several geometric properties due to which we got following different classes of the submanifolds.

- A submanifold is said to be totally geodesic submanifold with respect to $\overline{\nabla}^\circ$ (resp. $\overline{\nabla}$ or $\overline{\nabla}^*$), if the second fundamental form h° (resp. h or h^*) vanishes identically, that is $h^\circ = 0$ (resp. $h = 0$ or $h^* = 0$).
- A submanifold is said to be minimal submanifold with respect to $\overline{\nabla}^\circ$ (resp. $\overline{\nabla}$ or $\overline{\nabla}^*$), if the mean curvature vector H° (resp. H or H^*) vanishes identically, that is $H^\circ = 0$ (resp. $H = 0$ or $H^* = 0$).

Let $K(\pi)$ denotes the sectional curvature of a Riemannian manifold M of the plane section $\pi \subset T_pM$ at a point $p \in M$. If $\{e_1, \ldots, e_{m+1}\}$ be an orthonormal basis of T_pM and $\{e_{m+2}, \ldots, e_{2n+1}\}$ be an orthonormal basis of $T_p^\perp M$ at any $p \in M$, then

$$\tau(p) = \sum_{1 \leq i < j \leq m+1} K(e_i \wedge e_j),$$

where τ is the scalar curvature. The normalized scalar curvature ρ is defined as

$$2\tau = m(m+1)\rho.$$

We also put

$$h_{ij}^\gamma = g(h(e_i, e_j), e_\gamma), \quad h_{ij}^{*\gamma} = g(h^*(e_i, e_j), e_\gamma),$$

where $i, j \in \{1, \dots, m+1\}$, $\gamma \in \{m+2, \dots, 2n+1\}$.

The square norm P at $p \in M$ is defined as

$$||P||^2 = \sum_{i,j=1}^{m+1} g^2(e_i, \phi e_j).$$

Lemma 2.1 ([11]). *For $m+1 \geq 3$, $a_1, a_2, \dots, a_{m+1}$, $m+1$ real numbers*

$$\sum_{1 \leq i < j \leq m+1} a_i a_j - a_1 a_2 \leq \frac{m-1}{2m} \left(\sum_{i=1}^{m+1} a_i \right)^2 .$$

Moreover, equality holds if and only if $a_1 + a_2 = a_3 = \dots = a_{m+1}$.

3. Euler's Inequality for Cosymplectic Manifold

In this section we will prove the Euler's inequality for statistical submanifolds of Cosymplectic manifold. To be precise we will prove the following.

Theorem 3.1. *Let M be a statistical submanifold in a cosymplectic statistical manifold $\overline{M}(c)$. Then*

$$2\tau \geq \frac{c}{4}[m(m-1) + 3||P||^2] - ||h^\circ||^2 + (m+1)^2 g(H, H^*).$$

Further, equality case of the inequality holds if and only if $h = h^*$.

Proof. From (2.2), (2.3) and (2.5), we have

$$\begin{aligned} g(S(X,Y)Z,W) =& g(\overline{S}(X,Y)Z,W) - \frac{1}{2}[g(h(X,Z),h^*(Y,W)) \\ & - g(h^*(X,W),h(Y,Z)) + (h^*(X,Z),h(Y,W)) \\ & - g(h(X,W),h^*(Y,Z))] \\ =& \frac{c}{4}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ & + g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) \\ & + 2g(X,\phi Y)g(\phi Z,W) + \eta(X)\eta(Z)g(Y,W) \\ & - \eta(Y)\eta(Z)g(X,W) + g(X,Z)\eta(Y)g(\xi,W) \\ & - g(Y,Z)\eta(X)g(\xi,W)] \\ & - \frac{1}{2}[g(h(X,Z),h^*(Y,W)) - g(h^*(X,W),h(Y,Z)) \\ & + g(h^*(X,Z),h(Y,W)) - g(h(X,W),h^*(Y,Z))]. \end{aligned}$$

Put $X = W = e_i$, $Y = Z = e_j$, we have

$$\begin{aligned} g(S(e_i,e_j)e_j,e_i) =& \frac{c}{4}\{g(e_j,e_j)g(e_i,e_i) - g(e_i,e_j)g(e_j,e_i) \\ & + g(e_i,\phi e_j)g(\phi e_j,e_i) - g(e_j,\phi e_j)g(\phi e_i,e_i) \\ & + 2g(e_i,\phi e_j)g(\phi e_j,e_i) + \eta(e_i)\eta(e_j)g(e_j,e_i) \end{aligned}$$

$$
\begin{aligned}
&- \eta(e_j)\eta(e_j)g(e_i, e_i) + g(e_i, e_j)\eta(e_j)g(\xi, e_i)\\
&- g(e_j, e_j)\eta(e_i)g(\xi, e_i)\}\\
&- \frac{1}{2}\Big[g\Big(h(e_i, e_j), h^*(e_j, e_i)\Big) - g\Big(h^*(e_i, e_i), h(e_j, e_j)\Big)\\
&+ g\Big(h^*(e_i, e_j), h(e_j, e_i)\Big) - g\Big(h(e_i, e_i), h^*(e_j, e_j)\Big)\Big].
\end{aligned}
$$

Taking summation, we derive

$$
\begin{aligned}
2\tau =& \frac{c}{4}\Big[(m+1)^2 - (m+1)\\
&+ 3\sum_{i,j=1}^{m+1} g^2(e_i, \phi e_j) + 1 - (m+1) + 1 - (m+1)\Big]\\
&- \frac{1}{2}\sum_{i,j=1}^{m+1}\Big[g\Big(h(e_i, e_j), h^*(e_j, e_i)\Big) - g\Big(h^*(e_i, e_i), h(e_j, e_j)\Big)\\
&+ g\Big(h^*(e_i, e_j), h(e_j, e_i)\Big) - g\Big(h(e_i, e_i), h^*(e_j, e_j)\Big)\Big]\\
=& \frac{c}{4}\Big[m(m-1) + 3||P||^2\Big] - \frac{1}{2}\sum_{i,j=1}^{m+1}\Big[2g\Big(h(e_i, e_j), h^*(e_j, e_i)\Big)\\
&- g\Big(h^*(e_i, e_i), h(e_j, e_j)\Big) - g\Big(h(e_i, e_i), h^*(e_j, e_j)\Big)\Big]\\
=& \frac{c}{4}\Big[m(m-1) + 3||P||^2\Big]\\
&- \frac{1}{2}\Big[2g\Big(h(e_i, e_j), h^*(e_j, e_i)\Big) - 2(m+1)^2 g(H, H^*)\\
=& \frac{c}{4}\Big[m(m-1) + 3||P||^2\Big] - g\Big(h(e_i, e_j), h^*(e_j, e_i)\Big) + (m+1)^2 g(H, H^*)\\
=& \frac{c}{4}\Big[m(m-1) + 3||P||^2\Big] + (m+1)^2 g(H, H^*) - \sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1} h_{ij}^{\alpha} h_{ij}^{*\alpha}\\
=& \frac{c}{4}\Big[m(m-1) + 3||P||^2\Big] + (m+1)^2 g(H, H^*)\\
&- \frac{1}{4}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}\Big[(h_{ij}^{\alpha} + h_{ij}^{*\alpha})^2 - (h_{ij}^{\alpha} - h_{ij}^{*\alpha})^2\Big]\\
=& \frac{c}{4}\Big[m(m-1) + 3||P||^2\Big] + (m+1)^2 g(H, H^*) - ||h^{\circ}||^2\\
&+ \frac{1}{4}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}\Big[(h_{ij}^{\alpha} - h_{ij}^{*\alpha})^2\\
\geq& \frac{c}{4}\Big[m(m-1) - ||h^{\circ}||^2 + 3||P||^2\Big] + (m+1)^2 g(H, H^*),
\end{aligned}
$$

which is the required result. □

An immediate consequence of the Theorem 3.1 is the following result.

Corollary 3.1. *Let M be a statistical submanifold in a cosymplectic statistical manifold $\overline{M}(c)$. Then*

Angle between H and H^*	*Inequalities*
θ	$2\tau \geq \frac{c}{4}[m(m-1)+3\|P\|^2] - \|\|h^\circ\|\|^2 + (m+1)^2\|H\|\|H^*\|\cos\theta$
0°	$2\tau \geq \frac{c}{4}[m(m-1)+3\|P\|^2] - \|\|h^\circ\|\|^2 + (m+1)^2\|H\|\|H^*\|$
90°	$2\tau \geq \frac{c}{4}[m(m-1)+3\|P\|^2] - \|\|h^\circ\|\|^2$

4. Chen's Inequality for Cosymplectic Statistical Manifolds

This section is devoted to the main result of the article. Here, we obtain Chen's inequality for statistical submanifolds of Cosymplectic statistical manifolds with constant ϕ-sectional curvature.

Theorem 4.1. *Let M be a statistical submanifold in a cosymplectic statistical manifold $\overline{M}(c)$. Then*

$$\begin{aligned}\tau - K(\pi) \geq & 2\tau_\circ - k_\circ(\pi) + \frac{c}{4}[(1+m-m^2) - 3||P||^2 \\ & + 3\Theta(\pi) - \Phi(\pi)] - \frac{(m+1)^2(m-1)}{4m}(||H||^2 + ||H^*||^2),\end{aligned}$$

where $\Theta(\pi) = g^2(\phi e_1, e_2)$, $\Phi(\pi) = \eta^2(e_1) + \eta^2(e_2)$, $\pi = e_1 \wedge e_2$. Moreover, the equality holds if and only if

$$\begin{aligned}& h^\alpha_{11} + h^\alpha_{22} = h^\alpha_{33} = \cdots = h^\alpha_{m+1m+1}, \\ & h^{*\alpha}_{11} + h^{*\alpha}_{22} = h^{*\alpha}_{33} = \cdots = h^{*\alpha}_{m+1m+1}, \\ & h^\alpha_{ij} = h^{*\alpha}_{ij}, \quad 1 \leq i \neq j \leq m+1,\ (i,j) \neq (1,2),(2,1),\ \alpha \in \{m+2, \dots, 2n+1\}.\end{aligned} \tag{4.1}$$

Proof. From (2.2), (2.3) and (2.4), we have

$$\begin{aligned}2g(\overline{S}(X,Y)Z,W) = & 2g(S(X,Y)Z,W) + g(h(X,Z),h^*(Y,W)) \\ & - g(h^*(X,W),h(Y,Z)) + g(h^*(X,Z),h(Y,W)) \\ & - g(h(X,W),h^*(Y,Z)).\end{aligned} \tag{4.2}$$

Using (2.5) in (4.2) and putting $X = W = e_i$, $Y = Z = e_j$, we get

$$\begin{aligned}
2g(S(e_i, e_j)e_j, e_i) =& \frac{c}{2}[g(e_j, e_j)g(e_i, e_j) - g(e_i, e_j)g(e_j, e_i) \\
&+ g(e_i, \phi e_j)g(\phi e_j, e_i) - g(e_j, \phi e_j)g(\phi e_i, e_j) \\
&+ 2g(e_i, \phi e_j)g(\phi e_j, e_i) + \eta(e_i)\eta(e_j)g(e_j, e_i) \\
&- \eta(e_j)\eta(e_j)g(e_i, e_i) + g(e_i, e_j)\eta(e_j)g(\xi, e_i) \\
&- g(e_j, e_j)\eta(e_i)g(\xi, e_i)] - g(h(e_i, e_j), h^*(e_j, e_i)) \\
&+ g(h^*(e_i, e_i), h(e_j, e_j)) - g(h^*(e_i, e_j), h(e_j, e_i)) \\
&+ g(h(e_i, e_i), h^*(e_j, e_j)).
\end{aligned}$$

Applying summation over $i, j = 1, 2, \ldots, m+1$, we obtain

$$\begin{aligned}
\sum_{1 \le i < j \le m+1} 2g(S(e_i, e_j)e_j, e_i) =& \frac{c}{2}[(m+1)^2 - (m+1) \\
&+ 3\sum_{i,j=1}^{m+1} g^2(\phi e_j, e_i) + 1 - (m+1) + 1 - (m+1)] \\
&- g(h(e_i, e_j), h^*(e_j, e_i)) + g(h^*(e_i, e_i), h(e_j, e_j)) \\
&- g(h^*(e_i, e_j), h(e_j, e_i)) + g(h(e_i, e_i), h^*(e_j, e_j)) \\
=& \frac{c}{2}[m^2 - m + 3||P||^2] \\
&- g(h(e_i, e_j), h^*(e_j, e_i)) + g(h^*(e_i, e_i), h(e_j, e_j)) \\
&- g(h^*(e_i, e_j), h(e_j, e_i)) + g(h(e_i, e_i), h^*(e_j, e_j)),
\end{aligned}$$

which implies

$$\begin{aligned}
\tau =& \frac{3}{4}c||P||^2 + \frac{c}{4}m(m-1) + \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h_{ii}^{*\alpha}h_{jj}^{\alpha} + h_{ii}^{\alpha}h_{jj}^{*\alpha} - 2h_{ij}^{\alpha}h_{ij}^{*\alpha}) \\
=& \frac{3}{4}c||P||^2 + \frac{c}{4}m(m-1) \\
&+ \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}[(h_{ii}^{\alpha} + h_{ii}^{*\alpha})(h_{jj}^{\alpha} + h_{jj}^{*\alpha}) - h_{ii}^{\alpha}h_{jj}^{\alpha} - h_{ii}^{*\alpha}h_{jj}^{*\alpha} \\
&- (h_{ij}^{\alpha} + h_{ij}^{*\alpha})^2 + (h_{ij}^{\alpha})^2 + (h_{ij}^{*\alpha})^2] \\
=& \frac{3}{4}c||P||^2 + \frac{c}{4}m(m-1) \\
&+ \sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}\left\{2[h_{ii}^{\circ\alpha}h_{jj}^{\circ\alpha} - (h_{ij}^{\circ\alpha})^2] - \frac{1}{2}[h_{ii}^{\alpha}h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2] - \frac{1}{2}[h_{ii}^{*\alpha}h_{jj}^{*\alpha} - (h_{ij}^{*\alpha})^2]\right\}.
\end{aligned} \tag{4.3}$$

Further, with respect to Levi-Civita connection, we have

$$\tau_\circ = \frac{3}{4}c||P||^2 + \frac{c}{4}m(m-1) + \sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}[h_{ii}^{\circ\alpha}h_{jj}^{\circ\alpha} - (h_{ij}^{\circ\alpha})^2]. \tag{4.4}$$

We also have

$$\begin{aligned} K(\pi) =& \frac{1}{2}[g(R(e_1,e_2)e_2,e_1) + g(R^*(e_1,e_2)e_2,e_1)] \\ =& \frac{1}{2}[g(\overline{R}(e_1,e_2)e_2,e_1) + g(\overline{R}^*(e_1,e_2)e_2,e_1) \\ &- 2g(h^*(e_1,e_2),h(e_2,e_1)) + 2g(h^*(e_1,e_1),h(e_2,e_2))] \\ =& g(\overline{S}(e_1,e_2)e_2,e_1) + \sum_{\alpha=m+2}^{2n+1}\left[\frac{1}{2}h_{11}^{*\alpha}h_{22}^{\alpha} + \frac{1}{2}h_{11}^{\alpha}h_{22}^{*\alpha} - h_{12}^{*\alpha}h_{12}^{\alpha}\right] \\ =& g(\overline{S}(e_1,e_2)e_2,e_1) + \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}[(h_{11}^{\alpha}+h_{11}^{*\alpha})(h_{22}^{\alpha}+h_{22}^{*\alpha}) \\ &- h_{11}^{\alpha}h_{22}^{\alpha} - h_{11}^{*\alpha}h_{22}^{*\alpha} - (h_{12}^{\alpha}+h_{12}^{*\alpha})^2 + (h_{12}^{\alpha})^2 + (h_{12}^{*\alpha})^2] \\ =& g(\overline{S}(e_1,e_2)e_2,e_1) + \sum_{\alpha=m+2}^{2n+1}\Big[2\{h_{11}^{\circ\alpha}h_{22}^{\circ\alpha} - (h_{12}^{\circ\alpha})^2\} \\ &- \frac{1}{2}\{h_{11}^{\alpha}h_{22}^{\alpha} - (h_{12}^{\alpha})^2\} - \frac{1}{2}\{h_{11}^{*\alpha}h_{22}^{*\alpha} - (h_{12}^{*\alpha})^2\}\Big]. \end{aligned} \tag{4.5}$$

On the other hand

$$\begin{aligned} g(\overline{S}(e_1,e_2)e_2,e_1) =& \frac{c}{4}[g(e_2,e_2)g(e_1,e_1) - g(e_1,e_2)g(e_2,e_1) + g(e_1,\phi e_2)g(\phi e_2,e_1) \\ &- g(e_2,\phi e_2)g(\phi e_1,e_1) + 2g(e_1,\phi e_2)g(\phi e_2,e_1) \\ &+ \eta(e_1)\eta(e_2)g(e_2,e_1) - \eta(e_2)\eta(e_2)g(e_1,e_1) \\ &+ g(e_1,e_2)\eta(e_2)g(\xi,e_1) - g(e_2,e_2)\eta(e_1)g(\xi,e_1)] \\ =& \frac{c}{4}[1 + 3g^2(e_1,\phi e_2) - \eta(e_1)^2 - \eta(e_2)], \\ =& \frac{c}{4}[1 + 3\theta(\pi) - \Phi(\pi)]. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we get

$$\begin{aligned} K(\pi) =& \frac{c}{4}[1 + 3\theta(\pi) - \Phi(\pi)] + \sum_{\alpha=m+2}^{2n+1}[2\{h_{11}^{\circ\alpha}h_{22}^{\circ\alpha} - (h_{12}^{\circ\alpha})^2\} \\ &- \frac{1}{2}\{h_{11}^{\alpha}h_{22}^{\alpha} - (h_{12}^{\alpha})^2\} - \frac{1}{2}\{h_{11}^{*\alpha}h_{22}^{*\alpha} - (h_{12}^{*\alpha})^2\}]. \end{aligned} \tag{4.7}$$

Also, with respect to Levi-Civita connection, we get

$$K_\circ(\pi) = \frac{c}{4}[1 + 3\theta(\pi) - \Phi(\pi)] + \sum_{\alpha=m+2}^{2n+1}\{h_{11}^{\circ\alpha}h_{22}^{\circ\alpha} - (h_{12}^{\circ\alpha})^2\}. \tag{4.8}$$

Substract (4.7) from (4.3), we get

$$\begin{aligned}
\tau - K(\pi) =& \frac{3}{4}c||P||^2 + \frac{c}{4}m(m-1) - \frac{c}{4}[1 + 3\theta(\pi) - \Phi(\pi)] \\
&+ \sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}\Bigg\{2[h_{ii}^{\circ\alpha}h_{jj}^{\circ\alpha} - (h_{ij}^{\circ\alpha})^2] - \frac{1}{2}[h_{ii}^{\alpha}h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2] \\
&- \frac{1}{2}[h_{ii}^{*\alpha}h_{jj}^{*\alpha} - (h_{ij}^{*\alpha})^2]\Bigg\} - \sum_{\alpha=m+2}^{2n+1}\Big[2\{h_{11}^{\circ\alpha}h_{22}^{\circ\alpha} - (h_{12}^{\circ\alpha})^2\} \\
&- \frac{1}{2}\{h_{11}^{\alpha}h_{22}^{\alpha} - (h_{12}^{\alpha})^2\} - \frac{1}{2}\{h_{11}^{*\alpha}h_{22}^{*\alpha} - (h_{12}^{*\alpha})^2\}\Big] \\
=& \frac{c}{4}\{3||P||^2 + (m^2 - m) - 1 - 3\theta(\pi) + \Phi(\pi)\} \\
&+ \sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1} 2h_{ii}^{\circ\alpha}h_{jj}^{\circ\alpha} - 2\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h_{ij}^{\circ\alpha})^2 - \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1} h_{ii}^{\alpha}h_{jj}^{\alpha} \\
&+ \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h_{ij}^{\alpha})^2 - \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1} h_{ii}^{*\alpha}h_{jj}^{*\alpha} + \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h_{ij}^{*\alpha})^2 \\
&- 2\sum_{\alpha=m+2}^{2n+1} h_{11}^{\circ\alpha}h_{22}^{\circ\alpha} + 2\sum_{\alpha=m+2}^{2n+1}(h_{12}^{\circ\alpha})^2 + \frac{1}{2}\sum_{\alpha=m+2}^{2n+1} h_{11}^{\alpha}h_{22}^{\alpha} \\
&- \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}(h_{12}^{\alpha})^2 + \frac{1}{2}\sum_{\alpha=m+2}^{2n+1} h_{11}^{*\alpha}h_{22}^{*\alpha} - \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}(h_{12}^{*\alpha}) \\
\geq& \frac{c}{4}\{3||P||^2 + (m^2 - m) - 1 - 3\theta(\pi) + \Phi(\pi)\} \\
&- \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h_{ii}^{\alpha}h_{jj}^{\alpha} - h_{11}^{\alpha}h_{22}^{\alpha}) - \frac{1}{2}\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h_{ii}^{*\alpha}h_{jj}^{*\alpha} - h_{11}^{*\alpha}h_{22}^{*\alpha}) \\
&+ 2\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}[(h_{ii}^{\circ\alpha}h_{jj}^{\circ\alpha}) - h_{11}^{\circ\alpha}h_{22}^{\circ\alpha}] - 2\left[\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}(h_{ij}^{\circ\alpha})^2 - (h_{12}^{\circ\alpha})^2\right] \\
\geq& \frac{c}{4}\{3||P||^2 + (m^2 - m) - 1 - 3\theta(\pi) + \Phi(\pi)\} \\
&- \frac{(m+1)^2(m-1)}{4m}(||H||^2 + ||H^*||^2) \\
&+ 2\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}[h_{ii}^{\circ\alpha}h_{jj}^{\circ\alpha} - (h_{ij}^{\circ\alpha})] - 2\sum_{\alpha=m+2}^{2n+1}\sum_{i,j=1}^{m+1}[h_{11}^{\circ\alpha}h_{22}^{\circ\alpha} - (h_{12}^{\circ\alpha})].
\end{aligned}$$

Using (4.4) and (4.8), we have

$$\begin{aligned}
\tau - K(\pi) \geq& \frac{c}{4}\{3||P||^2 + (m^2 - m) - 1 - 3\theta(\pi) + \Phi(\pi)\} \\
&- \frac{(m+1)^2(m-1)}{4m}(||H||^2 + ||H^*||^2) + 2\left[\tau_\circ - \frac{3}{4}c||P||^2\right]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{c}{4}m(m-1) - 2\left[-\frac{c}{4}(1+3\theta(\pi) - \Phi(\pi) + K_\pi\right] \\
=&\frac{3}{4}||P||^2 + \frac{c}{4}(m^2-m) - \frac{c}{4} - \frac{3}{4}c\theta(\pi) + \frac{c}{4}\Phi(\pi) \\
&- \frac{(m+1)^2(m-1)}{4m}(||H||^2 + ||H^*||^2) + 2\tau_\circ - \frac{3}{2}c||P||^2 \\
&- \frac{c}{2}(m^2-m) + \frac{c}{2} + \frac{3}{2}c\theta(\pi) - \frac{c}{2}\Phi(\pi) - 2K_\circ(\pi) \\
\geq&\frac{c}{4} - \frac{3}{4}c||P||^2 - \frac{c}{4}(m^2-m) + \frac{3}{4}c\theta(\pi) - \frac{c}{4}\Phi(\pi) \\
&- \frac{(m+1)^2(m-1)}{4m}(||H||^2 + ||H^*||^2) + 2\tau_\circ - K_\circ(\pi) \\
\geq&\frac{c}{4}[(1+m-m^2) - 3||P||^2 + 3\theta(\pi) - \Phi(\pi)] \\
&- \frac{(m+1)^2(m-1)}{4m}(||H||^2 + ||H^*||^2) + 2\tau_\circ - K_\circ(\pi),
\end{aligned}
$$

which is the required inequality. Moreover, equality holds if and only if it satisfies (4.1). □

From the above theorem we have the following non-existence result of minimal statistical submanifolds in cosymplectic statistical manifold.

Corollary 4.1. *Let M be a statistical submanifold in a cosymplectic statistical manifold $\overline{M}(c)$ such that*

$$(\tau - K(\pi)) - (2\tau_\circ - K_\circ(\pi)) < \frac{c}{4}[(1+m-m^2) - 3||P||^2 + 3\theta(\pi) - \Phi(\pi)],$$

then M can not be minimally immersed in $\overline{M}(c)$ with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ symulteneously.

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