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THREE-WEIGHT AND FIVE-WEIGHT LINEAR CODES OVER FINITE FIELDS

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Abstract. Recently, linear codes constructed from defining sets have been studied extensively. For an odd prime p, let Tr_e^m be the trace function from $\mathbb{F}_{p^m}$ onto $\mathbb{F}_{p^e}$, where e is a divisor of m. In this paper, for the defining set $D = \{x \in \mathbb{F}_{p^m}^* : \mathrm{Tr}_e^m(x^2 + x) = 0\} = \{d_1, d_2, \dots, d_n\}$ (say), we define a p^e-ary linear code $\mathcal{C}_D$ by

$$\mathcal{C}_D = \{c_x = \left(\mathrm{Tr}_e^m(xd_1), \mathrm{Tr}_e^m(xd_2), \dots, \mathrm{Tr}_e^m(xd_n)\right) : x \in \mathbb{F}_{p^m}\}$$

and present three-weight and five-weight linear codes with their weight distributions. We show that each nonzero codeword of $\mathcal{C}_D$ is minimal for $\frac{m}{e} \geq 5$ and, thus, such codes are applicable in secret sharing schemes.

1. Introduction

Throughout this paper, let p be an odd prime, and let $\mathbb{F}_{p^m}$ be the finite field with p^m elements for any positive integer m. Denote by $\mathbb{F}_{p^m}^* = \mathbb{F}_{p^m} \setminus \{0\}$ the multiplicative group of $\mathbb{F}_{p^m}$.

An (n, M) code over $\mathbb{F}_{p^e}$, where $e \mid m$ and $\frac{m}{e} > 2$, is a subset of $\mathbb{F}_{p^e}^n$ of size M. Since linear codes are easier to describe, encode and decode than nonlinear codes, they have been an interesting topic in both theory and practice for many years. A linear code $\mathcal{C}$ over $\mathbb{F}_{p^e}$ is a subspace of $\mathbb{F}_{p^e}^n$. An $[n, k, d]$ linear code $\mathcal{C}$ is a k-dimensional subspace of $\mathbb{F}_{p^e}^n$ with minimum Hamming-distance d. The vectors in a linear code $\mathcal{C}$ are known as *codewords*. The number of nonzero coordinates in $c \in \mathcal{C}$ is called the Hamming-weight $\mathrm{wt}(c)$ of a codeword c. Let A_i denote the number of codewords with Hamming weight

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i in a linear code $\mathcal{C}$ of length n. The weight enumerator of $\mathcal{C}$ is defined by

$$1 + A_1 z + A_2 z^2 + \cdots + A_n z^n,$$

where $(1, A_1, \ldots, A_n)$ is called the *weight distribution* of $\mathcal{C}$. Throughout the paper, $\#\{\cdot\}$ denotes the cardinality of the set. If $\#\{i : A_i \neq 0, 1 \leq i \leq n\} = t$, then the code $\mathcal{C}$ is said to be t-weight code. Several classes of linear codes with various weights have been constructed in [3, 5, 6, 8, 19], and a lot of literature is present on the weight distributions of some special linear codes [1, 2, 14, 15].

Let $D = \{d_1, d_2, \ldots, d_n\} \subseteq F_{p^m}$. A linear code $\mathcal{C}_D$ of length n over $\mathbb{F}_p$ is defined by

$$\mathcal{C}_D = \{\Big(\mathrm{Tr}_1^m(xd_1), \mathrm{Tr}_1^m(xd_2), \ldots, \mathrm{Tr}_1^m(xd_n)\Big) : x \in \mathbb{F}_{p^m}\},$$

where Tr_1^m denotes the absolute trace function from $\mathbb{F}_{p^m}$ onto $\mathbb{F}_p$. The set D is known as the defining set of this code $\mathcal{C}_D$. Ding et al. introduced this construction (see [6, 7]), and many others used it to obtain linear codes with few weights [8, 17]. In [3, 6, 11, 14, 17, 19], the authors constructed the code $\mathcal{C}_D$ over $\mathbb{F}_p$ with few weights by considering certain defining sets with absolute trace function. In particular, the authors, in [11], give linear codes over $\mathbb{F}_p$ by employing Gauss sums and Pless Power Moments [10, page 260].

In this paper, we use Gauss sums and cyclotomic numbers to find linear codes over $\mathbb{F}_{p^e}$ by considering a new defining set obtained by replacing Tr by Tr_e^m in the defining set D given in [11]. Let m, s and e are positive integers with $s > 2$ and $m = es$. Now we define the trace function Tr_e^m from $\mathbb{F}_{p^m}$ onto $\mathbb{F}_{p^e}$ as follows:

$$\mathrm{Tr}_e^m(x) = \sum_{k=0}^{s-1} x^{p^{ke}}.$$

Now, set

$$D = \{x \in \mathbb{F}_{p^m}^* : \mathrm{Tr}_e^m(x^2 + x) = 0\} = \{d_1, d_2, \ldots, d_n\},$$
$$C_D = \{\boldsymbol{c}_x = (\mathrm{Tr}_e^m(xd_1), \mathrm{Tr}_e^m(xd_2), \ldots, \mathrm{Tr}_e^m(xd_n)) : x \in \mathbb{F}_{p^m}\}. \tag{1.1}$$

Then we present the weight distribution of the proposed linear code $\mathcal{C}_D$ of (1.1) in the Section 4.

2. Preliminaries

We begin with some preliminaries by introducing the concept of cyclotomic numbers. Let a be a primitive element of $\mathbb{F}_{p^m}$ and $p^m = Nh + 1$ for two positive integers $N > 1$, $h > 1$. The *cyclotomic classes* of order N in $\mathbb{F}_{p^m}$ are the cosets $\mathcal{C}_i^{(N,p^m)} = a^i \langle a^N \rangle$ for $i = 0, 1 \ldots, N-1$, where $\langle a^N \rangle$ denotes the subgroup of $\mathbb{F}_{p^m}^*$ generated by a^N. It is obvious that $\#\mathcal{C}_i^{(N,p^m)} = h$. For fixed i and j, we define the *cyclotomic number* $(i,j)^{(N,p^m)}$ to be the number of solutions of the equation

$$x_i + 1 = x_j, \quad x_i \in \mathcal{C}_i^{(N,p^m)}, x_j \in \mathcal{C}_j^{(N,p^m)},$$

where $1 = a^0$ is the multiplicative identity of $\mathbb{F}_{p^m}$. That is, $(i,j)^{(N,p^m)}$ is the number of ordered pairs (s,t) such that

$$a^{Ns+i} + 1 = a^{Nt+j}, \quad 0 \le s,t \le h-1.$$

Now, we present some notions and results about group characters and Gauss sums for later use (see [12] for details).

An additive character χ of $\mathbb{F}_{p^m}$ is a mapping from $\mathbb{F}_{p^m}$ into the multiplicative group of complex numbers of absolute value 1 with $\chi\ (g_1+g_2) = \chi\ (g_1)\chi(g_2)$ for all $g_1, g_2 \in \mathbb{F}_{p^m}$. By ([12], Theorem 5.7), for any $b \in \mathbb{F}_{p^m}$,

$$\chi_b(x) = \zeta_p^{\mathrm{Tr}_1^m(bx)}, \quad \text{for all } x \in \mathbb{F}_{p^m}, \tag{2.1}$$

defines an additive character of $\mathbb{F}_{p^m}$, where $\zeta_p = e^{\frac{2\pi\sqrt{-1}}{p}}$, and every additive character can be obtained in this way. An additive character defined by $\chi_0(x) = 1$ for all $x \in \mathbb{F}_{p^m}$ is called the trivial character while all other characters are called nontrivial characters. The character χ_1 in (2.1) is called the canonical additive character of $\mathbb{F}_{p^m}$.

The orthogonal property of additive characters of $\mathbb{F}_{p^m}$ can be found in ([12], Theorem 5.4) and is given as

$$\sum_{x\in\mathbb{F}_{p^m}} \chi(x) = \begin{cases} p^m, & \text{if } \chi \text{ trivial}, \\ 0, & \text{if } \chi \text{ non-trivial}. \end{cases} \tag{2.2}$$

Characters of the multiplicative group $\mathbb{F}_{p^m}^*$ of $\mathbb{F}_{p^m}$ are called multiplicative character of $\mathbb{F}_{p^m}$. By [12, Theorem 5.8], for each $j = 0, 1, \ldots, p^m-2$, the function ψ_j with

$$\psi_j(g^k) = e^{\frac{2\pi\sqrt{-1}jk}{p^m-1}}, \quad \text{for } k = 0, 1, \ldots, p^m-2$$

defines a multiplicative character of $\mathbb{F}_{p^m}$, where g is a generator of $\mathbb{F}_{p^m}^*$. For $j = \frac{p^m-1}{2}$, we have the quadratic character $\eta = \psi_{\frac{p^m-1}{2}}$ defined by

$$\eta(g^k) = \begin{cases} -1, & \text{if } 2 \nmid k, \\ 1, & \text{if } 2 \mid k. \end{cases}$$

Moreover, we extend this quadratic character by letting $\eta(0) = 0$.

The quadratic Gauss sum $G = G(\eta, \chi_1)$ over $\mathbb{F}_{p^m}$ is defined by

$$G(\eta, \chi_1) = \sum_{x\in\mathbb{F}_{p^m}^*} \eta(x)\chi_1(x).$$

Now, let $\overline{\eta}$ and $\overline{\chi}_1$ denote the quadratic and canonical character of $\mathbb{F}_{p^e}$ respectively. Then we define the quadratic Gauss sum $\overline{G} = G(\overline{\eta}, \overline{\chi}_1)$ over $\mathbb{F}_{p^e}$ by

$$G(\overline{\eta}, \overline{\chi}_1) = \sum_{x\in\mathbb{F}_{p^e}^*} \overline{\eta}(x)\overline{\chi}_1(x).$$

The explicit values of quadratic Gauss sums are given by the following lemma.

Lemma 2.1. ([12, Theorem 5.15]). *Let the symbols be the same as before. Then*

$$G(\eta,\chi_1)=(-1)^{m-1}\sqrt{-1}^{\frac{(p-1)^2m}{4}}\sqrt{p^m},\quad G(\overline{\eta},\overline{\chi}_1)=(-1)^{e-1}\sqrt{-1}^{\frac{(p-1)^2e}{4}}\sqrt{p^e}.$$

Lemma 2.2. ([13, Lemma 2]). *Let the symbols be the same as before. Then the following hold.*

1. *If $s\geq 2$ is even, then $\eta(y)=1$ for each $y\in\mathbb{F}_{p^e}^*$;*
2. *If s is odd, then $\eta(y)=\overline{\eta}(y)$ for each $y\in\mathbb{F}_{p^e}^*$.*

Lemma 2.3. ([16]). *When $N=2$, the cyclotomic numbers are given by*

1. *h even: $(0,0)^{(2,p^m)}=\frac{h-2}{2}$, $(0,1)^{(2,p^m)}=(1,0)^{(2,p^m)}=(1,1)^{(2,p^m)}=\frac{h}{2}$;*
2. *h odd: $(0,0)^{(2,p^m)}=(1,0)^{(2,p^m)}=(1,1)^{(2,p^m)}=\frac{h-1}{2}$, $(0,1)^{(2,p^m)}=\frac{h+1}{2}$.*

Lemma 2.4. ([12, Theorem 5.33]). *Let χ be a non-trivial additive character of $\mathbb{F}_{p^m}$, and let $f(x)=a_2x^2+a_1x+a_0\in\mathbb{F}_{p^m}[x]$ with $a_2\neq 0$. Then*

$$\sum_{x\in\mathbb{F}_{p^m}}\chi(f(x))=\chi(a_0-a_1^2(4a_2)^{-1})\eta(a_2)G(\eta,\chi).$$

Lemma 2.5. ([12, Theorem 2.26]). *Let Tr_1^m and Tr_1^e be absolute trace functions over $\mathbb{F}_{p^m}$ and $\mathbb{F}_{p^e}$ respectively, and let Tr_e^m be the trace function from $\mathbb{F}_{p^m}$ onto $\mathbb{F}_{p^e}$. Then*

$$\mathrm{Tr}_1^m(x)=\mathrm{Tr}_1^e(\mathrm{Tr}_e^m(x)),$$

for all $x\in\mathbb{F}_{p^m}$.

3. Basic Results

In this section, we provide some important results to establish our main results.

Lemma 3.1. *For each $\lambda\in\mathbb{F}_{p^e}$, set $S_\lambda=\#\{x\in\mathbb{F}_{p^m}:\mathrm{Tr}_e^m(x^2)=\lambda\}$. If s is odd, then*

$$S_\lambda=\begin{cases}p^{m-e}+p^{-e}\overline{\eta}(-1)\overline{\eta}(\lambda)G\overline{G}, & \text{if } \lambda\neq 0,\\ p^{m-e}, & \text{if } \lambda=0.\end{cases}$$

Proof. For each $\lambda\in\mathbb{F}_{p^e}$, we have

$$\begin{aligned}S_\lambda&=\frac{1}{p^e}\sum_{x\in\mathbb{F}_{p^m}}\left(\sum_{y\in\mathbb{F}_{p^e}}\zeta_p^{\mathrm{Tr}_1^e(y(\mathrm{Tr}_e^m(x^2)-\lambda))}\right)\\&=\frac{1}{p^e}\sum_{x\in\mathbb{F}_{p^m}}\left(1+\sum_{y\in\mathbb{F}_{p^e}^*}\zeta_p^{\mathrm{Tr}_e^m(yx^2)-\mathrm{Tr}_1^e(\lambda y)}\right)\\&=p^{m-e}+\sum_{y\in\mathbb{F}_{p^e}^*}\zeta_p^{-\mathrm{Tr}_1^e(\lambda y)}\sum_{x\in\mathbb{F}_{p^m}}\chi_1(yx^2)\\&=p^{m-e}+G\sum_{y\in\mathbb{F}_{p^e}^*}\overline{\chi}_1(-\lambda y)\eta(y).\end{aligned}$$

This completes the proof. □

Lemma 3.2. *For $\lambda, \mu \in \mathbb{F}_{p^e}$, define*

$$N(\lambda,\mu) = \#\{x \in \mathbb{F}_{p^m} : \mathrm{Tr}_e^m(x^2) = \lambda \text{ and } \mathrm{Tr}_e^m(x) = \mu\}.$$

Then the following assertions hold.

1. *If $2 \mid s$ and $p \mid s$, then*

$$N(\lambda,\mu) = \begin{cases} p^{m-2e} + p^{-e}(p^e-1)G, & \text{if } \lambda = 0 \text{ and } \mu = 0,\\ p^{m-2e}, & \text{if } \lambda = 0 \text{ and } \mu \neq 0,\\ p^{m-2e} - p^{-e}G, & \text{if } \lambda \neq 0 \text{ and } \mu = 0,\\ p^{m-2e}, & \text{if } \lambda \neq 0 \text{ and } \mu \neq 0.\end{cases}$$

2. *If $2 \mid s$ and $p \nmid s$, then*

$$N(\lambda,\mu) = \begin{cases} p^{m-2e}, & \text{if } \lambda = 0 \text{ and } \mu = 0,\\ p^{m-2e} + p^{-e}G, & \text{if } \lambda = 0 \text{ and } \mu \neq 0,\\ p^{m-2e}, & \text{if } \lambda \neq 0 \text{ and } \mu^2 - s\lambda = 0,\\ p^{m-2e} + \overline{\eta}(\mu^2 - s\lambda)p^{-e}G, & \text{if } \lambda \neq 0 \text{ and } \mu^2 - s\lambda \neq 0.\end{cases}$$

3. *If $2 \nmid s$ and $p \mid s$, then*

$$N(\lambda,\mu) = \begin{cases} p^{m-2e}, & \text{if } \lambda = 0,\\ p^{m-2e} + \overline{\eta}(-\lambda)p^{-e}G\overline{G}, & \text{if } \lambda \neq 0 \text{ and } \mu = 0,\\ p^{m-2e}, & \text{if } \lambda \neq 0 \text{ and } \mu \neq 0.\end{cases}$$

4. *If $2 \nmid s$ and $p \nmid s$, then*

$$N(\lambda,\mu) = \begin{cases} p^{m-2e} + \overline{\eta}(-s)p^{-2e}(p^e-1)G\overline{G}, & \text{if } \mu^2 - s\lambda = 0,\\ p^{m-2e} - \overline{\eta}(-s)p^{-2e}G\overline{G}, & \text{if } \mu^2 - s\lambda \neq 0.\end{cases}$$

Proof. By the properties of additive character and Lemma 2.4, we have

$$\begin{aligned} N(\lambda,\mu) &= p^{-2e}\sum_{x\in\mathbb{F}_{p^m}}\Big(\sum_{y\in\mathbb{F}_{p^e}}\zeta_p^{\mathrm{Tr}_1^e(y(\mathrm{Tr}_e^m(x^2)-\lambda))}\Big)\Big(\sum_{z\in\mathbb{F}_{p^e}}\zeta_p^{\mathrm{Tr}_1^e(z(\mathrm{Tr}_e^m(x)-\mu))}\Big)\\ &= p^{-2e}\sum_{x\in\mathbb{F}_{p^m}}\Big(1+\sum_{y\in\mathbb{F}_{p^e}^*}\zeta_p^{\mathrm{Tr}_1^e(y\mathrm{Tr}_e^m(x^2)-y\lambda)}\Big)\Big(1+\sum_{z\in\mathbb{F}_{p^e}^*}\zeta_p^{\mathrm{Tr}_1^e(z\mathrm{Tr}_e^m(x)-z\mu)}\Big)\\ &= p^{m-2e} + p^{-2e}(S_1+S_2+S_3), \end{aligned} \tag{3.1}$$

where

$$S_1 = \sum_{x\in\mathbb{F}_{p^m}}\sum_{z\in\mathbb{F}_{p^e}^*}\zeta_p^{\mathrm{Tr}_1^e(z\mathrm{Tr}_e^m(x)-z\mu)} = \sum_{z\in\mathbb{F}_{p^e}^*}\overline{\chi}_1(-z\mu)\sum_{x\in\mathbb{F}_{p^m}}\chi_1(zx) = 0,$$

$$S_2 = \sum_{x\in\mathbb{F}_{p^m}}\sum_{y\in\mathbb{F}_{p^e}^*}\zeta_p^{\mathrm{Tr}_1^e(y\mathrm{Tr}_e^m(x^2)-y\lambda)} = \sum_{y\in\mathbb{F}_{p^e}^*}\overline{\chi}_1(-y\lambda)\sum_{x\in\mathbb{F}_{p^m}}\chi_1(yx^2),$$

$$S_3 = \sum_{x\in\mathbb{F}_{p^m}}\sum_{y\in\mathbb{F}_{p^e}^*}\zeta_p^{\mathrm{Tr}_1^e(y\mathrm{Tr}_e^m(x^2)-y\lambda)}\sum_{z\in\mathbb{F}_{p^e}^*}\zeta_p^{\mathrm{Tr}_1^e(z\mathrm{Tr}_e^m(x)-z\mu)}$$

$$= \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-y\lambda) \sum_{z\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-z\mu) \sum_{x\in\mathbb{F}_{p^m}} \chi_1(yx^2+zx).$$

By Lemma 2.4, it is easy to prove that

$$S_2 = \begin{cases} G(p^e-1), & \text{if } \lambda = 0 \text{ and } 2 \mid s, \\ 0, & \text{if } \lambda = 0 \text{ and } 2 \nmid s, \\ -G, & \text{if } \lambda \neq 0 \text{ and } 2 \mid s, \\ \overline{\eta}(-\lambda)G\overline{G}, & \text{if } \lambda \neq 0 \text{ and } 2 \nmid s. \end{cases}$$

By Lemma 2.4, we have

$$\begin{aligned} S_3 &= \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-y\lambda) \sum_{z\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-z\mu) \sum_{x\in\mathbb{F}_{p^m}} \chi_1(yx^2+zx) \\ &= G \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y)\eta(y) \sum_{z\in\mathbb{F}_{p^e}^*} \overline{\chi}_1\left(-\frac{sz^2}{4y} - \mu z\right), \end{aligned}$$

and there are the following cases to consider.

Case I. Suppose that $2 \mid s$ and $p \mid s$. Then

$$S_3 = G \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y) \sum_{z\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\mu z) = \begin{cases} G(p^e-1)^2, & \text{if } \lambda = 0 \text{ and } \mu = 0, \\ -G(p^e-1), & \text{if } \lambda = 0 \text{ and } \mu \neq 0, \\ -G(p^e-1), & \text{if } \lambda \neq 0 \text{ and } \mu = 0, \\ G, & \text{if } \lambda \neq 0 \text{ and } \mu \neq 0. \end{cases}$$

Case II. We consider that $2 \mid s$ and $p \nmid s$. Then, from Lemma 2.4, we have

$$\begin{aligned} S_3 &= G \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y) \sum_{z\in\mathbb{F}_{p^e}^*} \overline{\chi}_1\left(-\frac{sz^2}{4y} - \mu z\right) \\ &= G \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1\left(\frac{\mu^2 - s\lambda}{s} y\right) \overline{\eta}\left(-\frac{s}{4y}\right) \overline{G} - G \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y) \\ &= \begin{cases} -G(p^e-1), & \text{if } \lambda = 0 \text{ and } \mu = 0, \\ G, & \text{if } \lambda = 0 \text{ and } \mu \neq 0, \\ G, & \text{if } \lambda \neq 0 \text{ and } \mu^2 - s\lambda = 0, \\ \left(\overline{\eta}(\mu^2 - s\lambda)p^e + 1\right)G, & \text{if } \lambda \neq 0 \text{ and } \mu^2 - s\lambda \neq 0. \end{cases} \end{aligned}$$

Case III. Assume that $2 \nmid s$ and $p \mid s$. Then

$$S_3 = G \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y)\overline{\eta}(y) \sum_{z\in\mathbb{F}_{p^e}^*} \overline{\chi}_1(-\mu z) = \begin{cases} 0, & \text{if } \lambda = 0, \\ \overline{\eta}(-\lambda)(p^e-1)G\overline{G}, & \text{if } \lambda \neq 0 \text{ and } \mu = 0, \\ -\overline{\eta}(-\lambda)G\overline{G}, & \text{if } \lambda \neq 0 \text{ and } \mu \neq 0. \end{cases}$$

Case IV. Suppose that $2 \nmid s$ and $p \nmid s$. Then, by Lemma 2.4, we have

$$\begin{aligned}
S_3 &= G \sum_{y \in \mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y)\overline{\eta}(y) \sum_{z \in \mathbb{F}_{p^e}^*} \overline{\chi}_1\left(-\frac{sz^2}{4y} - \mu z\right) \\
&= G\overline{G} \sum_{y \in \mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y)\overline{\eta}(y)\overline{\chi}_1\left(\frac{\mu^2 y}{s}\right)\overline{\eta}\left(-\frac{s}{4y}\right) - G \sum_{y \in \mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y)\overline{\eta}(y) \\
&= \overline{\eta}(-s)G\overline{G} \sum_{y \in \mathbb{F}_{p^e}^*} \overline{\chi}_1\left(\frac{\mu^2 - s\lambda}{s} y\right) - G \sum_{y \in \mathbb{F}_{p^e}^*} \overline{\chi}_1(-\lambda y)\overline{\eta}(y) \\
&= \begin{cases}
\overline{\eta}(-s)(p^e - 1)G\overline{G}, & \text{if } \lambda = 0 \text{ and } \mu = 0, \\
-\overline{\eta}(-s)G\overline{G}, & \text{if } \lambda = 0 \text{ and } \mu \neq 0, \\
\Big((p^e - 1)\overline{\eta}(-s) - \overline{\eta}(-\lambda)\Big)G\overline{G}, & \text{if } \lambda \neq 0 \text{ and } \mu^2 - s\lambda = 0, \\
-\Big(\overline{\eta}(-s) + \overline{\eta}(-\lambda)\Big)G\overline{G}, & \text{if } \lambda \neq 0 \text{ and } \mu^2 - s\lambda \neq 0.
\end{cases}
\end{aligned}$$

Combining (3.1) and the values of S_1, S_2 and S_3, we get the complete proof. □

Lemma 3.3. *Let the symbols be the same as before, and let*

$$\Omega_1 = \sum_{y \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\mathrm{Tr}_1^e(y \mathrm{Tr}_e^m(x^2 + x))}.$$

Then

$$\Omega_1 = \begin{cases}
(p^e - 1)G, & \text{if } 2 \mid s \text{ and } p \mid s, \\
-G, & \text{if } 2 \mid s \text{ and } p \nmid s, \\
0, & \text{if } 2 \nmid s \text{ and } p \mid s, \\
\overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \nmid s.
\end{cases}$$

Proof. By Lemmas 2.4 and 2.5, we have

$$\begin{aligned}
\Omega_1 &= \sum_{y \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \chi_1(yx^2 + yx) = G \sum_{y \in \mathbb{F}_{p^e}^*} \chi_1\left(-\frac{y}{4}\right)\eta(y) \\
&= G \sum_{y \in \mathbb{F}_{p^e}^*} \eta(y)\zeta_p^{\mathrm{Tr}_1^m(-\frac{y}{4})} = G \sum_{y \in \mathbb{F}_{p^e}^*} \eta(y)\zeta_p^{\mathrm{Tr}_1^e(-\frac{y}{4}\mathrm{Tr}_e^m(1))} \\
&= \begin{cases}
G \sum_{y \in \mathbb{F}_{p^e}^*} \eta(y), & \text{if } p \mid s, \\
G \sum_{y \in \mathbb{F}_{p^e}^*} \eta(y)\zeta_p^{\mathrm{Tr}_1^e(-\frac{ys}{4})}, & \text{if } p \nmid s.
\end{cases}
\end{aligned}$$

$$= \begin{cases} G \sum\limits_{y\in\mathbb{F}_{p^e}^*} 1, & \text{if } 2 \mid s \text{ and } p \mid s, \\ G \sum\limits_{y\in\mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(-\frac{ys}{4})}, & \text{if } 2 \mid s \text{ and } p \nmid s, \\ G \sum\limits_{y\in\mathbb{F}_{p^e}^*} \overline{\eta}(y), & \text{if } 2 \nmid s \text{ and } p \mid s, \\ G \sum\limits_{y\in\mathbb{F}_{p^e}^*} \overline{\eta}(y)\zeta_p^{\mathrm{Tr}_1^e(-\frac{ys}{4})}, & \text{if } 2 \nmid s \text{ and } p \nmid s, \end{cases}$$

$$= \begin{cases} (p^e-1)G, & \text{if } 2 \mid s \text{ and } p \mid s, \\ -G, & \text{if } 2 \mid s \text{ and } p \nmid s, \\ 0, & \text{if } 2 \nmid s \text{ and } p \mid s, \\ \overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \nmid s, \end{cases}$$

as required. ∎

Lemma 3.4. *For $b \in \mathbb{F}_{p^m}^*$ and $c \in \mathbb{F}_{p^e}^*$, let*

$$\Omega_3 = \sum_{z\in\mathbb{F}_{p^e}^*} \sum_{y\in\mathbb{F}_{p^e}^*} \sum_{x\in\mathbb{F}_{p^m}} \chi_1(yx^2 + yx + bzx).$$

Then we have the following statemets.

1. *If $\mathrm{Tr}_e^m(b^2) \neq 0$ and $\mathrm{Tr}_e^m(b) \neq 0$, then*

$$\Omega_3 = \begin{cases} \overline{\eta}(-1)G\overline{G}^2 - G(p^e-1), & \text{if } 2 \mid s \text{ and } p \mid s, \\ G, & \text{if } 2 \mid s,\ p \nmid s \text{ and } \left(\mathrm{Tr}_e^m(b)\right)^2 = s\mathrm{Tr}_e^m(b^2), \\ \overline{\eta}(s\mathrm{Tr}_e^m(b^2) - (\mathrm{Tr}_e^m(b)+2c)^2)G\overline{G}^2 + G, & \text{if } 2 \mid s,\ p \nmid s \text{ and } \left(\mathrm{Tr}_e^m(b)\right)^2 \neq s\mathrm{Tr}_e^m(b^2), \\ -\overline{\eta}(-\mathrm{Tr}_e^m(b^2))G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \mid s, \\ \overline{\eta}(-\mathrm{Tr}_e^m(b^2))G\overline{G}(p^e-1) - \overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s,\ p \nmid s \text{ and } \left(\mathrm{Tr}_e^m(b)\right)^2 = s\mathrm{Tr}_e^m(b^2), \\ -\overline{\eta}(-\mathrm{Tr}_e^m(b^2))G\overline{G} - \overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s,\ p \nmid s \text{ and } \left(\mathrm{Tr}_e^m(b)\right)^2 \neq s\mathrm{Tr}_e^m(b^2). \end{cases}$$

2. *If $\mathrm{Tr}_e^m(b^2) \neq 0$ and $\mathrm{Tr}_e^m(b) = 0$, then*

$$\Omega_3 = \begin{cases} -(p^e-1)G, & \text{if } 2 \mid s \text{ and } p \mid s, \\ \overline{\eta}(s\mathrm{Tr}_e^m(b^2))G\overline{G}^2 + G, & \text{if } 2 \mid s \text{ and } p \nmid s, \\ \overline{\eta}(-\mathrm{Tr}_e^m(b^2))(p^e-1)G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \mid s, \\ -\Big(\overline{\eta}(-\mathrm{Tr}_e^m(b^2)) + \overline{\eta}(-s)\Big)G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \nmid s. \end{cases}$$

3. *If $\mathrm{Tr}_e^m(b^2) = 0$ and $\mathrm{Tr}_e^m(b) \neq 0$, then*

$$\Omega_3 = \begin{cases} -(p^e-1)G, & \text{if } 2 \mid s \text{ and } p \mid s, \\ G, & \text{if } 2 \mid s \text{ and } p \nmid s, \\ 0, & \text{if } 2 \nmid s \text{ and } p \mid s, \\ -\overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \nmid s. \end{cases}$$

4. *If* $\mathrm{Tr}_e^m(b^2) = 0$ *and* $\mathrm{Tr}_e^m(b) = 0$, *then*

$$\Omega_3 = \begin{cases} (p^e-1)^2 G, & \text{if } 2 \mid s \text{ and } p \mid s, \\ -(p^e-1)G, & \text{if } 2 \mid s \text{ and } p \nmid s, \\ 0, & \text{if } 2 \nmid s \text{ and } p \mid s, \\ \overline{\eta}(-s)(p^e-1)G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \nmid s. \end{cases}$$

Proof. By Lemma 2.4, we have

$$\begin{aligned}
\Omega_3 &= \sum_{z\in\mathbb{F}_{p^e}^*} \sum_{y\in\mathbb{F}_{p^e}^*} \sum_{x\in\mathbb{F}_{p^m}} \chi_1(yx^2 + yx + bzx) \\
&= G \sum_{z\in\mathbb{F}_{p^e}^*} \sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\chi_1\left(-\frac{(y+bz)^2}{4y}\right) \\
&= G \sum_{z\in\mathbb{F}_{p^e}^*} \sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\chi_1\left(\frac{-y^2 - 2byz - b^2z^2}{4y}\right) \\
&= G \sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\chi_1\left(-\frac{y}{4}\right) \sum_{z\in\mathbb{F}_{p^e}^*} \chi_1\left(-\frac{bz}{2} - \frac{b^2z^2}{4y}\right) \\
&= G \sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\chi_1\left(-\frac{y}{4}\right) \sum_{z\in\mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^m(-\frac{bz}{2} - \frac{b^2z^2}{4y})} \\
&= G \sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\chi_1\left(-\frac{y}{4}\right) \sum_{z\in\mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e\left(-\frac{z}{2}\mathrm{Tr}_e^m(b) - \frac{z^2}{4y}\mathrm{Tr}_e^m(b^2)\right)}.
\end{aligned}$$

Note that, in the first part, $\mathrm{Tr}_e^m(b^2) \neq 0$ and $\mathrm{Tr}_e^m(b) \neq 0$. Therefore,

$$\begin{aligned}
\Omega_3 &= G \sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\chi_1(-\frac{y}{4}) \sum_{z\in\mathbb{F}_{p^e}^*} \overline{\chi}_1\left(-\frac{z^2}{4y}\mathrm{Tr}_e^m(b^2) - \frac{z}{2}\mathrm{Tr}_e^m(b)\right) \\
&= G \sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\chi_1\left(-\frac{y}{4}\right)\left(\sum_{z\in\mathbb{F}_{p^e}} \overline{\chi}_1\left(-\frac{z^2}{4y}\mathrm{Tr}_e^m(b^2) - \frac{z}{2}\mathrm{Tr}_e^m(b)\right) - 1\right) \\
&= G \sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\chi_1\left(-\frac{y}{4}\right) \sum_{z\in\mathbb{F}_{p^e}} \overline{\chi}_1\left(-\frac{z^2}{4y}\mathrm{Tr}_e^m(b^2) - \frac{z}{2}\mathrm{Tr}_e^m(b)\right) - G \sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\chi_1\left(-\frac{y}{4}\right) \\
&= G \sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\chi_1\left(-\frac{y}{4}\right) \overline{\chi}_1\left(\frac{y\left(\mathrm{Tr}_e^m(b)\right)^2}{4\mathrm{Tr}_e^m(b^2)}\right) \overline{\eta}\left(-y\mathrm{Tr}_e^m(b^2)\right)\overline{G} - G \sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\chi_1\left(-\frac{y}{4}\right) \\
&= \overline{\eta}\left(-\mathrm{Tr}_e^m(b^2)\right) G\overline{G} \sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\overline{\chi}_1\left(\frac{\left(\mathrm{Tr}_e^m(b)\right)^2 - s\mathrm{Tr}_e^m(b^2)}{4\mathrm{Tr}_e^m(b^2)} y\right)\overline{\eta}(y) - G \sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\chi_1\left(-\frac{y}{4}\right)
\end{aligned}$$

$$= \begin{cases} \overline{\eta}(-1)G\overline{G}^2 - G(p^e-1), & \text{if } 2 \mid s \text{ and } p \mid s,\\ G, & \text{if } 2 \mid s,\ p \nmid s \text{ and } \left(\mathrm{Tr}_e^m(b)\right)^2 = s\mathrm{Tr}_e^m(b^2),\\ \overline{\eta}(s\mathrm{Tr}_e^m(b^2) - (\mathrm{Tr}_e^m(b))^2)G\overline{G}^2 + G, & \text{if } 2 \mid s, p \nmid s \text{ and } \left(\mathrm{Tr}_e^m(b)\right)^2 \neq s\mathrm{Tr}_e^m(b^2),\\ -\overline{\eta}(-\mathrm{Tr}_e^m(b^2))G\overline{G}, & \text{if } 2 \nmid s \text{ and } p \mid s,\\ \overline{\eta}(-\mathrm{Tr}_e^m(b^2))G\overline{G}(p^e-1) - \overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s,\ p \nmid s \text{ and } \left(\mathrm{Tr}_e^m(b)\right)^2 = s\mathrm{Tr}_e^m(b^2),\\ -\overline{\eta}(-\mathrm{Tr}_e^m(b^2))G\overline{G} - \overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s, p \nmid s \text{ and } \left(\mathrm{Tr}_e^m(b)\right)^2 \neq s\mathrm{Tr}_e^m(b^2). \end{cases}$$

This completes the proof of the first part.

Following the similar arguments used in the first part, one can easily prove the remaining parts. □

Lemma 3.5. *For $\mu \in \mathbb{F}_{p^e}^*$, let $V = \#\{x \in \mathbb{F}_{p^m} : \mathrm{Tr}_e^m(x) = \mu$ and $\left(\mathrm{Tr}_e^m(x)\right)^2 = s\mathrm{Tr}_e^m(x^2)\}$. Then, for $p \nmid s$, we have*

$$V = \begin{cases} p^{m-2e}, & \textit{if } 2 \mid s,\\ p^{m-2e} + \overline{\eta}(-s)p^{-2e}(p^e-1)G\overline{G}, & \textit{if } 2 \nmid s. \end{cases}$$

Proof. We can rewrite V as

$$V = \#\left\{x \in \mathbb{F}_{p^m} : \mathrm{Tr}_e^m(x) = \mu \text{ and } \mathrm{Tr}_e^m(x^2) = \frac{\mu^2}{s}\right\}.$$

Then, by definition, we have

$$\begin{aligned} V &= p^{-2e} \sum_{x \in \mathbb{F}_{p^m}} \left(\sum_{y \in \mathbb{F}_{p^e}} \zeta_p^{\mathrm{Tr}_1^e\left(y(\mathrm{Tr}_e^m(x^2) - \frac{\mu^2}{s})\right)}\right) \left(\sum_{z \in \mathbb{F}_{p^e}} \zeta_p^{\mathrm{Tr}_1^e\left(z(\mathrm{Tr}_e^m(x) - \mu)\right)}\right)\\ &= p^{-2e} \sum_{x \in \mathbb{F}_{p^m}} \left(1 + \sum_{y \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(\mathrm{Tr}_e^m(yx^2) - \frac{y\mu^2}{s})}\right) \left(1 + \sum_{z \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(\mathrm{Tr}_e^m(zx) - z\mu)}\right)\\ &= p^{m-2e} + p^{-2e}(N_1 + N_2 + N_3), \end{aligned}$$

where

$$N_1 = \sum_{z \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\mathrm{Tr}_1^m(zx) - \mathrm{Tr}_1^e(z\mu)} = 0, \quad N_2 = \sum_{y \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\mathrm{Tr}_1^m(yx^2) - \mathrm{Tr}_1^e\left(\frac{y\mu^2}{s}\right)},$$

$$N_3 = \sum_{y \in \mathbb{F}_{p^e}^*} \sum_{z \in \mathbb{F}_{p^e}^*} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{\mathrm{Tr}_1^m(yx^2 + zx) - \mathrm{Tr}_1^e\left(\frac{y\mu^2}{s} + z\mu\right)}.$$

Now, by Lemma 2.4, we obtain

$$N_2 = \sum_{y \in \mathbb{F}_{p^e}^*} \zeta_p^{-\mathrm{Tr}_1^e\left(\frac{y\mu^2}{s}\right)} \chi(0)\eta(y)G$$

$$= \begin{cases} G \sum_{y\in\mathbb{F}_{p^e}^*} \zeta_p^{-\mathrm{Tr}_1^e\left(\frac{y\mu^2}{s}\right)}, & \text{if } 2 \mid s, \\ \overline{\eta}(-s)G \sum_{y\in\mathbb{F}_{p^e}^*} \overline{\eta}\left(-\frac{y\mu^2}{s}\right)\overline{\chi}_1\left(-\frac{y\mu^2}{s}\right), & \text{if } 2 \nmid s, \end{cases}$$

$$= \begin{cases} -G, & \text{if } 2 \mid s, \\ \overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s, \end{cases}$$

and

$$\begin{aligned} N_3 &= \sum_{z\in\mathbb{F}_{p^e}^*}\sum_{y\in\mathbb{F}_{p^e}^*} \zeta_p^{-\mathrm{Tr}_1^e\left(\frac{y\mu^2}{s}+z\mu\right)} \sum_{x\in\mathbb{F}_{p^m}} \chi_1(yx^2+zx) \\ &= \sum_{z\in\mathbb{F}_{p^e}^*}\sum_{y\in\mathbb{F}_{p^e}^*} \zeta_p^{-\mathrm{Tr}_1^e\left(\frac{y\mu^2}{s}+z\mu\right)} \chi_1\left(-\frac{z^2}{4y}\right)\eta(y)G \\ &= G\sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\overline{\chi}_1\left(-\frac{y\mu^2}{s}\right)\sum_{z\in\mathbb{F}_{p^e}^*}\overline{\chi}_1\left(-\frac{sz^2}{4y}-z\mu\right) \\ &= G\sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\overline{\chi}_1\left(-\frac{y\mu^2}{s}\right)\sum_{z\in\mathbb{F}_{p^e}}\overline{\chi}_1\left(-\frac{sz^2}{4y}-z\mu\right) - G\sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\overline{\chi}_1\left(-\frac{y\mu^2}{s}\right) \\ &= G\sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\overline{\chi}_1\left(-\frac{y\mu^2}{s}\right)\overline{\chi}_1\left(\frac{\mu^2}{s}y\right)\overline{\eta}\left(-\frac{s}{4y}\right)\overline{G} - G\sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\overline{\chi}_1\left(-\frac{y\mu^2}{s}\right) \\ &= \overline{\eta}(-s)G\overline{G}\sum_{y\in\mathbb{F}_{p^e}^*}\eta(y)\overline{\eta}(y) - G\sum_{y\in\mathbb{F}_{p^e}^*} \eta(y)\overline{\chi}_1\left(-\frac{y\mu^2}{s}\right) \\ &= \begin{cases} G, & \text{if } 2 \mid s, \\ \overline{\eta}(-s)(p^e-1)G\overline{G} - \overline{\eta}(-s)G\overline{G}, & \text{if } 2 \nmid s. \end{cases} \end{aligned}$$

Also

$$V = p^{m-2e} + p^{-2e}(N_1+N_2+N_3).$$

Thus, we get the desired result. □

Lemma 3.6. *Suppose that $\lambda, \mu \in \mathbb{F}_{p^e}^*$. For $i \in \{1,-1\}$, let K_i denote the number of pairs (λ,μ) such that $\overline{\eta}(\mu^2 - s\lambda) = i$. Then we have*

$$K_1 = \frac{1}{2}(p^e-1)(p^e-3), \quad K_{-1} = \frac{1}{2}(p^e-1)^2.$$

Proof. We can rewrite $\mu^2 - s\lambda \neq 0$ as

$$\frac{s\lambda}{\mu^2 - s\lambda} + 1 = \frac{\mu^2}{\mu^2 - s\lambda}. \tag{3.2}$$

Set $p^e = 2h+1$. Now, for any fixed $\overline{\mu}^2 - s\overline{\lambda}$ such that $\overline{\eta}(\overline{\mu}^2 - s\overline{\lambda}) = 1$, the number of the pairs (λ, μ^2) satisfying (3.2) is equal to

$$(0,0)^{(2,p^e)} + (1,0)^{(2,p^e)} = h - 1 \quad \text{(by Lemma 2.2)}.$$

Similarly, for a fixed $\overline{\mu}^2 - s\overline{\lambda}$ such that $\overline{\eta}(\overline{\mu}^2 - s\overline{\lambda}) = -1$, the number of pairs (λ, μ^2) satisfying (4.1) is equal to

$$(0,1)^{(2,p^e)} + (1,1)^{(2,p^e)} = h \quad \text{(from Lemma 2.2)}.$$

Consequently, the number of the pairs (λ, μ) such that $\overline{\eta}(\overline{\mu}^2 - s\overline{\lambda}) = 1$ (resp. $\overline{\eta}(\overline{\mu}^2 - s\overline{\lambda}) = -1$) is $2(h-1)$ (resp. $2h$). We conclude that $K_1 = (p^e-1)(h-1)$ (resp. $K_{-1} = (p^e-1)h$), and hence the result follows. ☐

Lemma 3.7. *Suppose that $\lambda, \mu \in \mathbb{F}_{p^e}^*$ and $\mu^2 - s\lambda \neq 0$. For $i \in \{1, -1\}$, let ψ_i denote the number of the pairs (λ, μ) such that $\overline{\eta}(-\lambda) = i$. Then we have*

$$\psi_1 = \begin{cases} \frac{1}{2}(p^e-1)(p^e-3), & \textit{if } \overline{\eta}(-s) = 1, \\ \frac{1}{2}(p^e-1)^2, & \textit{if } \overline{\eta}(-s) = -1, \end{cases}$$

and

$$\psi_{-1} = \begin{cases} \frac{1}{2}(p^e-1)^2, & \textit{if } \overline{\eta}(-s) = 1, \\ \frac{1}{2}(p^e-1)(p^e-3), & \textit{if } \overline{\eta}(-s) = -1. \end{cases}$$

Proof. The proof of the lemma is similar to the proof of Lemma 3.6 and is omitted here. ☐

4. Main Results

Our task in this section is to prove some lemmas needed to obtain a class of 3-weight and 5-weight linear codes over $\mathbb{F}_{p^e}$.

Now, let D be the defining set defined by

$$D = \{x \in \mathbb{F}_{p^m}^* : \mathrm{Tr}_e^m(x^2 + x) = 0\}.$$

Assume that $l_0 = |D| + 1$. Then

$$l_0 = \frac{1}{p^e} \sum_{x \in \mathbb{F}_{p^m}} \sum_{y \in \mathbb{F}_{p^e}} \zeta_p^{\mathrm{Tr}_1^e(y \mathrm{Tr}_e^m(x^2+x))} = p^{m-e} + \frac{1}{p^e} \sum_{x \in \mathbb{F}_{p^m}} \sum_{y \in \mathbb{F}_{p^e}^*} \zeta_p^{\mathrm{Tr}_1^e(y \mathrm{Tr}_e^m(x^2+x))}.$$

Define $N_b = \#\{x \in \mathbb{F}_{p^m} : \mathrm{Tr}_e^m(x^2+x) = 0 \text{ and } \mathrm{Tr}_e^m(bx) = 0\}$. Let $\mathrm{wt}(c_b)$ denote the Hamming-weight of the codeword $\boldsymbol{c}_b$ of the code $\mathcal{C}_D$. It is easy to verify that

$$\mathrm{wt}(c_b) = l_0 - N_b. \tag{4.1}$$

For $b \in \mathbb{F}_{p^m}^*$, we have

$$N_b = p^{-2e} \sum_{x \in \mathbb{F}_{p^m}} \left(\sum_{y \in \mathbb{F}_{p^e}} \zeta_p^{\mathrm{Tr}_1^e(y \mathrm{Tr}_e^m(x^2+x))} \right) \left(\sum_{z \in \mathbb{F}_{p^e}} \zeta_p^{\mathrm{Tr}_1^e(z \mathrm{Tr}_e^m(bx))} \right)$$

$$=p^{-2e}\sum_{x\in\mathbb{F}_{p^m}}\left(1+\sum_{y\in\mathbb{F}_{p^e}^*}\zeta_p^{\mathrm{Tr}_1^e(y\mathrm{Tr}_e^m(x^2+x))}\right)\left(1+\sum_{z\in\mathbb{F}_{p^e}^*}\zeta_p^{\mathrm{Tr}_1^e(z\mathrm{Tr}_e^m(bx))}\right)$$
$$=p^{m-2e}+p^{-2e}\sum_{y\in\mathbb{F}_{p^e}^*}\sum_{x\in\mathbb{F}_{p^m}}\zeta_p^{\mathrm{Tr}_1^e(\mathrm{Tr}_e^m(yx^2+yx))}+p^{-2e}\sum_{y\in\mathbb{F}_{p^e}^*}\sum_{x\in\mathbb{F}_{p^m}}\zeta_p^{\mathrm{Tr}_1^e(\mathrm{Tr}_e^m(zbx))}$$
$$+p^{-2e}\sum_{y\in\mathbb{F}_{p^e}^*}\sum_{z\in\mathbb{F}_{p^e}^*}\sum_{x\in\mathbb{F}_{p^m}}\zeta_p^{\mathrm{Tr}_1^e(\mathrm{Tr}_e^m(yx^2+yx+bzx))}$$
$$=p^{m-2e}+p^{-2e}\sum_{y\in\mathbb{F}_{p^e}^*}\sum_{x\in\mathbb{F}_{p^m}}\zeta_p^{\mathrm{Tr}_1^m(yx^2+yx)}+p^{-2e}\sum_{y\in\mathbb{F}_{p^e}^*}\sum_{z\in\mathbb{F}_{p^e}^*}\sum_{x\in\mathbb{F}_{p^m}}\zeta_p^{\mathrm{Tr}_1^m(yx^2+yx+bzx)}. \tag{4.2}$$

In this section, we calculate l_0, N_b and give the proofs of the main results.

4.1. **The first case of three-weight linear codes.** In this subsection, we consider that $2 \mid s$ and $p \mid s$. In order to determine the weight distribution of $\mathcal{C}_D$ of (1.1), we need the following lemma.

Lemma 4.1. *Let $b \in \mathbb{F}_{p^m}^*$ and the symbols be the same as before. Then*

$$N_b=\begin{cases} p^{m-2e}, & \text{if } \mathrm{Tr}_e^m(b^2)=0 \text{ and } \mathrm{Tr}_e^m(b)\neq 0,\\ & \text{or } \mathrm{Tr}_e^m(b^2)\neq 0 \text{ and } \mathrm{Tr}_e^m(b)=0,\\ p^{m-2e}+p^{-e}(p^e-1)G, & \text{if } \mathrm{Tr}_e^m(b^2)=0 \text{ and } \mathrm{Tr}_e^m(b)=0,\\ p^{m-2e}+p^{-e}G, & \text{if } \mathrm{Tr}_e^m(b^2)\neq 0 \text{ and } \mathrm{Tr}_e^m(b)\neq 0.\end{cases}$$

Proof. The proof of the lemma directly follows from (4.2), Lemmas 3.3 and 3.4. □

Theorem 4.1. *Let s be even and $p \mid s$. Then the code $\mathcal{C}_D$ of* (1.1) *is a $[p^{m-e}-1+p^{-e}(p^e-1)G, s]$ linear code with the weight distribution given in Table* 1*, where $G=-(-1)^{\frac{m(p-1)^2}{8}}p^{\frac{m}{2}}$.*

Table 1. The weight ditribution of the codes in Theorem 4.1

Weight w	Frequency A_w
0	1
$(p^e-1)p^{m-2e}$	$p^{m-2e}-1+p^{-e}(p^e-1)G$
$(p^e-1)p^{m-2e}+p^{-e}(p^e-1)G$	$2(p^e-1)p^{m-2e}-p^{-e}(p^e-1)G$
$(p^e-1)p^{m-2e}+p^{-e}(p^e-2)G$	$(p^e-1)^2p^{m-2e}$

Proof. By Lemma 3.3, we have

$$l_0=p^{m-e}+p^{-e}(p^e-1)G.$$

Combining (4.1) and Lemma 4.1, we have the following distinct cases.

Case I. If $\mathrm{Tr}_e^m(b^2)=0$ and $\mathrm{Tr}_e^m(b)\neq 0$ or $\mathrm{Tr}_e^m(b^2)\neq 0$ and $\mathrm{Tr}_e^m(b)=0$, then we obtain

$$\mathrm{wt}(c_b)=l_0-N_b=(p^e-1)p^{m-2e}+p^{-e}(p^e-1)G.$$

By Lemma 3.2, $\mathrm{wt}(c_b) = (p^e - 1)p^{m-2e} + p^{-e}(p^e - 1)G$ occurs $2(p^e - 1)p^{m-2e} - p^{-e}(p^e - 1)G$ times.

Case II. If $\mathrm{Tr}_e^m(b^2) = 0$ and $\mathrm{Tr}_e^m(b) = 0$, then we have $\mathrm{wt}(c_b) = l_0 - N_b = (p^e - 1)p^{m-2e}$. By Lemma 3.2, the frequency is $p^{m-2e} - 1 + p^{-e}(p^e - 1)G$.

Case III. If $\mathrm{Tr}_e^m(b^2) \neq 0$ and $\mathrm{Tr}_e^m(b) \neq 0$, then we have

$$\mathrm{wt}(c_b) = (p^e - 1)p^{m-2e} + p^{-e}(p^e - 2)G.$$

From Lemma 3.2, the frequency is $(p^e - 1)^2 p^{m-2e}$. Hence, the result is established. □

Example 4.1. Let $(p, m, s, e) = (3, 12, 6, 2)$. Then the corresponding code $\mathcal{C}_D$ has parameters $[58400, 6, 51840]$ and the weight enumerator $1 + 105624z^{51840} + 419904z^{51921} + 5912z^{52488}$.

4.2. **The second case of three-weight linear codes.** In this subsection, suppose $2 \mid s$ and $p \nmid s$. By (4.2), Lemmas 3.3 and 3.4, it is easy to get the following lemma.

Lemma 4.2. *Let $b \in \mathbb{F}_{p^m}^*$. Then*

$$N_b = \begin{cases} p^{m-2e}, & \text{if } \mathrm{Tr}_e^m(b^2) = 0 \text{ and } \mathrm{Tr}_e^m(b) \neq 0, \\ p^{m-2e} + \overline{\eta}(-s\mathrm{Tr}_e^m(b^2))p^{-e}G, & \text{if } \mathrm{Tr}_e^m(b^2) \neq 0 \text{ and } \mathrm{Tr}_e^m(b) = 0, \\ p^{m-2e} - p^{-e}G, & \text{if } \mathrm{Tr}_e^m(b^2) = 0 \text{ and } \mathrm{Tr}_e^m(b) = 0, \\ p^{m-2e}, & \text{if } \mathrm{Tr}_e^m(b^2) \neq 0,\ \mathrm{Tr}_e^m(b) \neq 0 \\ & \text{and } \left(\mathrm{Tr}_e^m(b)\right)^2 = s\mathrm{Tr}_e^m(b^2), \\ p^{m-2e} + \overline{\eta}\Big((\mathrm{Tr}_e^m(b))^2 - s\mathrm{Tr}_e^m(b^2)\Big)p^{-e}G, & \text{if } \mathrm{Tr}_e^m(b^2) \neq 0,\ \mathrm{Tr}_e^m(b) \neq 0 \\ & \text{and } \left(\mathrm{Tr}_e^m(b)\right)^2 \neq (s\mathrm{Tr}_e^m(b^2)). \end{cases}$$

Theorem 4.2. *Let $2 \mid s$ and $p \nmid s$. Then the code $\mathcal{C}_D$ of (1.1) is a $[p^{m-e} - p^{-e}G - 1, s]$ linear code with the weight distribution given in Table 2, where $G = -(-1)^{\frac{m(p-1)^2}{8}} p^{\frac{m}{2}}$.*

TABLE 2. The weight distribution for the codes in Theorem 4.2

Weight w	Frequency A_w
0	1
$(p^e - 1)p^{m-2e} - p^{-e}G$	$(p^e - 1)(2p^{m-2e} + p^{-e}G)$
$(p^e - 1)p^{m-2e}$	$\frac{1}{2}(p^e - 1)(p^{m-e} - G) + p^{m-2e} - 1$
$(p^e - 1)p^{m-2e} - 2p^{-e}G$	$\frac{1}{2}(p^e - 1)(p^e - 2)(p^{m-2e} + p^{-e}G)$

Proof. If $2 \mid s$ and $p \nmid s$, then by Lemma 3.3, we have

$$l_0 = p^{m-e} - p^{-e}G.$$

By (4.2) and Lemma 4.2, we have following distinct cases to consider.

Case I. If $\mathrm{Tr}_e^m(b^2) = 0$ and $\mathrm{Tr}_e^m(b) \neq 0$ or $\mathrm{Tr}_e^m(b^2) \neq 0$ and $\Big(\mathrm{Tr}_e^m(b)\Big)^2 = s\mathrm{Tr}_e^m(b^2)$, then we can acquire

$$\mathrm{wt}(c_b) = l_0 - N_b = (p^e - 1)p^{m-2e} - p^{-e}G.$$

By Lemmas 3.2 and 3.5, the frequency is $(p^e - 1)(2p^{m-2e} + p^{-e}G)$.

Case II. If $\mathrm{Tr}_e^m(b^2) \neq 0$, $\mathrm{Tr}_e^m(b) = 0$ and $\overline{\eta}(-s\mathrm{Tr}_e^m(b^2)) = 1$ or $\mathrm{Tr}_e^m(b^2) \neq 0, \mathrm{Tr}_e^m(b) \neq 0$, $\Big(\mathrm{Tr}_e^m(b)\Big)^2 \neq s\mathrm{Tr}_e^m(b^2)$ and $\overline{\eta}\Big((\mathrm{Tr}_e^m(b))^2 - s\mathrm{Tr}_e^m(b^2)\Big) = 1$, then we have

$$\mathrm{wt}(c_b) = l_0 - N_b = (p^e - 1)p^{m-2e} - 2p^{-e}G.$$

From Lemmas 3.2 and 3.5, the frequency is $\frac{1}{2}(p^e - 1)(p^e - 2)(p^{m-2e} + p^{-e}G)$.

Case III. If $\mathrm{Tr}_e^m(b^2) = 0$ and $\mathrm{Tr}_e^m(b) = 0$ or $\mathrm{Tr}_e^m(b^2) \neq 0$, $\mathrm{Tr}_e^m(b) = 0$ and $\overline{\eta}(-s\mathrm{Tr}_e^m(b^2)) = -1$ or $\mathrm{Tr}_e^m(b^2) \neq 0$, $\Big(\mathrm{Tr}_e^m(b)\Big)^2 \neq s\mathrm{Tr}_e^m(b^2)$ and $\overline{\eta}\Big((\mathrm{Tr}_e^m(b))^2 - s\mathrm{Tr}_e^m(b^2)\Big) = -1$, then

$$\mathrm{wt}(c_b) = l_0 - N_b = (p^e - 1)p^{m-2e}.$$

It follows from Lemmas 3.2 and 3.5 that $\mathrm{wt}(c_b) = (p^e - 1)p^{m-2e} - 2p^{-e}G$ occurs $\frac{1}{2}(p^e - 1)(p^{m-e} - G) + p^{m-2e} - 1$ times. Thus, the proof is completed. □

Example 4.2. Let $(p, m, s, e) = (3, 8, 4, 2)$. Then the corresponding code $\mathcal{C}_D$ has parameters $[737, 4, 648]$ and the weight enumerator $1 + 3320z^{648} + 1224z^{657} + 2016z^{666}$.

4.3. **The first case of 5-weight linear codes.** In this subsection, we assume that $2 \nmid s$ and $p \mid s$. By (4.2), Lemma 3.3 and Lemma 3.4, we get the following lemma.

Lemma 4.3. *For $b \in \mathbb{F}_{p^m}^*$ and $\mathrm{Tr}_e^m(b^2) \neq 0$, we have*

$$N_b = \begin{cases} p^{m-2e} - p^{-2e}\overline{\eta}(-1)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b) \neq 0 \text{ and } \overline{\eta}\Big(\mathrm{Tr}_e^m(b^2)\Big) = 1, \\ p^{m-2e} + p^{-2e}\overline{\eta}(-1)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b) \neq 0 \text{ and } \overline{\eta}\Big(\mathrm{Tr}_e^m(b^2)\Big) = -1, \\ p^{m-2e} + p^{-2e}\overline{\eta}(-1)(p^e - 1)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b) = 0 \text{ and } \overline{\eta}\Big(\mathrm{Tr}_e^m(b^2)\Big) = 1, \\ p^{m-2e} - p^{-2e}\overline{\eta}(-1)(p^e - 1)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b) = 0 \text{ and } \overline{\eta}\Big(\mathrm{Tr}_e^m(b^2)\Big) = -1. \end{cases}$$

Moreover, if $\mathrm{Tr}_e^m(b^2) = 0$, then $N_b = p^{m-2e}$.

Theorem 4.3. *Let $2 \nmid s$ and $p \mid s$. Then the linear code $\mathcal{C}_D$ of* (1.1) *has parameters $[p^{m-e} - 1, s]$ and weight distribution in Table* 3, *where $G\overline{G} = (-1)^{m+e-2}(-1)^{\frac{(p-1)^2(m+e)}{8}}p^{\frac{(m+e)}{2}}$.*

Proof. Note that $2 \nmid s$ and $p \mid s$. By Lemma 3.3, we have $l_0 = p^{m-e}$, which gives the length of the code $\mathcal{C}_D$. It follows from (4.1) and Lemma 4.3 that $\mathrm{wt}(c_b)$ has five distinct values under following cases.

Case I. If $\mathrm{Tr}_e^m(b^2) = 0$, then we have $\mathrm{wt}(c_b) = l_0 - N_b = (p^e - 1)p^{m-2e}$. By Lemma 3.1, the frequency of such codewords is $p^{m-e} - 1$.

Case II. If $\mathrm{Tr}_e^m(b^2) \neq 0$, $\mathrm{Tr}_e^m(b) \neq 0$ and $\overline{\eta}(\mathrm{Tr}_e^m(b^2)) = 1$, then we can acquire

$$\mathrm{wt}(c_b) = l_0 - N_b = (p^e - 1)p^{m-2e} + p^{-2e}\overline{\eta}(-1)G\overline{G}.$$

From Lemma 3.2, the frequency is $\frac{1}{2}(p^e-1)^2p^{m-2e}$.

Case III. If $\mathrm{Tr}_e^m(b^2)\neq 0$, $\mathrm{Tr}_e^m(b)\neq 0$ and $\overline{\eta}(\mathrm{Tr}_e^m(b^2))=-1$, then we can obtain

$$\mathrm{wt}(c_b)=l_0-N_b=(p^e-1)p^{m-2e}-p^{-2e}\overline{\eta}(-1)G\overline{G}.$$

It follows from Lemma 3.2 that the frequency is $\frac{1}{2}(p^e-1)^2p^{m-2e}$.

Case IV. If $\mathrm{Tr}_e^m(b^2)\neq 0$, $\mathrm{Tr}_e^m(b)=0$ and $\overline{\eta}(\mathrm{Tr}_e^m(b^2))=1$, then we can obtain

$$\mathrm{wt}(c_b)=l_0-N_b=(p^e-1)p^{m-2e}-p^{-2e}\overline{\eta}(-1)(p^e-1)G\overline{G}.$$

By Lemma 3.2, the frequency is $\frac{1}{2}(p^e-1)(p^{m-2e}+p^{-e}\overline{\eta}(-1)G\overline{G})$.

Case V. If $\mathrm{Tr}_e^m(b^2)\neq 0$, $\mathrm{Tr}_e^m(b)=0$ and $\overline{\eta}(\mathrm{Tr}_e^m(b^2))=-1$, then we have

$$\mathrm{wt}(c_b)=l_0-N_b=(p^e-1)p^{m-2e}+p^{-2e}\overline{\eta}(-1)(p^e-1)G\overline{G}.$$

From Lemma 3.2, the frequency is $\frac{1}{2}(p^e-1)(p^{m-2e}-p^{-e}\overline{\eta}(-1)G\overline{G})$. Hence, the result is established. □

Example 4.3. Let $(p,m,s,e)=(3,6,3,2)$. Then the corresponding code $\mathcal{C}_D$ has parameters $[80,3,64]$ and the weight enumerator $1+72z^{64}+288z^{71}+80z^{72}+288z^{73}$. By Table 3, $\mathcal{C}_D$ in Theorem 4.3 is a four-weight linear code if and only if $p=s=3$.

Table 3. The weight distribution of the codes in Theorem 4.3

Weight w	Frequency A_w
0	1
$(p^e-1)p^{m-2e}$	$p^{m-e}-1$
$(p^e-1)p^{m-2e}+p^{-2e}\overline{\eta}(-1)G\overline{G}$	$\frac{1}{2}(p^e-1)^2p^{m-2e}$
$(p^e-1)p^{m-2e}-p^{-2e}\overline{\eta}(-1)G\overline{G}$	$\frac{1}{2}(p^e-1)^2p^{m-2e}$
$(p^e-1)p^{m-2e}-p^{-2e}\overline{\eta}(-1)(p^e-1)G\overline{G}$	$\frac{1}{2}(p^e-1)(p^{m-2e}+p^{-e}\overline{\eta}(-1)G\overline{G})$
$(p^e-1)p^{m-2e}+p^{-2e}\overline{\eta}(-1)(p^e-1)G\overline{G}$	$\frac{1}{2}(p^e-1)(p^{m-2e}-p^{-e}\overline{\eta}(-1)G\overline{G})$

Example 4.4. Let $(p,m,s,e)=(5,10,5,2)$. Then the corresponding code $\mathcal{C}_D$ has parameters $[5^8-1,5,24\times 5^6-600]$ and the weight enumerator $1+A_{w_1}z^{w_1}+A_{w_2}z^{w_2}+A_{w_3}z^{w_3}+A_{w_4}z^{w_4}+A_{w_5}z^{w_5}$, where the values of A_{w_i} and w_i for $1\leq i\leq 5$, are given in Table 4.

Table 4. The weight distribution of the code in Theorem 4.3 for $(p,m,s,e)=(5,10,5,2)$

Weight	Frequency
$w_1=24\times 5^6-600$	$A_{w_1}=12(5^6+5^4)$
$w_2=24\times 5^6-25$	$A_{w_2}=12\times 24\times 5^6$
$w_3=24\times 5^6$	$A_{w_3}=5^8-1$
$w_4=24\times 5^6+25$	$A_{w_4}=12\times 24\times 5^6$
$w_5=24\times 5^6+600$	$A_{w_5}=12(5^6-5^4)$

4.4. **The second case of five-weight linear codes.** In this subsection, suppose $2 \nmid s$ and $p \nmid s$. The auxiliary result that we need is the following.

Lemma 4.4. *Let $b \in \mathbb{F}_{p^m}^*$ and the symbols be the same as before. Then*

$$N_b = \begin{cases} p^{m-2e}, & \text{if } \mathrm{Tr}_e^m(b^2) = 0 \text{ and } \mathrm{Tr}_e^m(b) \neq 0, \\ p^{m-2e} + p^{-e}\overline{\eta}(-s)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b^2) = 0 \text{ and } \mathrm{Tr}_e^m(b) = 0, \\ p^{m-2e} - p^{-2e}\overline{\eta}\Big(-\mathrm{Tr}_e^m(b^2)\Big)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b^2) \neq 0 \text{ and } \mathrm{Tr}_e^m(b) = 0, \\ p^{m-2e} + p^{-2e}\overline{\eta}(-s)(p^e-1)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b^2) \neq 0, \mathrm{Tr}_e^m(b) \neq 0 \text{ and} \\ & \Big(\mathrm{Tr}_e^m(b)\Big)^2 = s\mathrm{Tr}_e^m(b^2), \\ p^{m-2e} - p^{-2e}\overline{\eta}\Big(-\mathrm{Tr}_e^m(b^2)\Big)G\overline{G}, & \text{if } \mathrm{Tr}_e^m(b^2) \neq 0,\ \mathrm{Tr}_e^m(b) \neq 0 \text{ and} \\ & \Big(\mathrm{Tr}_e^m(b)\Big)^2 \neq s\mathrm{Tr}_e^m(b^2). \end{cases}$$

Proof. The proof of the lemma follows from (4.2), Lemmas 3.3 and 3.4. □

Theorem 4.4. *Let s be odd with $p \nmid s$. Then the linear code C_D of* (1.1) *has parameters $[p^{m-e} + p^{-e}\overline{\eta}(-s)G\overline{G} - 1, s]$ and weight distribution in Tables* 5 *and* 6*, where $G\overline{G} = (-1)^{m+e-2}(-1)^{\frac{(p-1)^2(m+e)}{8}}p^{\frac{(m+e)}{2}}$.*

Proof. Firstly, we assume that $\overline{\eta}(-s) = 1$. For $2 \nmid s$ and $p \nmid s$, by Lemma 3.3, we have

$$l_0 = p^{m-e} + p^{-e}G\overline{G}.$$

It follows from (4.1) and Lemma 4.4 that $\mathrm{wt}(c_b)$ has five distinct values under following cases.

Case I. If $\mathrm{Tr}_e^m(b^2) = 0$ and $\mathrm{Tr}_e^m(b) \neq 0$, then we have

$$\mathrm{wt}(c_b) = l_0 - N_b = (p^e-1)p^{m-2e} + p^{-e}G\overline{G}.$$

By Lemma 3.2, the frequency is $(p^e-1)(p^{m-2e} - p^{-2e}G\overline{G})$.

Case II. If $\mathrm{Tr}_e^m(b^2) = 0$ and $\mathrm{Tr}_e^m(b) = 0$, then $\mathrm{wt}(c_b) = l_0 - N_b = (p^e-1)p^{m-2e}$. From Lemma 3.2, the frequency is $p^{m-2e} + p^{-2e}(p^e-1)G\overline{G} - 1$.

Case III. If $\mathrm{Tr}_e^m(b^2) \neq 0$, $\mathrm{Tr}_e^m(b) \neq 0$ and $\Big(\mathrm{Tr}_e^m(b)\Big)^2 = s\mathrm{Tr}_e^m(b^2)$, then we can obtain

$$\mathrm{wt}(c_b) = l_0 - N_b = (p^e-1)p^{m-2e} + p^{-2e}G\overline{G}.$$

It follows from Lemmas 3.2 and 3.5 that the frequency of such codewords is $(p^e-1)p^{m-2e} + p^{-2e}(p^e-1)^2G\overline{G}$.

Case IV. If $\mathrm{Tr}_e^m(b^2) \neq 0$, $\mathrm{Tr}_e^m(b) = 0$ and $\overline{\eta}(-\mathrm{Tr}_e^m(b^2)) = 1$ or $\mathrm{Tr}_e^m(b^2) \neq 0$, $\mathrm{Tr}_e^m(b) \neq 0$, $\Big(\mathrm{Tr}_e^m(b)\Big)^2 \neq s\mathrm{Tr}_e^m(b^2)$ and $\overline{\eta}(-\mathrm{Tr}_e^m(b^2)) = 1$, then we can obtain

$$\mathrm{wt}(c_b) = l_0 - N_b = (p^e-1)p^{m-2e} + p^{-2e}(p^e+1)G\overline{G}.$$

By Lemmas 3.2 and 3.7, the frequency is $\frac{1}{2}(p^e-1)(p^{m-e} - p^{-e}G\overline{G})$.

Case V. If $\mathrm{Tr}_e^m(b^2) \neq 0$, $\mathrm{Tr}_e^m(b) = 0$ and $\overline{\eta}(-\mathrm{Tr}_e^m(b^2)) = -1$ or $\mathrm{Tr}_e^m(b^2) \neq 0$, $\mathrm{Tr}_e^m(b) \neq 0$, $\left(\mathrm{Tr}_e^m(b)\right)^2 \neq s\mathrm{Tr}_e^m(b^2)$ and $\overline{\eta}(-\mathrm{Tr}_e^m(b^2)) = -1$, then we have

$$\mathrm{wt}(c_b) = l_0 - N_b = (p^e-1)p^{m-2e} + p^{-2e}(p^e-1)G\overline{G}.$$

From Lemmas 3.2 and 3.7, the frequency is $\frac{1}{2}(p^e-1)(p^e-2)(p^{m-2e} - p^{-2e}G\overline{G})$, which completes the Table 5. Similarly, we can complete the Table 6 by taking $\overline{\eta}(-s) = -1$. □

Table 5. The weight distribution of the codes in Theorem 4.4 with $\overline{\eta}(-s) = 1$

Weight w	Frequency A_w
0	1
$(p^e-1)p^{m-2e} + p^{-e}G\overline{G}$	$(p^e-1)(p^{m-2e} - p^{-2e}G\overline{G})$
$(p^e-1)p^{m-2e}$	$p^{m-2e} + p^{-2e}(p^e-1)G\overline{G} - 1$
$(p^e-1)p^{m-2e} + p^{-2e}G\overline{G}$	$(p^e-1)p^{m-2e} + p^{-2e}(p^e-1)^2G\overline{G}$
$(p^e-1)p^{m-2e} + p^{-2e}(p^e+1)G\overline{G}$	$\frac{1}{2}(p^e-1)(p^{m-e} - p^{-e}G\overline{G})$
$(p^e-1)p^{m-2e} + p^{-2e}(p^e-1)G\overline{G}$	$\frac{1}{2}(p^e-1)(p^e-2)(p^{m-2e} - p^{-2e}G\overline{G})$

Table 6. The weight distribution of the codes in Theorem 4.4 with $\overline{\eta}(-s) = -1$

Weight w	Frequency A_w
0	1
$(p^e-1)p^{m-2e} - p^{-e}G\overline{G}$	$(p^e-1)(p^{m-2e} + p^{-2e}G\overline{G})$
$(p^e-1)p^{m-2e}$	$p^{m-2e} - p^{-2e}(p^e-1)G\overline{G} - 1$
$(p^e-1)p^{m-2e} - p^{-2e}G\overline{G}$	$(p^e-1)p^{m-2e} - p^{-2e}(p^e-1)^2G\overline{G}$
$(p^e-1)p^{m-2e} - p^{-2e}(p^e-1)G\overline{G}$	$\frac{1}{2}(p^e-1)(p^{m-e} + p^{-e}G\overline{G})$
$(p^e-1)p^{m-2e} - p^{-2e}(p^e+1)G\overline{G}$	$\frac{1}{2}(p^e-1)(p^e-2)(p^{m-2e} + p^{-2e}G\overline{G})$

Example 4.5. Let $(p, m, s, e) = (5, 6, 3, 2)$. Then the corresponding code $\mathcal{C}_D$ has parameters $[649, 3, 600]$ and the weight enumerator as $1 + 48z^{600} + 1176z^{601} + 6624z^{624} + 576z^{625} + 7200z^{626}$.

5. Concluding Remarks

In this paper, we have presented a class of three-weight and five-weight linear codes. A number of three-weight and five-weight codes were discussed in [1,3,4,6,9,14,19,20].

Let w_0 and w_∞ denote the minimum and maximum non-zero weight of a linear code $\mathcal{C}_D$, respectively. The linear code $\mathcal{C}_D$ with $\frac{w_0}{w_\infty} > \frac{(p^e-1)}{p^e}$ can be used to construct a secret sharing scheme with interesting access structures (see [18]).

For the linear code $\mathcal{C}_D$ in Theorem 4.1, we have

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e} - (p^e-1)p^{\frac{m-2e}{2}}}{(p^e-1)p^{m-2e}} \quad \text{or} \quad \frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e}}{(p^e-1)p^{m-2e} + (p^e-1)p^{\frac{m-2e}{2}}}.$$

Let $\frac{m}{e} > 4$. Then by simple computation, we have

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e}}{(p^e-1)p^{m-2e} + (p^e-1)p^{\frac{m-2e}{2}}} > \frac{(p^e-1)p^{m-2e} - (p^e-1)p^{\frac{m-2e}{2}}}{(p^e-1)p^{m-2e}} > \frac{(p^e-1)}{p^e}.$$

For the linear code $\mathcal{C}_D$ of Theorem 4.2, we have

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e} - 2p^{\frac{m-2e}{2}}}{(p^e-1)p^{m-2e}} \quad \text{or} \quad \frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e}}{(p^e-1)p^{m-2e} + 2p^{\frac{m-2e}{2}}}.$$

Then it can easily be checked that

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e}}{(p^e-1)p^{m-2e} + 2p^{\frac{m-2e}{2}}} > \frac{(p^e-1)p^{m-2e} - 2p^{\frac{m-2e}{2}}}{(p^e-1)p^{m-2e}} > \frac{(p^e-1)}{p^e}, \quad \text{for } \frac{m}{e} > 4.$$

For the linear code $\mathcal{C}_D$ of Theorem 4.3, we have

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e} - (p^e-1)p^{\frac{m-3e}{2}}}{(p^e-1)p^{m-2e} + (p^e-1)p^{\frac{m-3e}{2}}} > \frac{(p^e-1)}{p^e}, \quad \text{for } \frac{m}{e} \geq 5.$$

For the linear code $\mathcal{C}_D$ of Theorem 4.4, we have

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e} - (p^e+1)p^{\frac{m-3e}{2}}}{(p^e-1)p^{m-2e}} \quad \text{or} \quad \frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e}}{(p^e-1)p^{m-2e} + (p^e+1)p^{\frac{m-3e}{2}}}.$$

Let $\frac{m}{e} \geq 5$. Then by simple calculation, we can show that

$$\frac{w_0}{w_\infty} = \frac{(p^e-1)p^{m-2e}}{(p^e-1)p^{m-2e} + (p^e+1)p^{\frac{m-3e}{2}}} > \frac{(p^e-1)p^{m-2e} - (p^e+1)p^{\frac{m-3e}{2}}}{(p^e-1)p^{m-2e}} > \frac{(p^e-1)}{p^e}.$$

Consequently, one can easily see that the codewords of the linear code $\mathcal{C}_D$ are minimal for $\frac{m}{e} \geq 5$. These linear codes can be used in secret sharing schemes.

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References

[1] S. T. Choi, J. S. No and J. Y. Kim, *Weight distribution of some cyclic codes*, in: 2012 *IEEE International Symposium on Information Theory Proceedings* IEEE, Cambridge, MA, USA, 2012, 2911–2913. `https://doi.org/10.1109/ISIT.2012.6284056`

[2] C. Ding, T. Kløve and F. Sica, *Two classes of ternary codes and their weight distributions*, Discrete Appl. Math. **111**(1–2) (2001), 37–53. `https://doi.org/10.1016/S0166-218X(00)00343-7`

[3] C. Ding, *A class of three-weight and four-weight codes*, in: Y. M. Chee, C. Li, S. Ling, H. Wang and C. Xing (Eds), *Proceedings of Second International Workshop on Coding Theory and Cryptography*, Lecture Notes in Computer Science **5557**, Springer-Verlag, Berlin, Heidelberg, 2009, 34–42. https://doi.org/10.1007/978-3-642-01877-0_4

[4] C. Ding, Y. Gao and Z. Zhou, *Five families of three-weight ternary cyclic codes and their duals*, IEEE Trans. Inform. Theory **59**(12) (2013), 7940–7946. https://doi.org/10.1109/TIT.2013.2281205

[5] C. Ding and J. Yang, *Hamming weights in irreducible cyclic codes*, Discrete Math. **313**(4) (2013), 434–446. https://doi.org/10.1016/j.disc.2012.11.009

[6] K. Ding and C. Ding, *Binary linear codes with three weights*, IEEE Communications Letters **18**(11) (2014), 1879–1882. https://doi.org/10.1109/LCOMM.2014.2361516

[7] C. Ding, *Linear codes from some* $2-$*designs*, IEEE Trans. Inform. Theory **61**(6) (2015), 3265–3275. https://doi.org/10.1109/TIT.2015.2420118

[8] K. Ding and C. Ding, *A class of two-weight and three-weight codes and their applications in secret sharing*, IEEE Trans. Inform. Theory **61**(11) (2015), 5828–5842. https://doi.org/10.1109/TIT.2015.2473861

[9] C. Ding, C. Li, N. Li and Z. Zhou, *Three-weight cyclic codes and their weight distributions*, Discrete Math. **339**(2) (2016), 415–427. https://doi.org/10.1016/j.disc.2015.09.001

[10] W. C. Huffman and V. Pless, *Fundamentals of Error-Correcting Codes*, Cambridge University Press, Cambridge, 2003.

[11] F. Li, Q. Wang and D. Lin, *A class of three-weight and five-weight linear codes*, Discrete Appl. Math. **241** (2018), 25–38. https://doi.org/10.1016/j.dam.2016.11.005

[12] R. Lidl. and H. Niederreiter, *Finite Fields*, Cambridge University Press, New York, 1997.

[13] Y. Liu and Z. Liu, *Complete weight enumerators of a new class of linear codes*, Discrete Math. **341**(7) (2018), 1959–1972. https://doi.org/10.1016/j.disc.2018.03.025

[14] G. Luo, X. Cao, S. Xu and J. Mi, *Binary linear codes with two or three weights from niho exponents*, Cryptogr. Commun. **10**(2) (2018), 301–318. https://doi.org/10.1007/s12095-017-0220-2

[15] A. Sharma and G. K. Bakshi, *The weight distribution of some irreducible cyclic codes*, Finite Fields Appl. **18**(1) (2012), 144–159. https://doi.org/10.1016/j.ffa.2011.07.002

[16] T. Storer, *Cyclotomy and Difference Sets*, Markham Publishing Company, Markham, Chicago, 1967.

[17] Q. Wang, K. Ding and R. Xue, *Binary linear codes with two-weights*, IEEE Communications Letters **19**(7) (2015), 1097–1100. https://doi.org/10.1109/LCOMM.2015.2431253

[18] J. Yuan and C. Ding, *Secret sharing schemes from three classes of linear codes*, IEEE Trans. Inform. Theory **52**(1) (2006), 206–212. https://doi.org/10.1109/TIT.2005.860412

[19] Z. Zhou and C. Ding, *A class of three-weight cyclic codes*, Finite Fields App. **25** (2014), 79–93. https://doi.org/10.1016/j.ffa.2013.08.005

[20] Z. Zhou, N. Li, C. Fan and T. Helleseth, *Linear codes with two or three weights from quadratic bent functions*, Des. Codes Cryptogr. **81**(2) (2016), 283–295. https://link.springer.com/article/10.1007/s10623-015-0144-9

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